CSE-481
Robotics Capstone

SLAM Using
Gaussian filters

SLAM:
Simultaneous Localization and Mapping

- Full SLAM:

$$p(x_{1:t}, m \mid z_{1:t}, u_{1:t})$$

- Online SLAM:

$$p(x_t, m \mid z_{1:t}, u_{1:t}) = \int\int\int p(x_{1:t}, m \mid z_{1:t}, u_{1:t})\; dx_1, dx_2, \ldots, dx_{t-1}$$

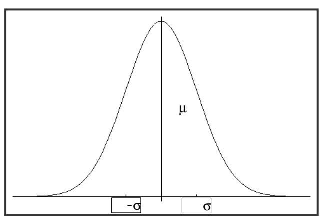
Integrations typically done one at a time

Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

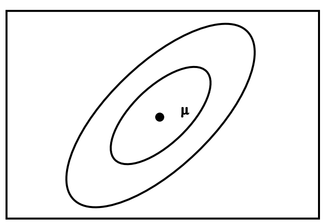
Univariate



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

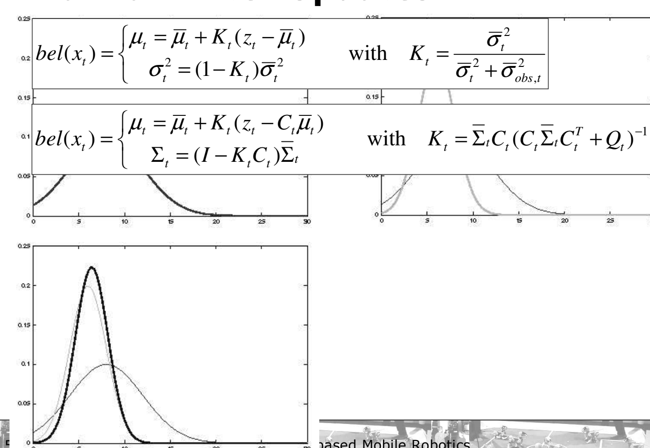
Multivariate



Kalman Filter Updates in 1D

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$



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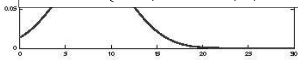
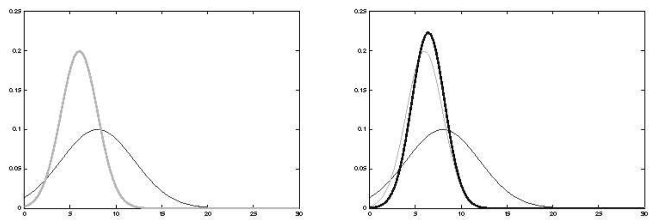
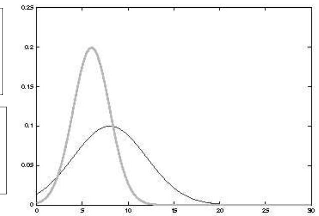


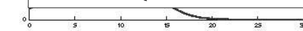


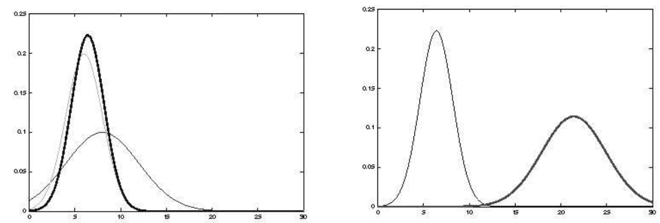
Kalman Filter Updates in 1D

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

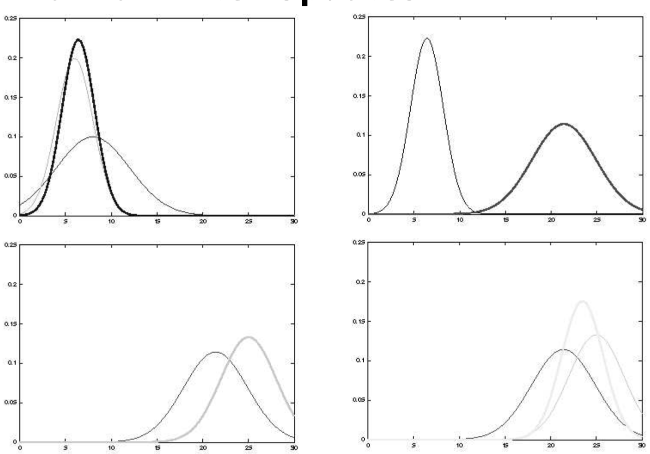
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$







Kalman Filter Updates



Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$bel(x_0) = N\left(x_0; \mu_0, \Sigma_0\right)$$

Linear Gaussian Systems: Dynamics

- Dynamics are linear function of state and control plus additive noise:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right)$$

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

Linear Gaussian Systems: Observations

- Observations are linear function of state plus additive noise:

$$z_t = C_t x_t + \delta_t$$

$$p(z_t \mid x_t) = N\left(z_t; C_t x_t, Q_t\right)$$

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(z_t; C_t x_t, Q_t\right) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):
2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$
5. Correction:
6. $K_t = \bar{\Sigma}_t C_t (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$
9. Return μ_t, Σ_t

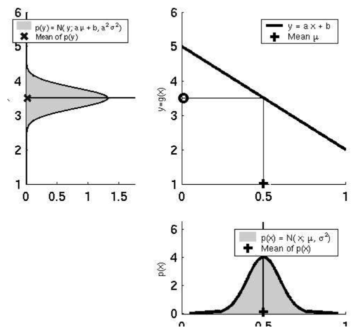
Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

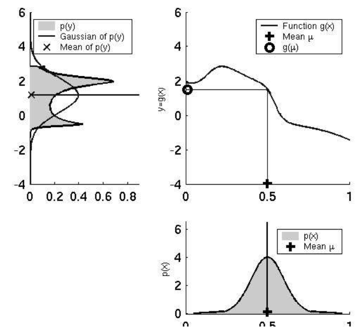
$$x_t = g(u_t, x_{t-1})$$

$$z_t = h(x_t)$$

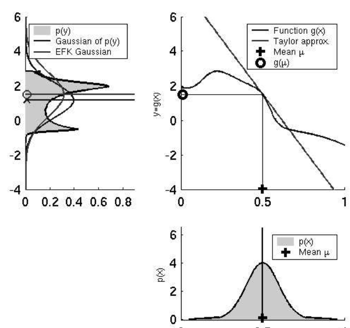
Linearity Assumption Revisited



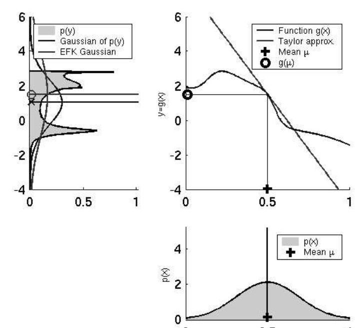
Non-linear Function



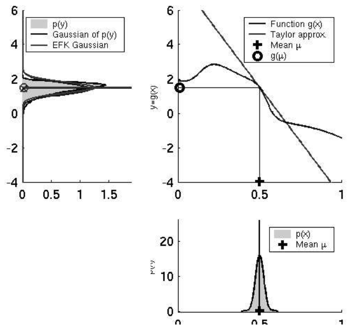
EKF Linearization (1)



EKF Linearization (2)



EKF Linearization (3)



EKF Algorithm

1. **Extended_Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

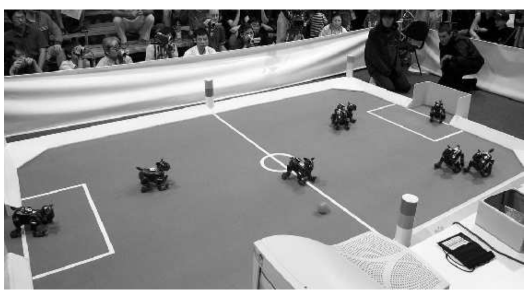
2. Prediction:
3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ ⟵ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$ ⟵ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t H_t (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$ ⟵ $K_t = \bar{\Sigma}_t C_t (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ ⟵ $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ ⟵ $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

$$H_t = \frac{\partial h(\bar{\mu}_t)}{x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{x_{t-1}}$$

Landmark-based Localization



1. **EKF_localization** (μ_{t-1}, Σ_{t-1}, u_t, z_t, m):

Prediction:

2. $G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$ Jacobian of g w.r.t location

3. $V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$ Jacobian of g w.r.t control

4. $M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$ Motion noise

5. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ Predicted mean
6. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T$ Predicted covariance

1. **EKF_localization** (μ_{t-1}, Σ_{t-1}, u_t, z_t, m):

   Correction:

2. $\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$ Predicted measurement mean

3. $H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$ Jacobian of h w.r.t location

4. $Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$

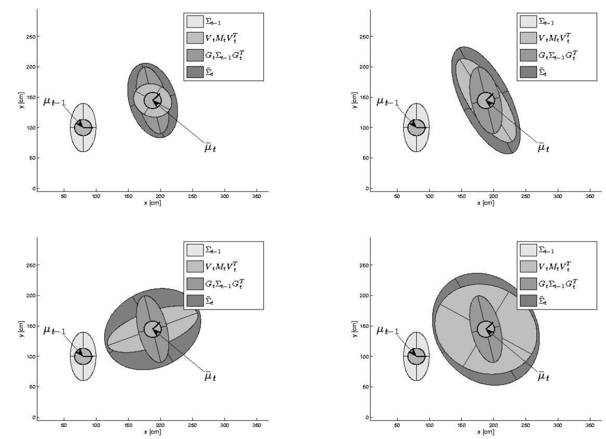
5. $S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$ Pred. measurement covariance

6. $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$ Kalman gain

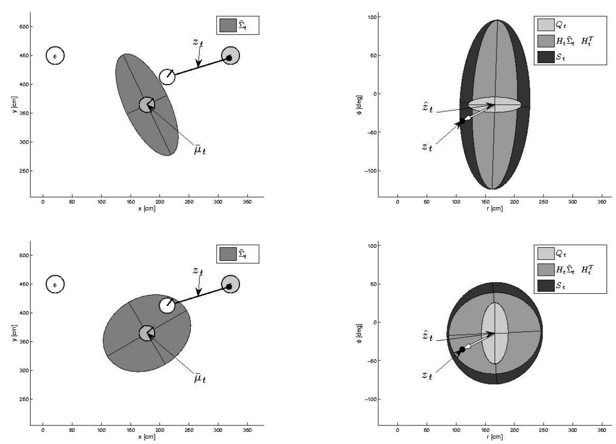
7. $\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$ Updated mean

8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ Updated covariance

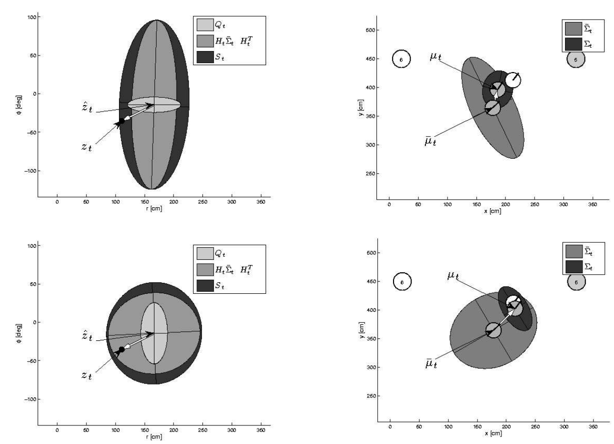
EKF Prediction Step



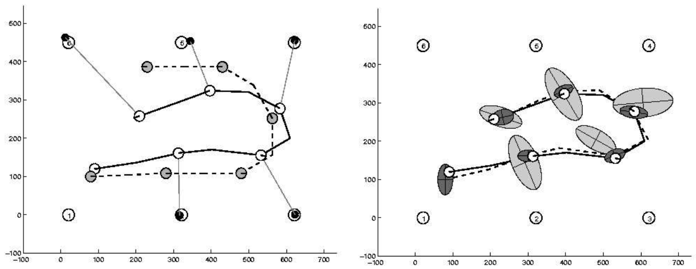
EKF Observation Prediction Step



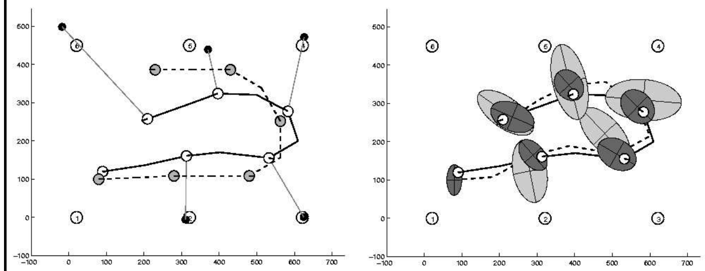
EKF Correction Step



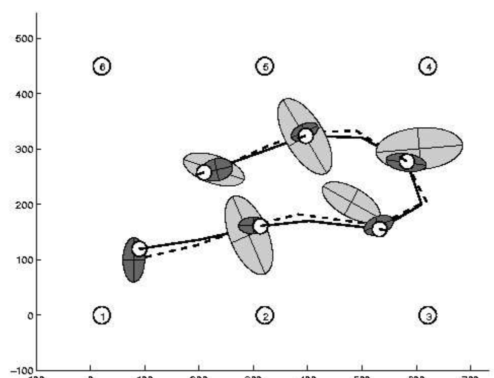
Estimation Sequence (1)



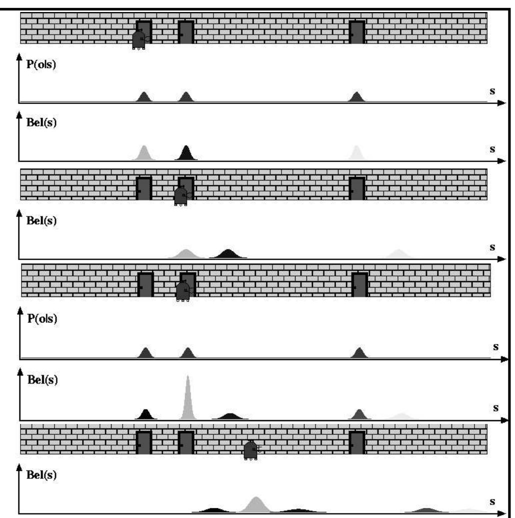
Estimation Sequence (2)



Comparison to GroundTruth



Multi-hypothesis Tracking



EKF Summary

- Highly efficient: Polynomial in measurement dimensionality *k* and state dimensionality *n*:
  $O(k^{2.376} + n^2)$
- Not optimal!
- Can diverge if nonlinearities are large!
- Works surprisingly well even when all assumptions are violated!

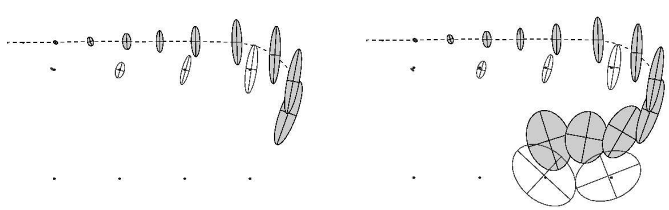
SLAM: Mapping with Kalman Filters

- Map with N landmarks:(2N+3)-dimensional Gaussian

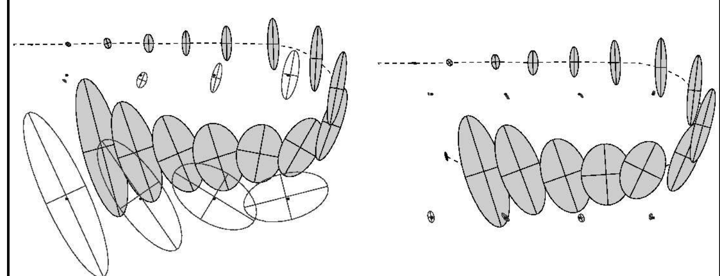
$$Bel(x_t, m_t) = \left\langle \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \\ x \\ y \\ \theta \end{pmatrix}, \begin{pmatrix} \sigma_{l_1}^2 & \sigma_{l_1 l_2} & \cdots & \sigma_{l_1 l_N} & \sigma_{l_1 x} & \sigma_{l_1 y} & \sigma_{l_1 \theta} \\ \sigma_{l_1 l_2} & \sigma_{l_2}^2 & \cdots & \sigma_{l_2 l_N} & \sigma_{l_2 x} & \sigma_{l_2 y} & \sigma_{l_2 \theta} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{l_1 l_N} & \sigma_{l_2 l_N} & \cdots & \sigma_{l_N}^2 & \sigma_{l_N x} & \sigma_{l_N y} & \sigma_{l_N \theta} \\ \sigma_{l_1 x} & \sigma_{l_2 x} & \cdots & \sigma_{l_N x} & \sigma_x^2 & \sigma_{xy} & \sigma_{x\theta} \\ \sigma_{l_1 y} & \sigma_{l_2 y} & \cdots & \sigma_{l_N y} & \sigma_{xy} & \sigma_y^2 & \sigma_{y\theta} \\ \sigma_{l_1 \theta} & \sigma_{l_2 \theta} & \cdots & \sigma_{l_N \theta} & \sigma_{x\theta} & \sigma_{y\theta} & \sigma_\theta^2 \end{pmatrix} \right\rangle$$

- Can handle hundreds of dimensions

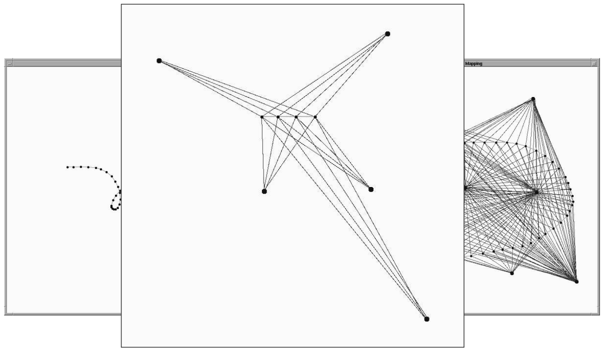
SLAM: Mapping with Kalman Filters



SLAM: Mapping with Kalman Filters



Bearings-only Mapping



[M. Deans 2002]

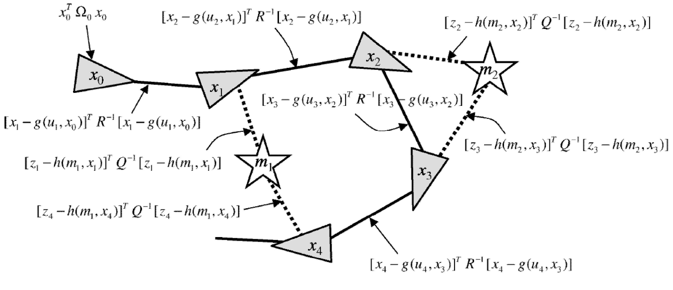
Approximations for SLAM

- Local submaps [Leonard et al.99, Bosse et al. 02, Newman et al. 03]
- Sparse links (correlations) [Lu & Milios 97, Guivant & Nebot 01]
- Sparse extended information filters [Frese et al. 01, Thrun et al. 02]
- Thin junction tree filters [Paskin 03]
- Rao-Blackwellisation (FastSLAM) [Murphy 99, Montemerlo et al. 02, Eliazar et al. 03, Haehnel et al. 03]

Graph-SLAM

- Full SLAM technique
- Generates probabilistic links
- Computes map only occasionally
- Based on Information Filter form

Graph-SLAM Idea



Sum of all constraints:

$$J_{\text{GraphSLAM}} = x_0^T \Omega_0 x_0 + \sum_t [x_t - g(u_t, x_{t-1})]^T R^{-1} [x_t - g(u_t, x_{t-1})] + \sum_t [z_t - h(m_{c_t}, x_t)]^T Q^{-1} [z_t - h(m_{c_t}, x_t)]$$

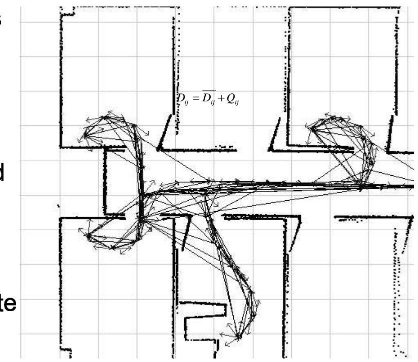
Robot Poses and Scans [Lu and Milios 1997]

- Successive robot poses connected by odometry
- Sensor readings yield constraints between poses
- Constraints represented by Gaussians

$$D_{ij} = \overline{D_{ij}} + Q_{ij}$$

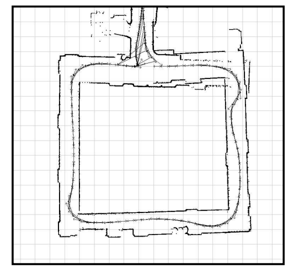
- Globally optimal estimate

$$\arg\max_{X_i}\left[P(D_{ij} \mid \overline{D_{ij}})\right]$$

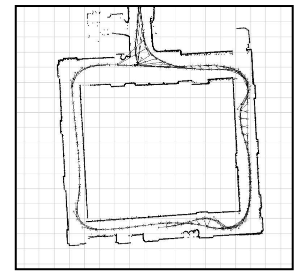


Loop Closure

- Use scan patches to detect loop closure
- Add new position constraints
- Deform the network based on covariances of matches



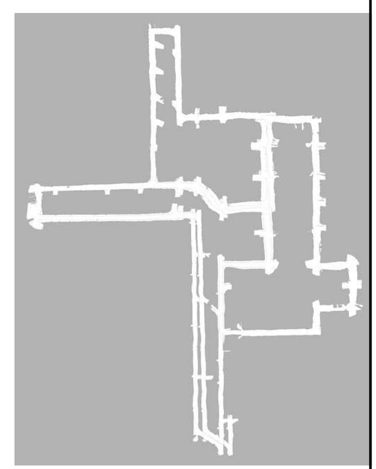
Before loop closure



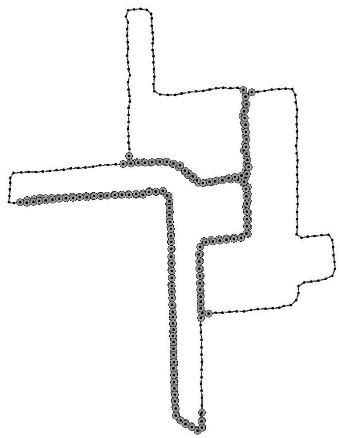
After loop closure

Mapping the Allen Center

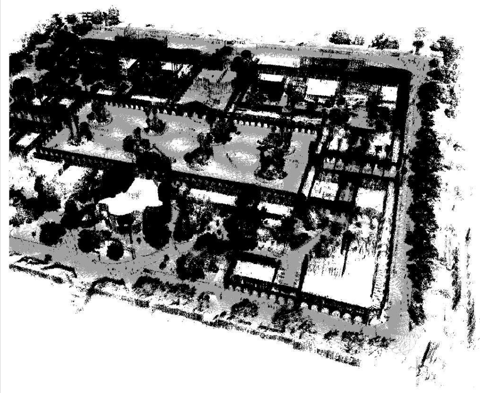
Mine Mapping



Mine Mapping: Data Associations

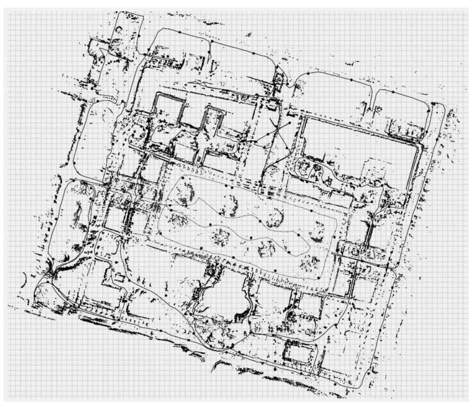


3D Outdoor Mapping

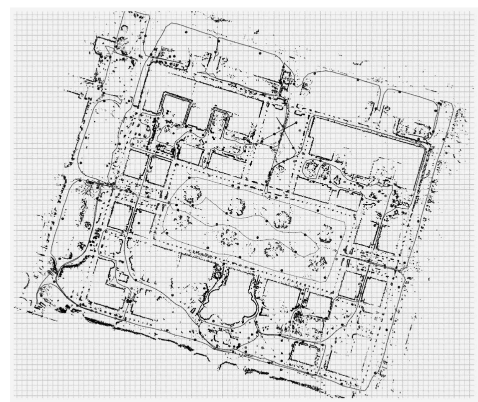


10^8 features, 10^5 poses, only few secs using cg.

Map Before Optimization



Map After Optimization



EKF and Graph-SLAM Summary

- Estimate posterior over robot and landmark locations
- Landmark estimates are correlated
- Graph-SLAM constructs link graph between poses and poses/landmarks
- Map extracted via marginalization or maximization of link graph