Autonomous Robotics

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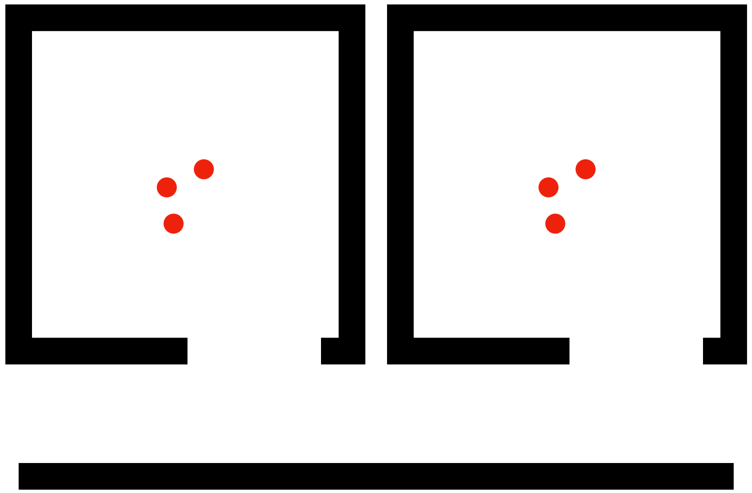
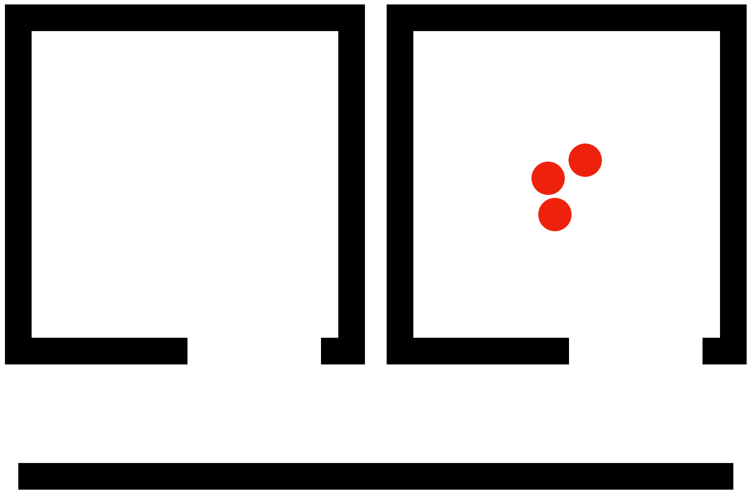
Recap

Problem 1: Two Room Challenge

Particles begin equally distributed, no motion or observation





All particles migrate to one room!

Reason: Resampling Increases Variance

50% prob. of resampling particle from Room 1 vs Room 2
31% prob. of preserving 50-50 particle split

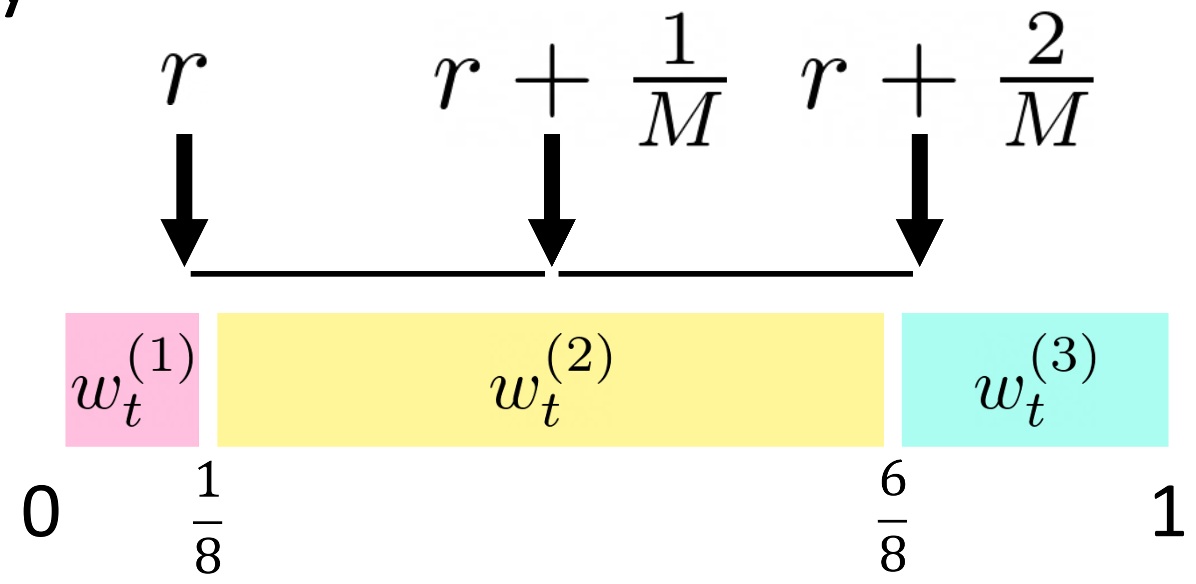
All particles migrate to one room!

Idea 1: Judicious Resampling

- Key idea: resample less often! (e.g., if the robot is stopped, don’t resample). Too often may lose particle diversity, infrequently may waste particles
- Common approach: don’t resample if weights have low variance
- Can be implemented in several ways: don’t resample when…
  - …all weights are equal
  - …weights have high entropy
  - …ratio of max to min weights is low

Idea 2: Low-Variance Resampling

- Sample one random number $r \sim [0, \frac{1}{M}]$
- Covers space of samples more systematically (and more efficiently)
- If all samples have same importance weight, won't lose particle diversity



Other Practical Concerns

- How many particles is enough?
  - Typically need more particles at the beginning (to cover possible states)
    - KLD Sampling (Fox, 2001) adaptively increases number of particles when state uncertainty is high, reduces when state uncertainty is low
- Particle filtering with overconfident sensor models
  - Squash sensor model prob. with power of 1/m
  - Sample from better proposal distribution than motion model
    - Manifold Particle Filter (Koval et al., 2017) for contact sensors
- Particle starvation: no particles near current state

MuSHR Localization Project

- Implement kinematic car motion model
- Implement different factors of single-beam sensor model
- Combine motion and sensor model with the Particle Filter algorithm

Lecture Outline

Recap

↓

Kalman Filtering

Can we get closed form updates for Bayesian Filtering?

Need to choose form of probability distributions

- Dynamics (Prediction)

$$\overline{Bel}(x_t) = \int p(x_t|u_t, x_{t-1})Bel(x_{t-1})dx_{t-1}$$

- Measurement (Correction)

$$Bel(x_t) = \eta P(z_t|x_t)\overline{Bel}(x_t)$$

Tractable computation of Bayesian posteriors

Solution: Linear Gaussian Models

- Dynamics (Prediction)

$$\overline{Bel}(x_t) = \int p(x_t|u_t, x_{t-1})Bel(x_{t-1})dx_{t-1}$$

- Measurement (Correction)

$$Bel(x_t) = \eta P(z_t|x_t)\overline{Bel}(x_t)$$

Model as Linear Gaussian

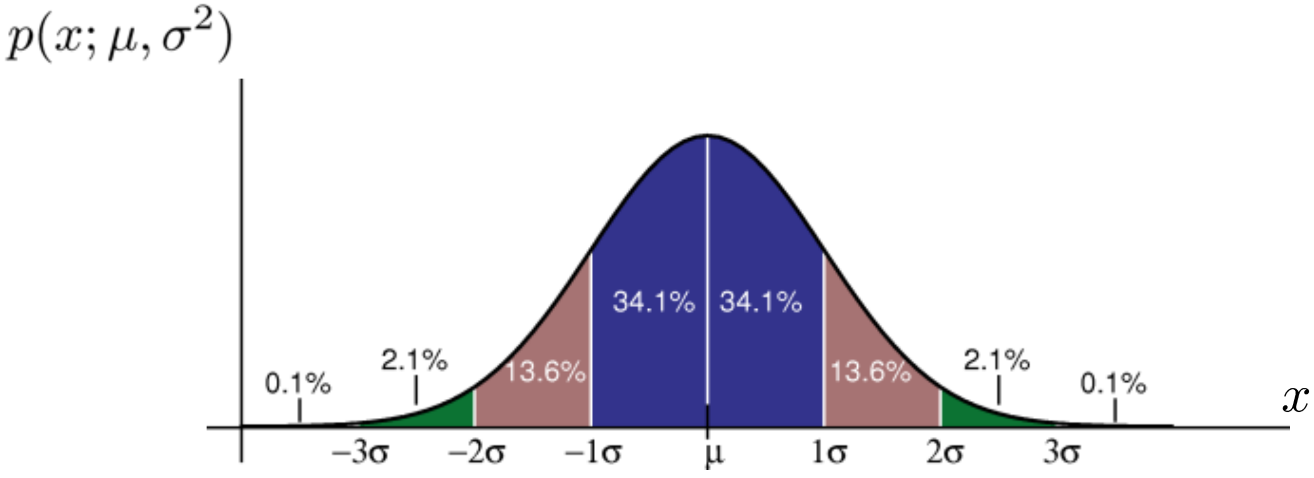
Let’s take a little Gaussian detour

Gaussians (1D)

- Gaussian with mean (μ) and standard deviation (σ)

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$p(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$



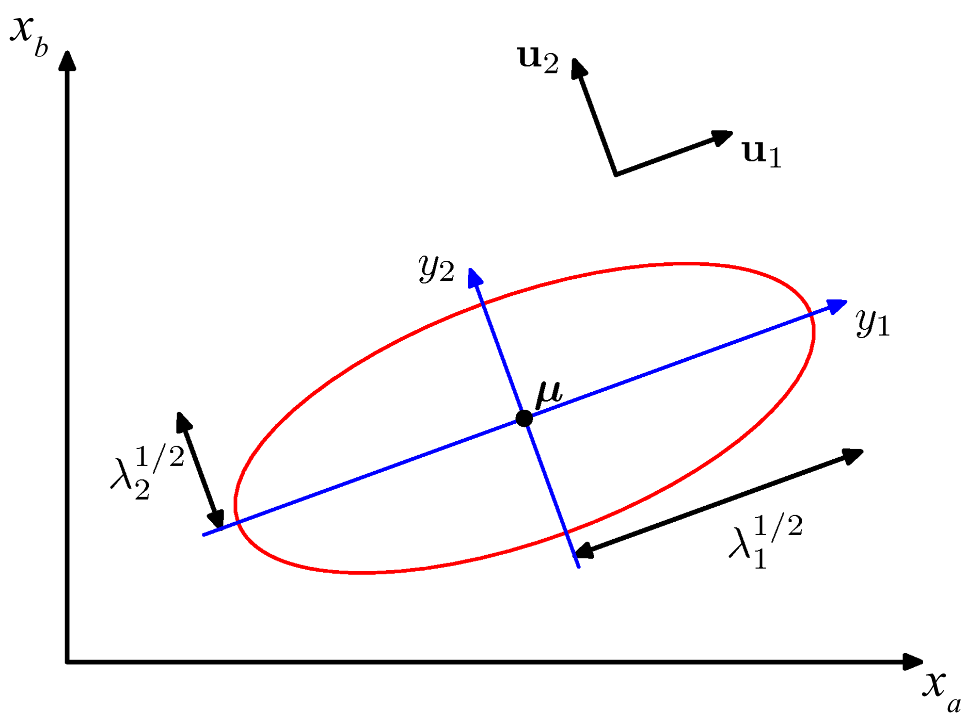
Gaussians (2D) – we won't get too deep into this!

$$p(\mathbf{x}) = \mathrm{N}(\mu, \Sigma)$$

$$\mathbf{x} = \begin{pmatrix} x_a \\ x_b \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$$

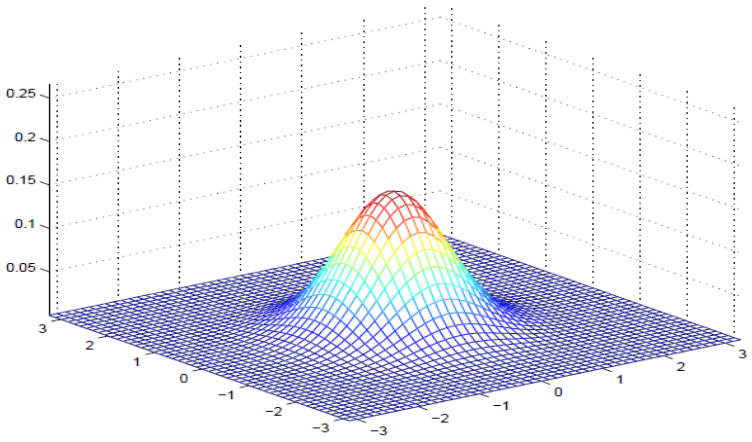
$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)}$$

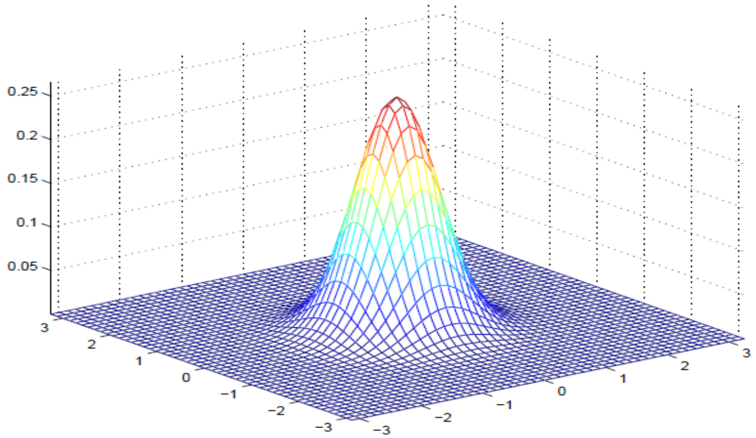


2D examples

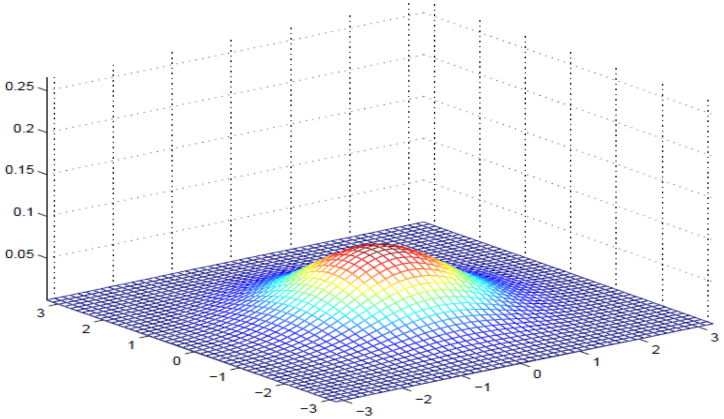
Slide from Pieter Abbeel



- μ = [0; 0]
- Σ = [1 0 ; 0 1]



- μ = [0; 0]
- Σ = [.6 0 ; 0 .6]



- μ = [0; 0]
- Σ = [2 0 ; 0 2]

Important Identities: Gaussians

Forward propagation

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = AX + B + \epsilon \\ \epsilon \sim \mathcal{N}(0, Q) \end{cases} \implies Y \sim \mathcal{N}(A\mu + B, A\Sigma A^T + Q)$$

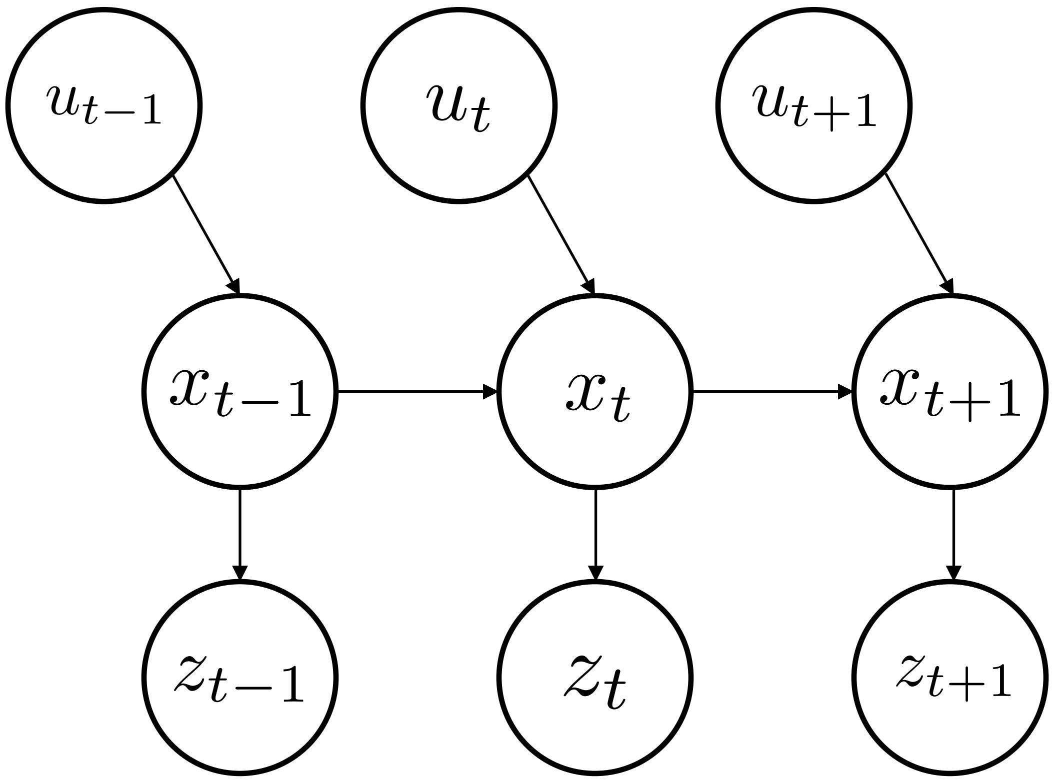
Conditioning

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = CX + B + \delta \\ \delta \sim \mathcal{N}(0, R) \end{cases} \implies X|Y = y_0 \sim \mathcal{N}(\mu + K(y_0 - C\mu), (I - KC)\Sigma)$$

- Marginalization and conditioning in Gaussians results in Gaussians
- We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations.

Discrete Kalman Filter

Kalman filter = Bayes filter with Linear Gaussian dynamics and sensor models



Discrete Kalman Filter: Scalar Version

Estimates the state ***x*** of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = ax_{t-1} + bu_t + \epsilon_t$$

$$\epsilon_t \sim \mathcal{N}(0, q)$$

with a measurement

$$z_t = cx_t + \delta_t$$

$$\delta_t \sim \mathcal{N}(0, r)$$

Linear Gaussian

Discrete Kalman Filter: Matrix Version

Estimates the state ***x*** of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = Ax_{t-1} + Bu_t + \epsilon_t$$

$$\epsilon_t \sim \mathcal{N}(0, Q)$$

with a measurement

$$z_t = Cx_t + \delta_t$$

$$\delta_t \sim \mathcal{N}(0, R)$$

Linear Gaussian

Components of a Kalman Filter

A Matrix (n x n) that describes how the state evolves from ***t-1*** to ***t*** without controls or noise.

B Matrix (n x l) that describes how the control $\boldsymbol{u_{t-1}}$ changes the state from ***t-1*** to ***t***

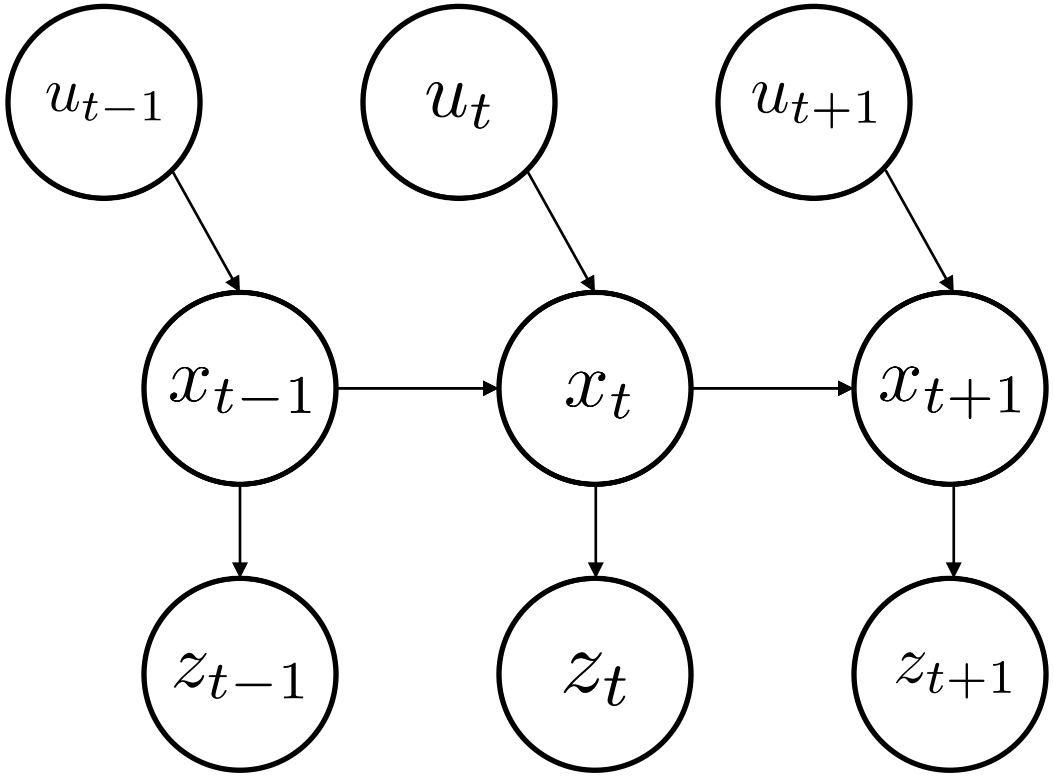
C Matrix (k x n) that describes how to map the state $\boldsymbol{x_t}$ to an observation $\boldsymbol{z_t}$.

ϵ_t

δ_t

Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance ***R*** and ***Q*** respectively.

Goal of the Kalman Filter: Same as Bayes Filter



Belief

$$p(x_t | z_{0:t}, u_{0:t})$$

Idea: recursive update

$$\propto p(z_t|x_t) \int p(x_t|x_{t-1}, u_t) p(x_{t-1}|z_{0:t-1}, u_{0:t-1})$$

Measurement

Dynamics

Recursive Belief

2 step process:

- Dynamics update (incorporate action)
- Measurement update (incorporate sensor reading)

Bayes Filters

Key Idea: Apply Markov to get a recursive update!

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \sum P(x_t|u_t, x_{t-1})bel(x_{t-1})$$

Linear Gaussian

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$Bel(x_0) = \mathcal{N}(\mu_0, \Sigma_0)$$

- $Bel(x_t)$ at any step t is: $\mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$
- $\overline{Bel}(x_t)$ at any step t is: $\mathcal{N}(\mu_{t|0:t-1}, \Sigma_{t|0:t-1})$

Linear Gaussian Systems: Prediction

- Integrate the effect of one action under the dynamics, before measurement comes in

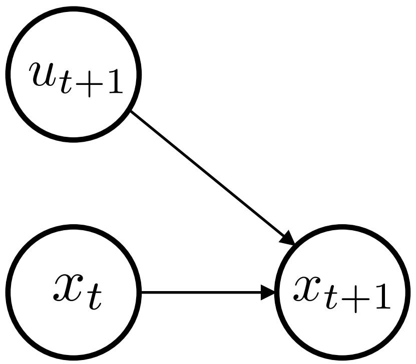
$$x_{t+1} = Ax_t + Bu_{t+1} + \epsilon_{t+1} \qquad \epsilon_{t+1} \sim \mathcal{N}(0, Q_{t+1})$$

$$p(x_{t+1}|x_t, u_{t+1}) = \mathcal{N}(Ax_t + Bu_{t+1}, Q_{t+1})$$

$$\overline{Bel}(x_{t+1}) \qquad\qquad Bel(x_t)$$

$$p(x_{t+1}|z_{0:t}, u_{0:t+1}) = \int p(x_t|u_{0:t}, z_{0:t})p(x_{t+1}|u_{t+1}, x_t)dx_t$$

Gaussian, easy!



Linear Gaussian Systems: Prediction

- Integrate the effect of one action under the dynamics, before measurement comes in

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$$\underbrace{p(x_{t+1}|z_{0:t}, u_{0:t+1})}_{\overline{Bel}(x_{t+1})} = \int \underbrace{p(x_t|u_{0:t}, z_{0:t})}_{Bel(x_t)} p(x_{t+1}|u_{t+1}, x_t) dx_t$$

Gaussian, easy!

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = AX + B + \epsilon \implies Y \sim \mathcal{N}(A\mu + B, A\Sigma A^T + Q) \\ \epsilon \sim \mathcal{N}(0, Q) \end{cases}$$

Linear Gaussian Systems: Prediction

- Integrate the effect of one action under the dynamics, before measurement comes in

$$p(x_t|u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$$

$$x_{t+1} = Ax_t + Bu_{t+1} + \epsilon_{t+1}$$

$$\epsilon_{t+1} \sim \mathcal{N}(0, Q_{t+1})$$

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = AX + B + \epsilon \\ \epsilon \sim \mathcal{N}(0, C) \end{cases} \implies Y \sim \mathcal{N}(A\mu + B, A\Sigma A^T + Q)$$

Previous belief $$p(x_t|u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$$

Belief Update $$p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(A\mu_{t|0:t} + Bu_{t+1}, A\Sigma_{t|0:t}A^T + Q_{t+1})$$

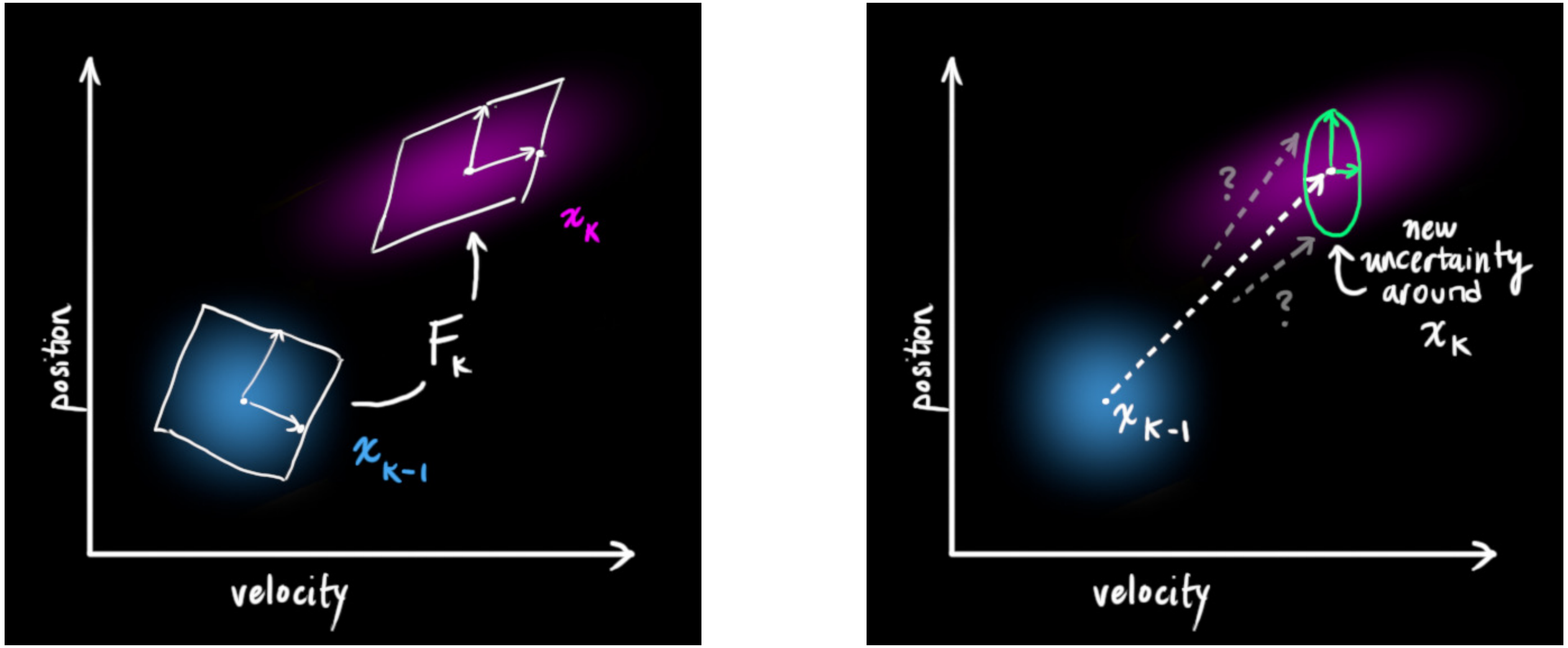
Intuition: Scale and shift the mean according to dynamics, uncertainty grows quadratically!

Linear Gaussian Systems: Prediction

Previous belief $p(x_t | u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$

Belief Update $p(x_{t+1} | u_{0:t+1}, z_{0:t}) = \mathcal{N}(A\mu_{t|0:t} + Bu_{t+1}, A\Sigma_{t|0:t}A^T + Q_{t+1})$

Intuition: Scale and shift the mean according to dynamics, uncertainty grows!

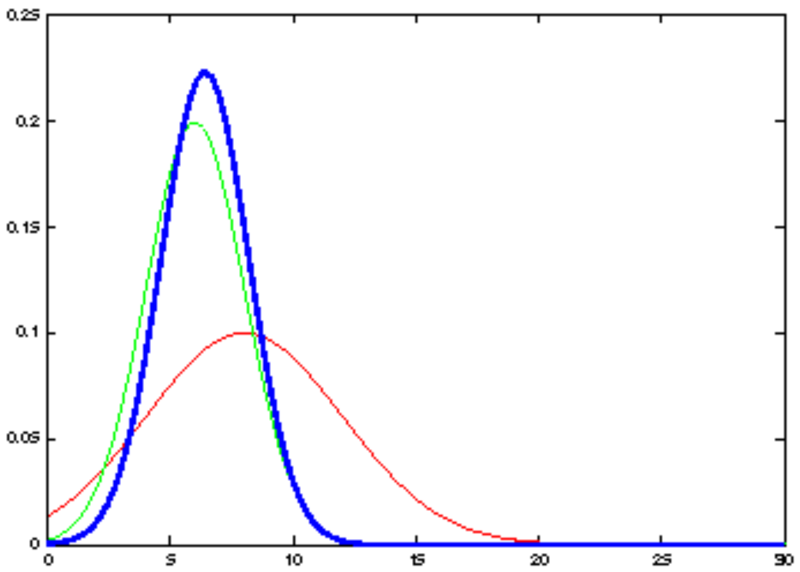


Intuition Behind Prediction Step

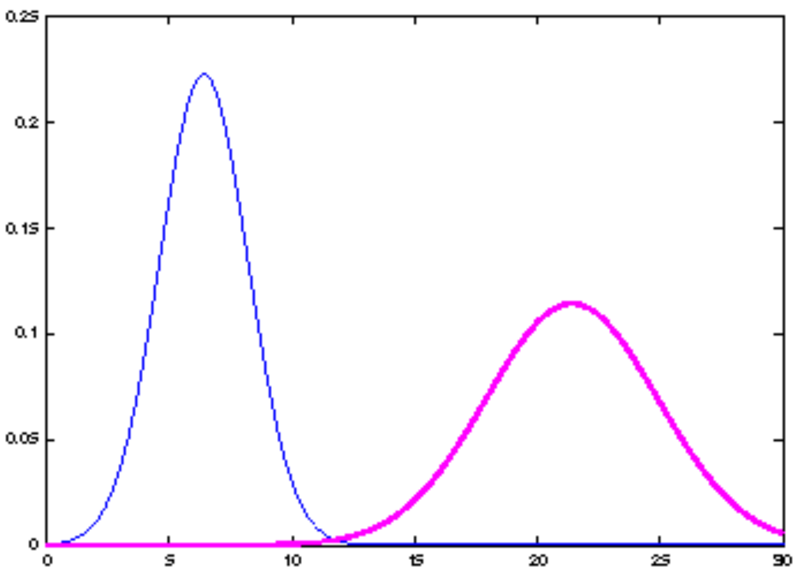
Previous belief $p(x_t | u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$

Belief Update $p(x_{t+1} | u_{0:t+1}, z_{0:t}) = \mathcal{N}(A\mu_{t|0:t} + Bu_{t+1}, A\Sigma_{t|0:t}A^T + Q_{t+1})$

Intuition: Scale and shift the mean according to dynamics, uncertainty grows!



Belief at x_t



Belief post dynamics → shifted mean, scaled and shifted variance

Linear Gaussian Systems: Observations

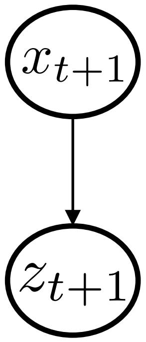
- Integrate the effect of an observation using sensor model, after dynamics

$$z_{t+1} = Cx_{t+1} + \delta_{t+1} \qquad \delta_{t+1} \sim \mathcal{N}(0, R_{t+1})$$

$$p(z_{t+1}|x_{t+1}) = \mathcal{N}(Cx_{t+1}, R_{t+1})$$

$Bel(x_{t+1})$ $\qquad$ $\overline{Bel}(x_{t+1})$

$$p(x_{t+1}|u_{0:t+1}, z_{0:t+1}) \propto p(z_{t+1}|x_{t+1})p(x_{t+1}|u_{0:t+1}, z_{0:t})$$



Gaussian, easy to normalize

Slightly harder than the dynamics step!

Linear Gaussian Systems: Observations

- Integrate the effect of an observation using sensor model, after dynamics

$$z_{t+1} = Cx_{t+1} + \delta_{t+1} \qquad \delta_{t+1} \sim \mathcal{N}(0, R_{t+1})$$

$$p(z_{t+1}|x_{t+1}) = \mathcal{N}(Cx_{t+1}, R_{t+1})$$

$$\underbrace{p(x_{t+1}|u_{0:t+1}, z_{0:t+1})}_{Bel(x_{t+1})} \propto p(z_{t+1}|x_{t+1})\underbrace{p(x_{t+1}|u_{0:t+1}, z_{0:t})}_{\overline{Bel}(x_{t+1})}$$

Conditioning

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = CX + B + \delta \\ \delta \sim \mathcal{N}(0, R) \end{cases} \implies X|Y = y_0 \sim \mathcal{N}(\mu + K(y_0 - C\mu), (I - KC)\Sigma)$$

$$K = \Sigma C^T (C\Sigma C^T + R)^{-1}$$

Linear Gaussian Systems: Observations

- Integrate the effect of an observation using sensor model, after dynamics

$$p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$$
$$z_{t+1} = Cx_{t+1} + \delta_{t+1}$$
$$\delta_{t+1} \sim \mathcal{N}(0, R_{t+1})$$

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = CX + B + \delta \\ \delta \sim \mathcal{N}(0, R) \end{cases} \implies \begin{matrix} X|Y = y_0 \sim \mathcal{N}(\mu + K(y_0 - C\mu), (I - KC)\Sigma) \\ K = \Sigma C^T (C\Sigma C^T + R)^{-1} \end{matrix}$$

Previous belief $\quad p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$ Computed from dynamics step

Updated belief $\quad p(x_{t+1}|u_{0:t+1}, z_{0:t+1})$

$$= \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

$$K_{t+1} = \Sigma_{t+1|0:t}C^T(C\Sigma_{t+1|0:t}C^T + R)^{-1}$$

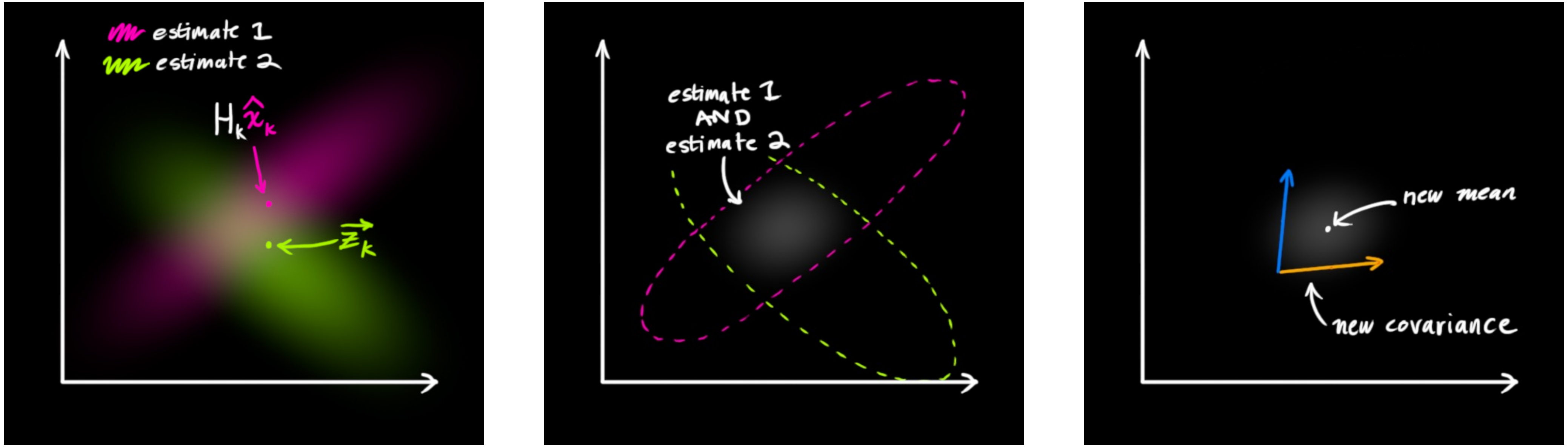
Linear Gaussian Systems: Observations

Previous belief $p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$ Computed from dynamics step

Updated belief $p(x_{t+1}|u_{0:t+1}, z_{0:t+1})$

$$= \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

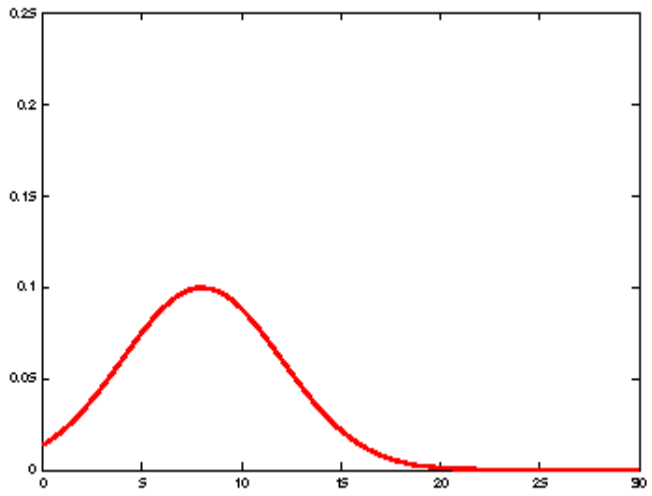
Intuition: Correct the update linearly according to measurement error from expectation, shrink uncertainty accordingly

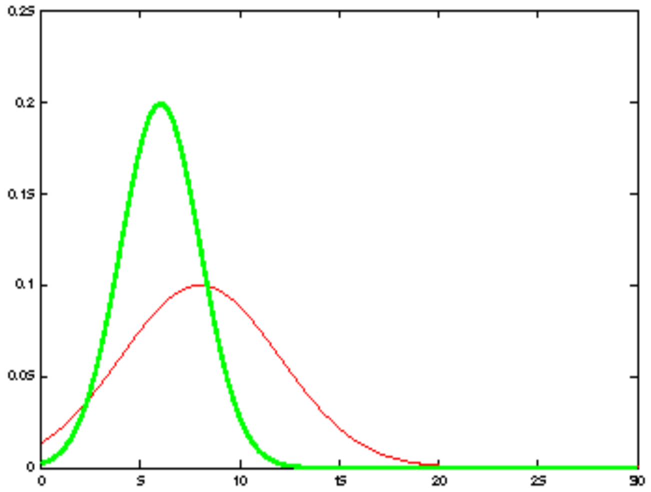


Intuition Behind Correction Step

Previous belief

New Measurement





$$p(x_{t+1}|u_{0:t+1}, z_{0:t+1}) = \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

$$K_{t+1} = \Sigma_{t+1|0:t}C^T(C\Sigma_{t+1|0:t}C^T + R)^{-1}$$

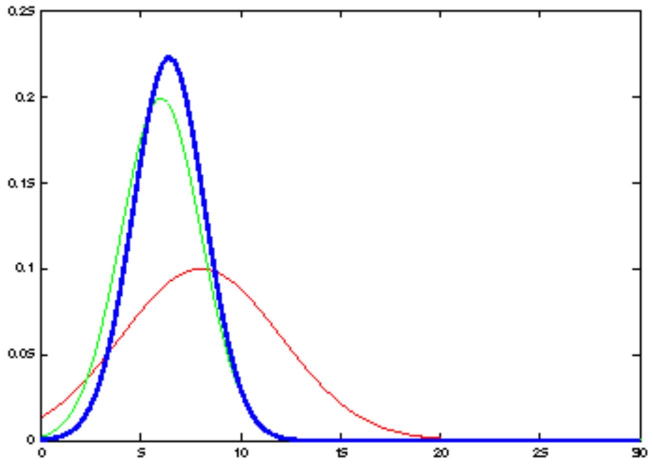
For the sake of simplicity, let's say C = I

$$K_{t+1} = \frac{\Sigma_{t+1|0:t}}{\Sigma_{t+1|0:t} + R}$$

Corrects belief based on measurement

→ Average between mean and measurement based on K

→ Scale down uncertainty based on K



Unpacking the Kalman Gain

Previous belief $$p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$$ Computed from dynamics step

Updated belief $$p(x_{t+1}|u_{0:t+1}, z_{0:t+1})$$

$$= \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

$$K_{t+1} = \Sigma_{t+1|0:t}C^T(C\Sigma_{t+1|0:t}C^T + R)^{-1}$$

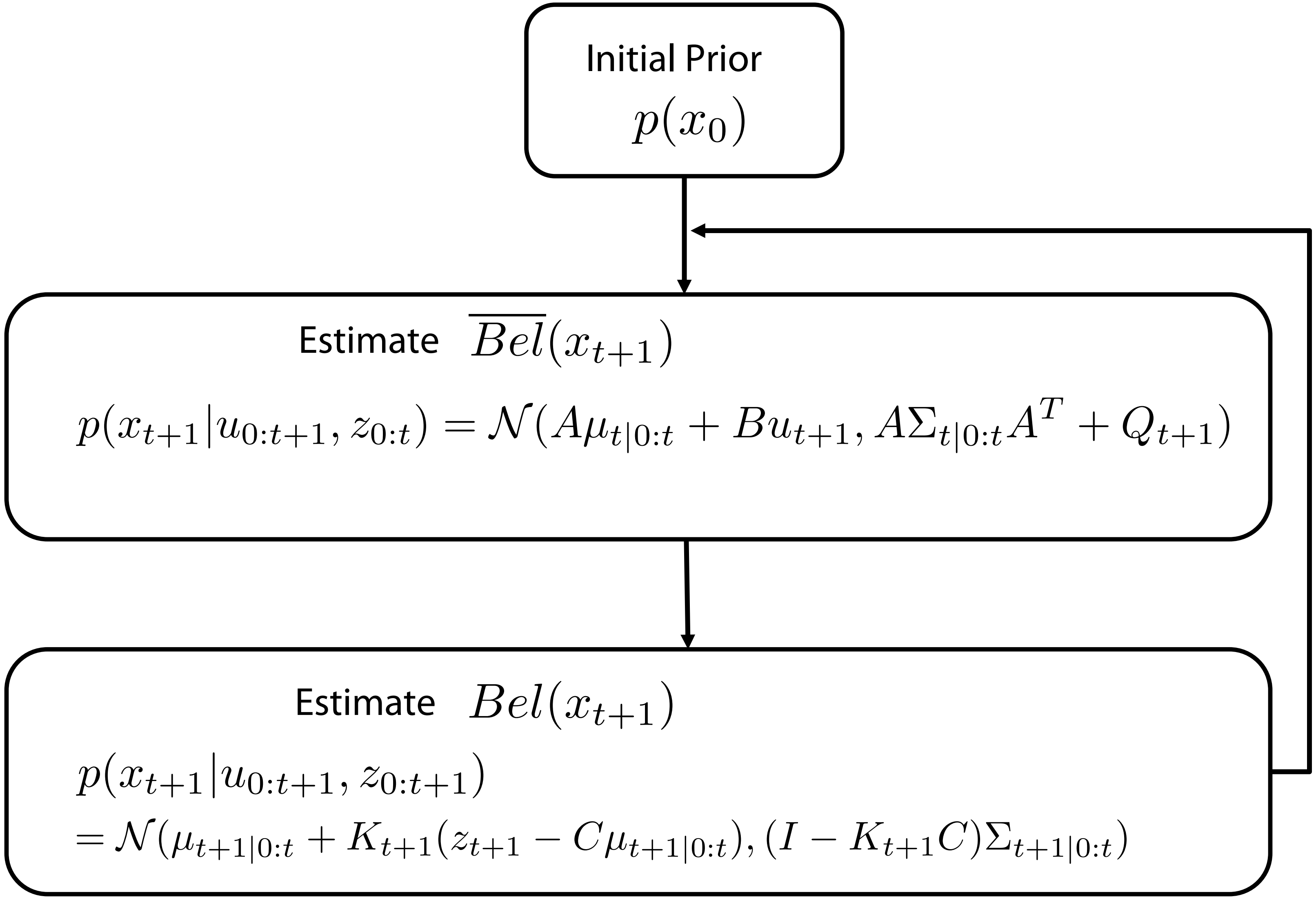
Case 1: Very noisy sensor, R>>Σ

For the sake of simplicity, let's say C = I

$$K_{t+1} = \frac{\Sigma_{t+1|0:t}}{\Sigma_{t+1|0:t} + R}$$

Case 2: Deterministic sensor, R = 0

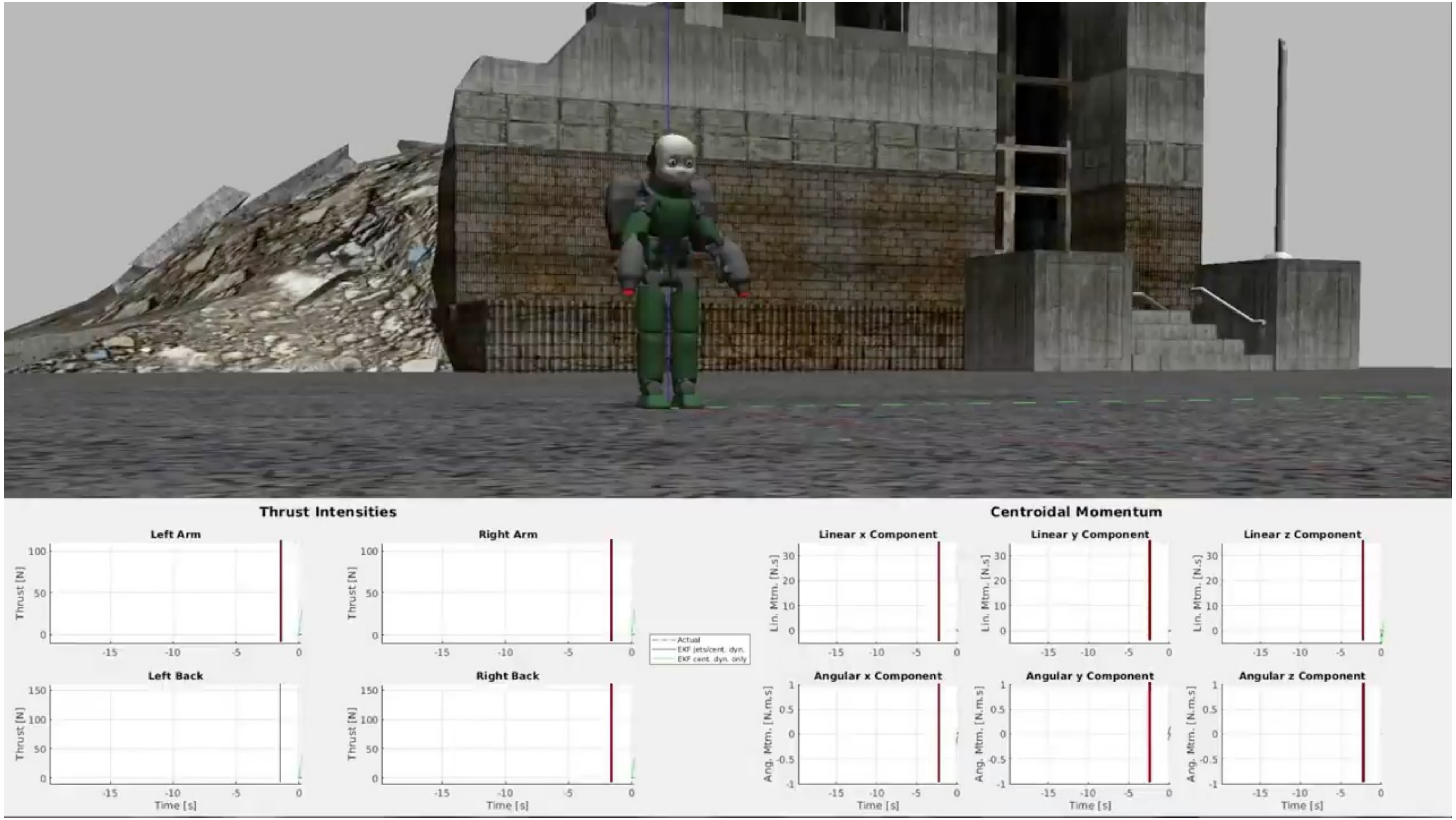
Kalman Filter Algorithm



Dynamics/Prediction
(given some u)

Measurement/Correction
(given some z)

Kalman Filter in Action



Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality ***k*** and state dimensionality ***n***:

  $O(k^{2.376} + n^2)$

Matrix Inversion (Correction)

$$K_{t+1} = \Sigma_{t+1|0:t} C^T (C \Sigma_{t+1|0:t} C^T + R_{t+1})^{-1}$$

Matrix Multiplication (Prediction)

$$p(x_{t+1}|z_{0:t}, u_{0:t+1}) \sim \mathcal{N}(A\mu_{t|0:t} + Bu_t, A\Sigma_{t|0:t}A^T + Q_t)$$

- Optimal for linear Gaussian systems!

- Most robotics systems are nonlinear!

Why should we care?

Still a very widely used technique for estimation/localization/mapping in real problems