Asymmetric Cryptography

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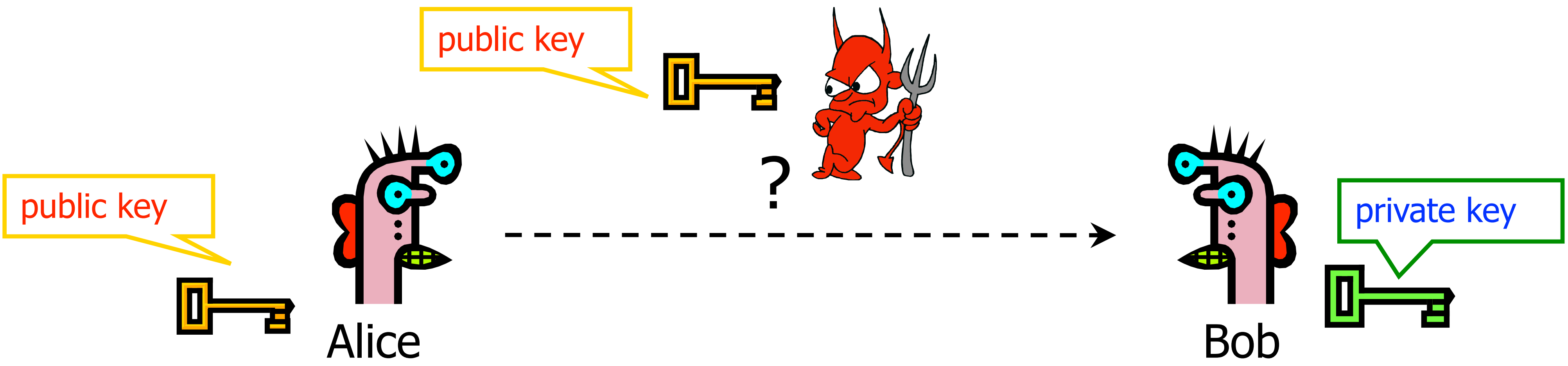
Thanks to Dan Boneh, Dieter Gollmann, John Manferdelli, John Mitchell, Vitaly Shmatikov, Bennet Yee, and many others for sample slides and materials ...

(Reminder:) Symmetric Cryptography

- **1 secret key**, shared between sender/receiver
- Repeat fast and simple operations lots of times (rounds) to mix up key and ciphertext
- **Why do we think it is secure?** (simplistic)
  - If we do lots and lots and lots of mixing, no simple formula (and reversible) describing the whole process (cryptographic weakness).
  - Mix in ways we think it's hard to short-circuit all the rounds. Especially non-linear mixing, e.g., S-boxes.
  - Some math gives us confidence in these assumptions

Public Key Cryptography

Basic Problem



<u>Given</u>: Everybody knows Bob's public key

Only Bob knows the corresponding private key

<u>Goals</u>: 1. Alice wants to send a secret message to Bob

2. Bob wants to authenticate himself

Public-Key Cryptography

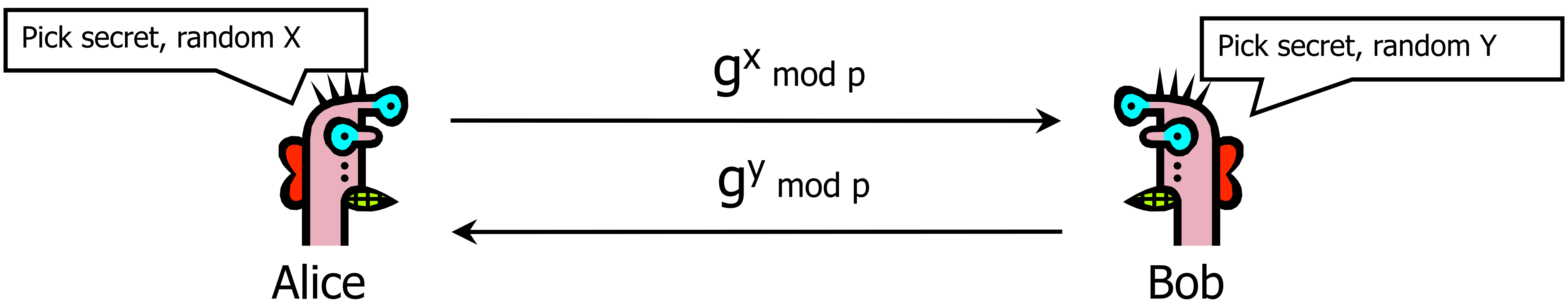
- Everyone has **1 private key and 1 public key**
- Mathematical relationship between private and public keys
- **Why do we think it is secure?** (simplistic)
  - Relies entirely on **problems we believe are "hard"**

Applications of Public-Key Crypto

- Encryption for confidentiality
  - Anyone can encrypt a message
    - With symmetric crypto, must know secret key to encrypt
  - Only someone who knows private key can decrypt
  - Key management is simpler (or at least different)
    - Secret is stored only at one site: good for open environments
- Digital signatures for authentication
  - Can “sign” a message with your private key
- Session key establishment
  - Exchange messages to create a secret session key
  - Then switch to symmetric cryptography (why?)

Diffie-Hellman Protocol (1976)

- Alice and Bob never met and share no secrets
- Public info: p and g
  - p is a large prime number, g is a generator of Z_p^*
    - $Z_p^* = \{1, 2 \ldots p-1\}$; $\forall a \in Z_p^* \; \exists i$ such that $a = g^i \bmod p$
    - Modular arithmetic: numbers "wrap around" after they reach p



Compute $k = (g^y)^x = g^{xy} \bmod p$

Compute $k = (g^x)^y = g^{xy} \bmod p$

Why Is Diffie-Hellman Secure?

- Discrete Logarithm (DL) problem:

  given g^x mod p, it's hard to extract x
  - There is no known efficient algorithm for doing this
  - This is not enough for Diffie-Hellman to be secure!
- Computational Diffie-Hellman (CDH) problem:

  given g^x and g^y, it's hard to compute g^{xy} mod p
  - … unless you know x or y, in which case it's easy
- Decisional Diffie-Hellman (DDH) problem:

  given g^x and g^y, it's hard to tell the difference between g^{xy} mod p and g^r mod p where r is random

Properties of Diffie-Hellman

- Assuming DDH problem is hard, Diffie-Hellman protocol is a secure key establishment protocol against <u>passive</u> attackers
  - Eavesdropper can't tell the difference between established key and a random value
  - Can use new key for symmetric cryptography
    - Approx. 1000 times faster than modular exponentiation
- Diffie-Hellman protocol (by itself) does not provide authentication

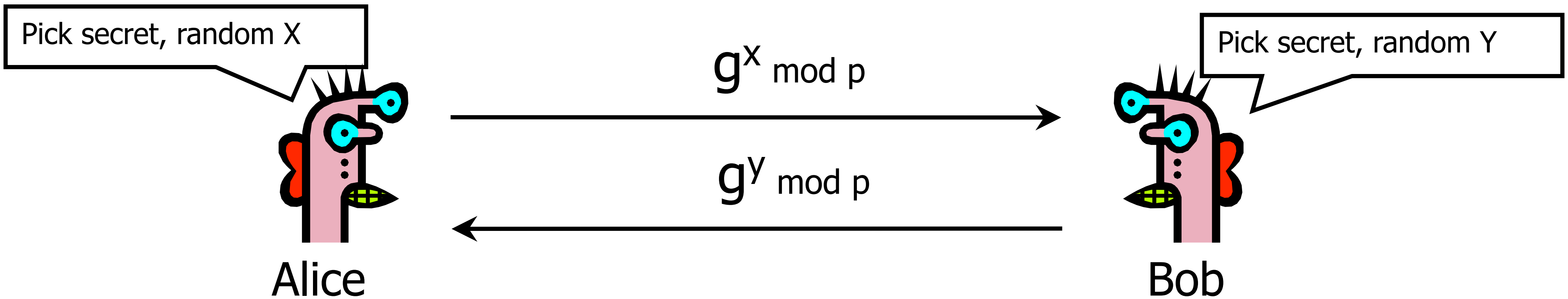
Properties of Diffie-Hellman

- DDH: not true for integers mod p, but true for other groups
- DL problem in p can be broken down into DL problems for subgroups, if factorization of p-1 is known.
- Common recommendation:
  - Choose p = 2q+1 where q is also a large prime.
  - Pick a g that generates a subgroup of order q in Z_p*
    - DDH is hard for this group
    - (OK to not know all the details of why for this course.)

  - Hash output of DH key exchange to get the key

Diffie-Hellman Protocol (1976)

- Alice and Bob never met and share no secrets
- Public info: p and g
  - p, q are large prime numbers, p=2q+1, g a generator for the subgroup of order q
    - Modular arithmetic: numbers "wrap around" after they reach p



Compute $k=H((g^y)^x)=H(g^{xy} \bmod p)$ Compute $k=H((g^x)^y)=H(g^{xy} \bmod p)$

Requirements for Public-Key Encryption

- Key generation: computationally easy to generate a pair (public key PK, private key SK)
  - Computationally infeasible to determine private key SK given only public key PK
- Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to compute M from C without SK
  - Even infeasible to learn partial information about M
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

Some Number Theory Facts

- Euler totient function $\varphi(n)$ where $n \geq 1$ is the number of integers in the $[1,n]$ interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
- Euler's theorem:

  if $a \in Z_n^*$, then $a^{\varphi(n)} = 1 \bmod n$

  Z_n^*: multiplicative group of integers mod n (integers relatively prime to n)
- Special case: <u>Fermat's Little Theorem</u>

  if p is prime and $\gcd(a,p)=1$, then $a^{p-1} = 1 \bmod p$

RSA Cryptosystem

[Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q
    - Say, 1024 bits each (need primality testing, too)
  - Compute $n=pq$ and $\varphi(n)=(p-1)(q-1)$
  - Choose small e, relatively prime to $\varphi(n)$
    - Typically, $e=3$ or $e=2^{16}+1=65537$ (why?)
  - Compute unique d such that $ed = 1 \bmod \varphi(n)$
  - Public key = (e,n); private key = (d,n)
- Encryption of m: $c = m^e \bmod n$
  - Modular exponentiation by repeated squaring
- Decryption of c: $c^d \bmod n = (m^e)^d \bmod n = m$