CSE 484 / CSE M 584: Computer Security and Privacy

Cryptography

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Admin

• HW2: Due Nov 7, 4:30pm

• Looking ahead, rough plan:
  • Lab 2 out ~Nov 5, due ~Nov 19 (Quiz Section on Nov 8)
  • HW 3 out ~Nov 19, due ~Nov 30
  • Lab 3 out ~Nov 26, due Dec 7 (Quiz Section on Nov 29)

• HW1s were awesome
Public Key Encryption
Requirements for Public Key Encryption

- **Key generation:** computationally easy to generate a pair (public key \( PK \), private key \( SK \))

- **Encryption:** given plaintext \( M \) and public key \( PK \), easy to compute ciphertext \( C=E_{PK}(M) \)

- **Decryption:** given ciphertext \( C=E_{PK}(M) \) and private key \( SK \), easy to compute plaintext \( M \)
  - Infeasible to learn anything about \( M \) from \( C \) without \( SK \)
  - Trapdoor function: \( \text{Decrypt}(SK, Encrypt(PK, M)) = M \)
Some Number Theory Facts

• Euler totient function $\varphi(n)$ ($n \geq 1$) is the number of integers in the $[1,n]$ interval that are relatively prime to $n$
  – Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  – Easy to compute for primes: $\varphi(p) = p-1$
  – Note that $\varphi(ab) = \varphi(a) \varphi(b)$
RSA Cryptosystem [Rivest, Shamir, Adleman 1977]

- **Key generation:**
  - Generate large primes $p$, $q$
    - Say, 1024 bits each (need primality testing, too)
  - Compute $n=pq$ and $\varphi(n)=(p-1)(q-1)$
  - Choose small $e$, relatively prime to $\varphi(n)$
    - Typically, $e=3$ or $e=2^{16}+1=65537$
  - Compute unique $d$ such that $ed \equiv 1 \mod \varphi(n)$
    - Modular inverse: $d \equiv e^{-1} \mod \varphi(n)$
  - Public key = $(e,n)$; private key = $(d,n)$

- **Encryption** of $m$ ($m$ a number between 0 and $n-1$): $c = m^e \mod n$
- **Decryption** of $c$: $c^d \mod n = (m^e \mod n)^d \mod n = m$
Why Decryption Works (FYI)

- Decryption of \( c \): \( c^d \mod n = (m^e \mod n)^d \mod n = (m^e)^d \mod n = m \)
- Recall \( n=pq \) and \( \varphi(n)=(p-1)(q-1) \) and \( ed \equiv 1 \mod \varphi(n) \)

- Chinese Remainder Theorem: To show \( m^{ed} \mod n \equiv m \mod n \), sufficient to show:
  - \( m^{ed} \mod p \equiv m \mod p \)
  - \( m^{ed} \mod q \equiv m \mod q \)

- If \( m \equiv 0 \mod p \) \( \rightarrow m^{ed} \equiv 0 \mod p \)

- Else \( m^{ed} = m^{ed-1}m = m^{k(q-1)(p-1)}m = m^{h(p-1)}m \) for some \( k \), and \( h=k(q-1) \). Why? Recall how \( d \) was chosen and the definition of \( \mod \).
- Fermat Little Theorem: \( m^{(p-1)h}m \equiv 1^hm \mod p \equiv m \mod p \)
Why is RSA Secure?

- **RSA problem:** given $c$, $n=pq$, and $e$ such that $\gcd(e, \varphi(n))=1$, find $m$ such that $m^e = c \mod n$
  - In other words, recover $m$ from ciphertext $c$ and public key $(n,e)$ by taking $e$th root of $c$ modulo $n$
  - There is no known efficient algorithm for doing this

- **Factoring problem:** given positive integer $n$, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$

- If factoring is easy, then RSA problem is easy (knowing factors means you can compute $d = \text{inverse of } e \mod (p-1)(q-1)$)
  - It may be possible to break RSA without factoring $n$ -- but if it is, we don’t know how
RSA Encryption Caveats

• Encrypted message needs to be interpreted as an integer less than $n$
• Don’t use RSA directly for privacy – output is deterministic! Need to pre-process input somehow
• Plain RSA also does not provide integrity
  – Can tamper with encrypted messages

In practice, OAEP is used: instead of encrypting $M$, encrypt $M \text{xor } G(r); r \text{xor } H(M \text{xor } G(r))$
  – $r$ is random and fresh, $G$ and $H$ are hash functions
More on RSA + OAEP

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Question: How do you decrypt a message encrypted with RSA + OAEP?
OAEP as a Figure

- \( M \oplus G(r) \oplus r \oplus H(M \oplus G(r)) \)

- Do you see how to invert? (Side note, similar to DES internals)
Digital Signatures
Digital Signatures: Basic Idea

**Given:** Everybody knows Bob’s **public key**
Only Bob knows the corresponding **private key**

**Goal:** Bob sends a “digitally signed” message
1. To compute a signature, must know the private key
2. To verify a signature, only the public key is needed
RSA Signatures

• Public key is \((n,e)\), private key is \((n,d)\)
• To sign message \(m\): \(s = m^d \mod n\)
  – Signing & decryption are same underlying operation in RSA
  – It’s infeasible to compute \(s\) on \(m\) if you don’t know \(d\)
• To verify signature \(s\) on message \(m\):
  verify that \(s^e \mod n = (m^d)^e \mod n = m\)
  – “Just like encryption” (for RSA primitive)
  – Anyone who knows \(n\) and \(e\) (public key) can verify signatures produced with \(d\) (private key)
• “Just like encryption” in quotes!
  – In practice, also need padding & hashing
  – Standard padding/hashing schemes exist for RSA signatures
DSS Signatures

- Digital Signature Standard (DSS)
- Public key: \((p, q, g, y=g^x \mod p)\), private key: \(x\)
- Security of DSS requires hardness of discrete log
  - If could solve discrete logarithm problem, would extract \(x\) (private key) from \(g^x \mod p\) (public key)

- Important Note: We have discussed discrete logs modulo integers.
- Significant advantages in using elliptic curve groups – groups with some similar mathematical properties (i.e., are “groups”) but have better security and performance (size) properties
Stepping Back
Cryptography Summary

• Goal: Privacy
  – Symmetric keys:
    • One-time pad, Stream ciphers
    • Block ciphers (e.g., DES, AES) \(\rightarrow\) modes: EBC, CBC, CTR
  – Public key crypto (e.g., Diffie-Hellman, RSA)

• Goal: Integrity
  – MACs, often using hash functions (e.g., MD5, SHA-256)

• Goal: Privacy and Integrity
  – Encrypt-then-MAC

• Goal: Authenticity (and Integrity)
  – Digital signatures (e.g., RSA, DSS)