CSE 484 / CSE M 584: Computer Security and Privacy

# Cryptography:
## Asymmetric Cryptography
[continued]

Spring 2015

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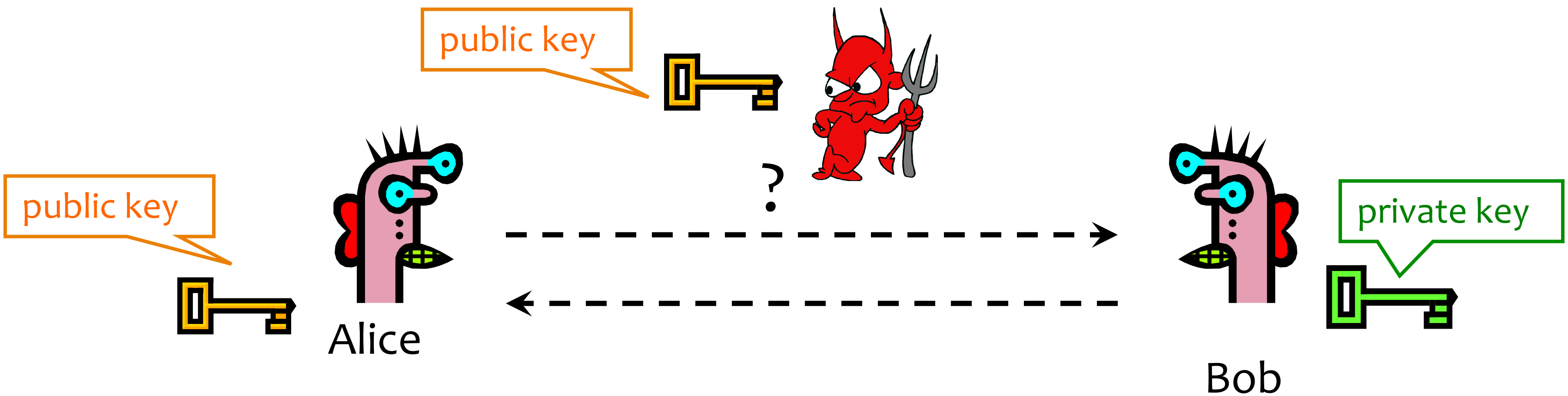
Thanks to Dan Boneh, Dieter Gollmann, Dan Halperin, Yoshi Kohno, John Manferdelli, John Mitchell, Vitaly Shmatikov, Bennet Yee, and many others for sample slides and materials ...

# Public Key Crypto: Basic Problem

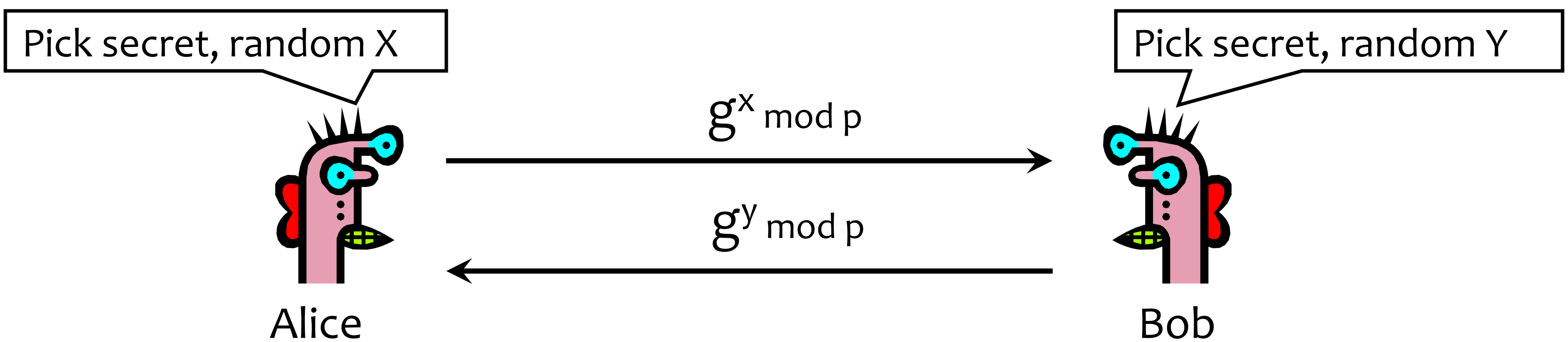



Given: Everybody knows Bob's public key
Only Bob knows the corresponding private key

Goals: 1. Alice wants to send a secret message to Bob
2. Bob wants to authenticate himself

# Diffie-Hellman Protocol (1976)

- Alice and Bob never met and share no secrets
- Public info: $p$ and $g$
  - $p$ is a large prime number, $g$ is a generator of $Z_p^*$
    - $Z_p^* = \{1, 2 \ldots p\text{-}1\}$; $\forall a \in Z_p^* \; \exists i$ such that $a = g^i \bmod p$
    - Modular arithmetic: numbers "wrap around" after they reach p




Compute $k = (g^y)^x = g^{xy} \bmod p$

Compute $k = (g^x)^y = g^{xy} \bmod p$

# Why is Diffie-Hellman Secure?

- Discrete Logarithm (DL) problem:
  given $g^x$ *mod p*, it's hard to extract x
  - There is no known efficient algorithm for doing this
  - This is not enough for Diffie-Hellman to be secure!
- Computational Diffie-Hellman (CDH) problem:
  given $g^x$ and $g^y$, it's hard to compute $g^{xy}$ *mod p*
  - … unless you know x or y, in which case it's easy
- Decisional Diffie-Hellman (DDH) problem:
  given $g^x$ and $g^y$, it's hard to tell the difference between $g^{xy}$ *mod p* and $g^r$ *mod p* where r is random

# Properties of Diffie-Hellman

- Assuming DDH problem is hard (depends on choice of parameters!), Diffie-Hellman protocol is a secure key establishment protocol against <u>passive</u> attackers
  - Eavesdropper can't tell the difference between the established key and a random value
  - Can use the new key for symmetric cryptography
- Diffie-Hellman protocol (by itself) does not provide authentication

# Requirements for Public Key Encryption

- Key generation: computationally easy to generate a pair (public key PK, private key SK)
- Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to learn anything about M from C without SK
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

# Some Number Theory Facts

- Euler totient function $\varphi(n)$ ($n \geq 1$) is the number of integers in the [1,n] interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  - Easy to compute for primes: $\varphi(p) = p-1$
  - Note that $\varphi(ab) = \varphi(a)\,\varphi(b)$

- Euler's theorem: if $a \in Z_n^*$, then $a^{\varphi(n)} = 1 \bmod n$

  $Z_n^*$: integers relatively prime to n

# RSA Cryptosystem [Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q
    - Say, 1024 bits each (need primality testing, too)
  - Compute $n=pq$ and $\varphi(n)=(p-1)(q-1)$
  - Choose small e, relatively prime to $\varphi(n)$
    - Typically, $e=3$ (can be vulnerable) or $e=2^{16}+1=65537$
  - Compute unique d such that $ed = 1 \bmod \varphi(n)$
    - Modular inverse: $d = e^{-1} \bmod \varphi(n)$
  - Public key = (e,n); private key = (d,n)
- Encryption of m: $c = m^e \bmod n$
- Decryption of c: $c^d \bmod n = (m^e)^d \bmod n = m$

# Why RSA Decryption Works

$e \cdot d = 1 \bmod \varphi(n)$, thus $e \cdot d = 1 + k \cdot \varphi(n)$ for some $k$

Let m be any integer in $Z_n$* (not all of $Z_n$)

$c^d \bmod n = (m^e)^d \bmod n = m^{1+k \cdot \varphi(n)} \bmod n$

$= (m \bmod n) * (m^{k \cdot \varphi(n)} \bmod n)$

Recall: Euler's theorem: if $a \in Z_n^*$, then $a^{\varphi(n)} = 1 \bmod n$

$c^d \bmod n = (m \bmod n) * (1 \bmod n)$

$= m \bmod n$

Proof omitted: True for all m in $Z_n$, not just m in $Z_n$*

# Why is RSA Secure?

- RSA problem: given c, n=pq, and e such that $\gcd(e, \varphi(n))=1$, find m such that $m^e=c \bmod n$
  - In other words, recover m from ciphertext c and public key (n,e) by taking $e^{th}$ root of c modulo n
  - There is no known efficient algorithm for doing this
- Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$
- If factoring is easy, then RSA problem is easy (knowing factors means you can compute d = inverse of e mod (p-1)(q-1))
  - It may be possible to break RSA without factoring n -- but if it is, we don't know how

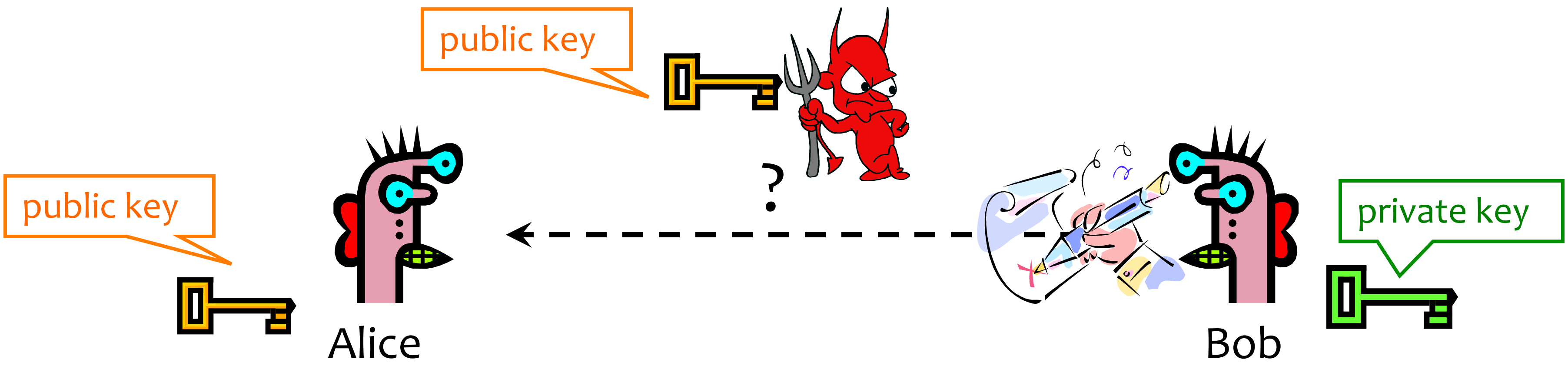
# RSA Encryption Caveats

- Encrypted message needs to be interpreted as an integer less than n
- Don't use RSA **directly** for privacy – output is deterministic! Need to pre-process input somehow
- Plain RSA also does <u>not</u> provide integrity
  - Can tamper with encrypted messages

In practice, OAEP is used: instead of encrypting M, encrypt $M \oplus G(r)\ ;\ r \oplus H(M \oplus G(r))$

- r is random and fresh, G and H are hash functions

# Digital Signatures: Basic Idea




Given: Everybody knows Bob's public key
Only Bob knows the corresponding private key

Goal: Bob sends a "digitally signed" message

1. To compute a signature, must know the private key
2. To verify a signature, only the public key is needed

# RSA Signatures

- Public key is (n,e), private key is (n,d)
- To sign message m: $s = m^d \bmod n$
  - Signing & decryption are same **underlying** operation in RSA
  - It's infeasible to compute s on m if you don't know d
- To verify signature s on message m:

  verify that $s^e \bmod n = (m^d)^e \bmod n = m$
  - Just like encryption
  - Anyone who knows n and e (public key) can verify signatures produced with d (private key)
- In practice, also need padding & hashing
  - Standard padding/hashing schemes exist for RSA signatures

# DSS Signatures

- Digital Signature Standard (DSS)
  – U.S. government standard (1991, most recent rev. 2013)
- Public key: (p, q, g, $y=g^x \bmod p$), private key: x
- Security of DSS requires hardness of discrete log
  – If could solve discrete logarithm problem, would extract x (private key) from $g^x \bmod p$ (public key)

# Advantages of Public Key Crypto

- Confidentiality without shared secrets
  - Very useful in open environments
  - Can use this for key establishment, with fewer "chicken-or-egg" problems
    - With symmetric crypto, two parties must share a secret before they can exchange secret messages
- Authentication without shared secrets
  - Use digital signatures to prove the origin of messages
- Encryption keys are public, but must be sure that Alice's public key is really *her* public key
  - This is a hard problem…

# Disadvantages of Public Key Crypto

- Calculations are 2-3 orders of magnitude slower
  - Modular exponentiation is an expensive computation
  - Typical usage: use public-key cryptography to establish a shared secret, then switch to symmetric crypto
    - E.g., IPsec, SSL, SSH, ...
- Keys are longer
  - 1024+ bits (RSA) rather than 128 bits (AES)
- Relies on unproven number-theoretic assumptions
  - What if factoring is easy?
    - Factoring is *believed* to be neither P, nor NP-complete
  - (Of course, symmetric crypto also rests on unproven assumptions…)