CSE 484 / CSE M 584: Computer Security and Privacy

Cryptography:
Asymmetric Cryptography (finish)

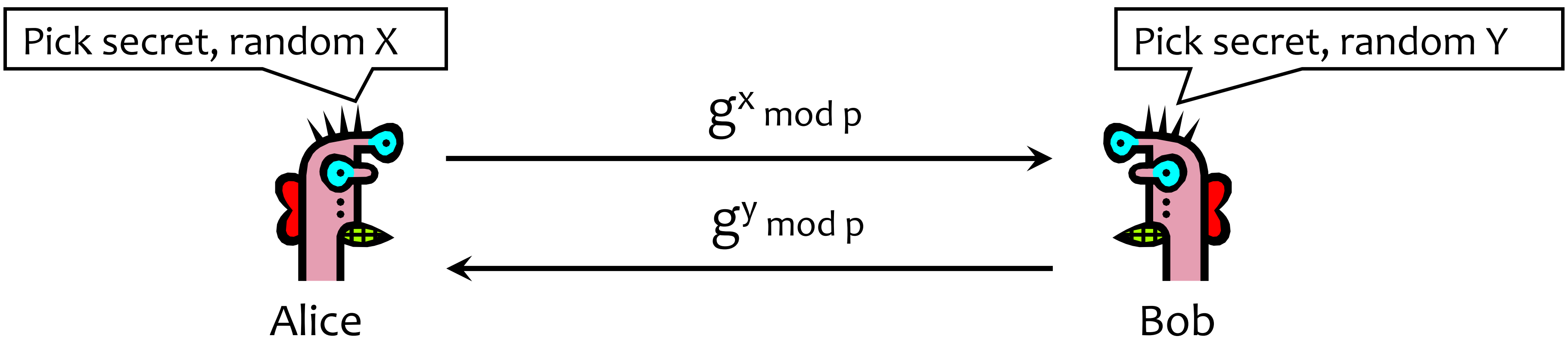
Spring 2016

Franziska (Franzi) Roesner
franzi@cs.washington.edu

Thanks to Dan Boneh, Dieter Gollmann, Dan Halperin, Yoshi Kohno, John Manferdelli, John Mitchell, Vitaly Shmatikov, Bennet Yee, and many others for sample slides and materials ...

Diffie-Hellman Protocol (1976)

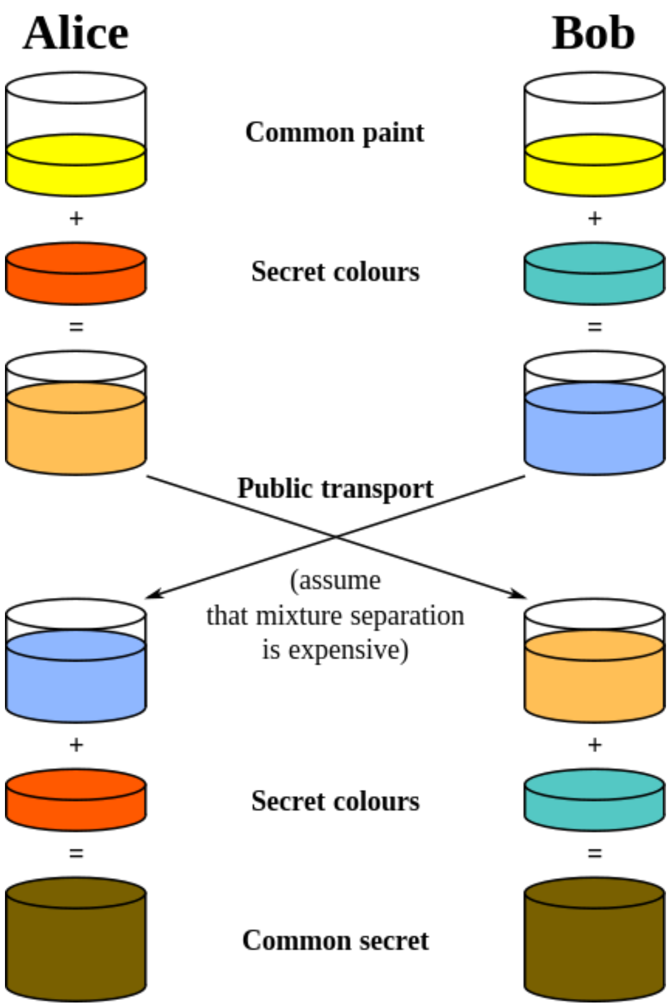
- Alice and Bob never met and share no secrets
- Public info: p and g
  - p is a large prime number, g is a generator of Z_p*
    - $Z_p^* = \{1, 2 \ldots p-1\}$; $\forall a \in Z_p^*$ $\exists i$ such that $a = g^i \bmod p$
    - Modular arithmetic: numbers "wrap around" after they reach p



Compute $k=(g^y)^x=g^{xy} \bmod p$

Compute $k=(g^x)^y=g^{xy} \bmod p$

Diffie-Hellman: Conceptually



Common paint: p and g

Secret colors: x and y

Send over public transport:
$g^x \bmod p$
$g^y \bmod p$

Common secret: $g^{xy} \bmod p$

[from Wikipedia]

Why is Diffie-Hellman Secure?

- Discrete Logarithm (DL) problem:
  given g^x *mod p*, it's hard to extract x
  - There is no known efficient algorithm for doing this
  - This is not enough for Diffie-Hellman to be secure!
- Computational Diffie-Hellman (CDH) problem:
  given g^x and g^y, it's hard to compute g^{xy} *mod p*
  - … unless you know x or y, in which case it's easy
- Decisional Diffie-Hellman (DDH) problem:

  given g^x and g^y, it's hard to tell the difference between g^{xy} *mod p* and g^r *mod p* where r is random

Properties of Diffie-Hellman

- Assuming DDH problem is hard (depends on choice of parameters!), Diffie-Hellman protocol is a secure key establishment protocol against <u>passive</u> attackers
  - Eavesdropper can't tell the difference between the established key and a random value
  - Can use the new key for symmetric cryptography
- Diffie-Hellman protocol (by itself) does not provide authentication

Requirements for Public Key Encryption

- Key generation: computationally easy to generate a pair (public key PK, private key SK)
- Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to learn anything about M from C without SK
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

Some Number Theory Facts

- Euler totient function $\varphi(n)$ ($n \geq 1$) is the number of integers in the $[1,n]$ interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  - Easy to compute for primes: $\varphi(p) = p-1$
  - Note that $\varphi(ab) = \varphi(a)\,\varphi(b)$

Some Number Theory Facts

- Euler totient function $\varphi(n)$ ($n \geq 1$) is the number of integers in the $[1,n]$ interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  - Easy to compute for primes: $\varphi(p) = p-1$
  - Note that $\varphi(ab) = \varphi(a)\,\varphi(b)$
- Euler's theorem: if $a \in Z_n^*$, then $a^{\varphi(n)} = 1 \bmod n$

  Z_n^*: integers relatively prime to n

RSA Cryptosystem [Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q
    - Say, 1024 bits each (need primality testing, too)
  - Compute **n**=pq and **φ(n)**=(p-1)(q-1)
  - Choose small e, relatively prime to φ(n)
    - Typically, $e=3$ or $e=2^{16}+1=65537$
  - Compute unique d such that ed = 1 mod φ(n)
    - Modular inverse: $d = e^{-1} \bmod \varphi(n)$
  - Public key = (e,n); private key = (d,n)
- Encryption of m: $c = m^e \bmod n$
- Decryption of c: $c^d \bmod n = (m^e)^d \bmod n = m$

Why RSA Decryption Works

$e \cdot d = 1 \bmod \varphi(n)$, thus $e \cdot d = 1 + k \cdot \varphi(n)$ for some k

Let m be any integer in Z_n^* (not all of Z_n)

$c^d \bmod n = (\mathbf{m}^e)^d \bmod n = \mathbf{m}^{1+k \cdot \varphi(n)} \bmod n$

$= (\mathbf{m} \bmod n) * (\mathbf{m}^{k \cdot \varphi(n)} \bmod n)$

Recall: Euler's theorem: if $a \in Z_n^*$, then $a^{\varphi(n)} = 1 \bmod n$

$c^d \bmod n = (\mathbf{m} \bmod n) * (\mathbf{1} \bmod n)$

$= \mathbf{m} \bmod n$

Proof omitted: True for all m in Z_n, not just m in Z_n^*

Why is RSA Secure?

- RSA problem: given c, $n=pq$, and e such that $\gcd(e, \varphi(n))=1$, find m such that $m^e=c \mod n$
  - In other words, recover m from ciphertext c and public key (n,e) by taking e^{th} root of c modulo n
  - There is no known efficient algorithm for doing this
- Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$
- If factoring is easy, then RSA problem is easy (knowing factors means you can compute d = inverse of e mod (p-1)(q-1))
  - It may be possible to break RSA without factoring n -- but if it is, we don't know how

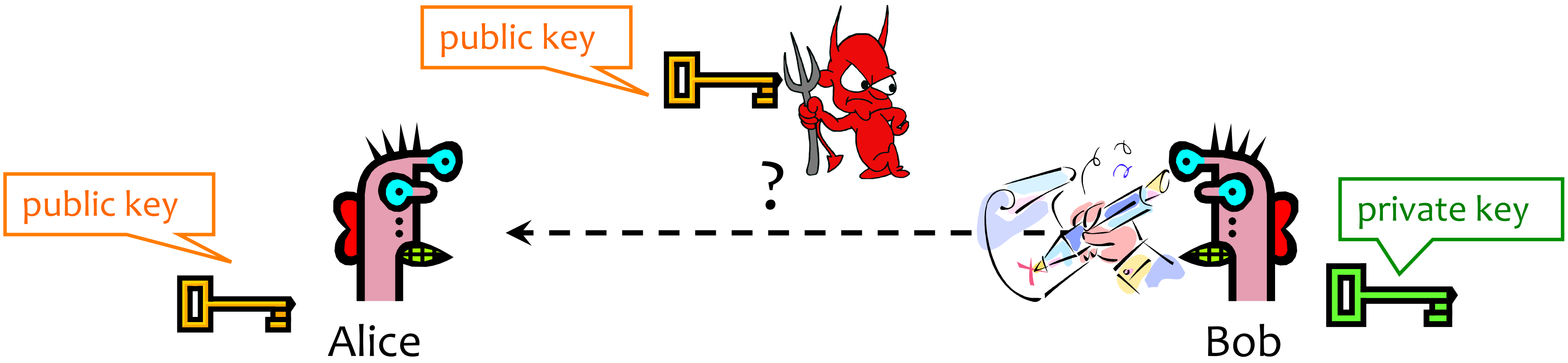
RSA Encryption Caveats

- Encrypted message needs to be interpreted as an integer less than n
- Don't use RSA **directly** for privacy – output is deterministic! Need to pre-process input somehow
- Plain RSA also does not provide integrity
  - Can tamper with encrypted messages

In practice, OAEP is used: instead of encrypting M, encrypt $M \oplus G(r)\ ;\ r \oplus H(M \oplus G(r))$

- r is random and fresh, G and H are hash functions

Digital Signatures: Basic Idea



Given: Everybody knows Bob's public key
Only Bob knows the corresponding private key

Goal: Bob sends a "digitally signed" message

1. To compute a signature, must know the private key
2. To verify a signature, only the public key is needed

RSA Signatures

- Public key is **(n,e)**, private key is **(n,d)**
- To sign message m: s = m^d mod n
  - Signing & decryption are same **underlying** operation in RSA
  - It's infeasible to compute **s** on **m** if you don't know **d**
- To verify signature s on message m:

  verify that s^e mod n = $(m^d)^e$ mod n = m
  - Just like encryption (for RSA primitive)
  - Anyone who knows **n** and **e** (public key) can verify signatures produced with d (private key)
- In practice, also need padding & hashing
  - Standard padding/hashing schemes exist for RSA signatures

DSS Signatures

- Digital Signature Standard (DSS)
  - U.S. government standard (1991, most recent rev. 2013)
- Public key: (p, q, g, $y=g^x \bmod p$), private key: x
- Security of DSS requires hardness of discrete log
  - If could solve discrete logarithm problem, would extract x (private key) from $g^x \bmod p$ (public key)

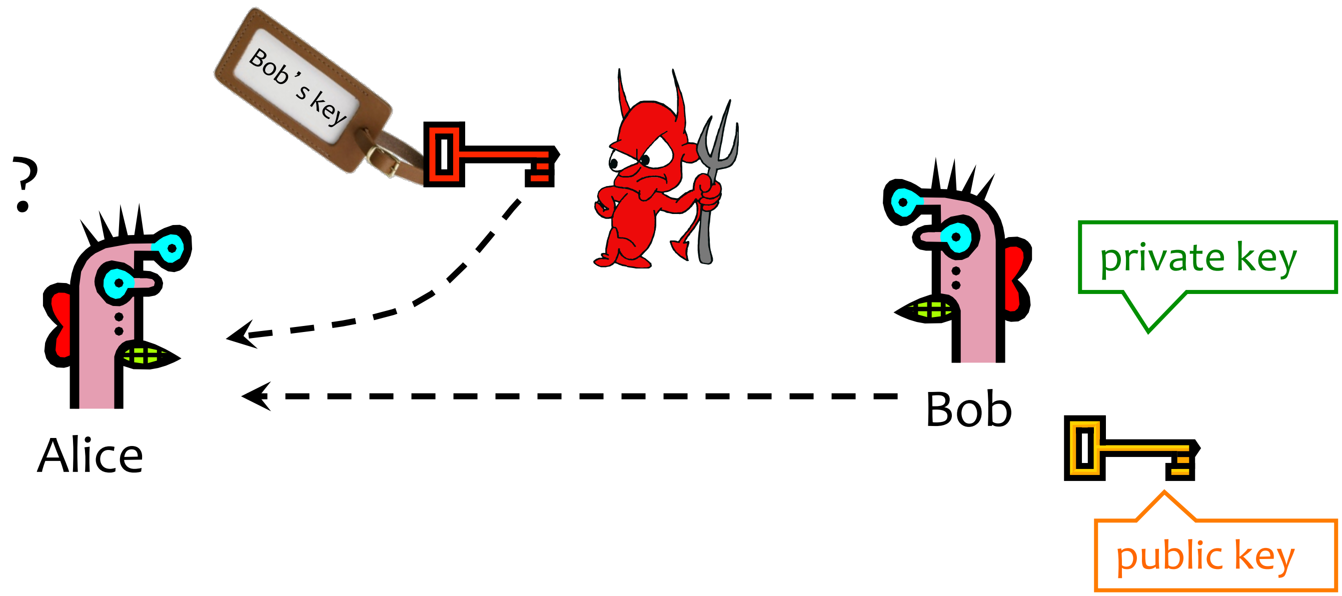
Advantages of Public Key Crypto

- Confidentiality without shared secrets
  - Very useful in open environments
  - Can use this for key establishment, with fewer "chicken-or-egg" problems
    - With symmetric crypto, two parties must share a secret before they can exchange secret messages
- Authentication without shared secrets
  - Use digital signatures to prove the origin of messages
- Encryption keys are public, but must be sure that Alice's public key is really *her* public key
  - This is a hard problem…

Disadvantages of Public Key Crypto

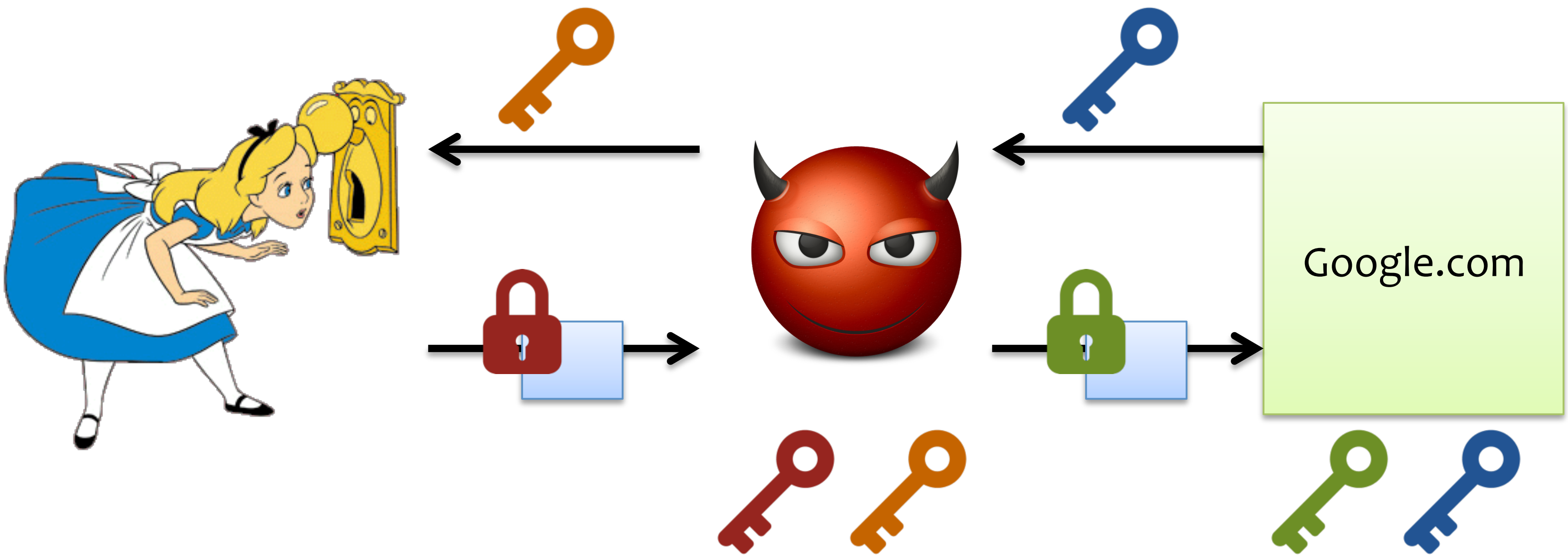
- Calculations are 2-3 orders of magnitude slower
  - Modular exponentiation is an expensive computation
  - Typical usage: use public-key cryptography to establish a shared secret, then switch to symmetric crypto
    - E.g., IPsec, SSL, SSH, ...
- Keys are longer
  - 1024+ bits (RSA) rather than 128 bits (AES)
- Relies on unproven number-theoretic assumptions
  - What if factoring is easy?
    - Factoring is *believed* to be neither P, nor NP-complete
  - (Of course, symmetric crypto also rests on unproven assumptions…)

Authenticity of Public Keys



Problem: How does Alice know that the public key she received is really Bob's public key?

Threat: Man-In-The-Middle (MITM)



Distribution of Public Keys

- Public announcement or public directory
  - Risks: forgery and tampering
- Public-key certificate
  - Signed statement specifying the key and identity
    - sig_{CA}("Bob", PK_B)
- Common approach: certificate authority (CA)
  - Single agency responsible for certifying public keys
  - After generating a private/public key pair, user proves his identity and knowledge of the private key to obtain CA's certificate for the public key (offline)
  - Every computer is pre-configured with CA's public key