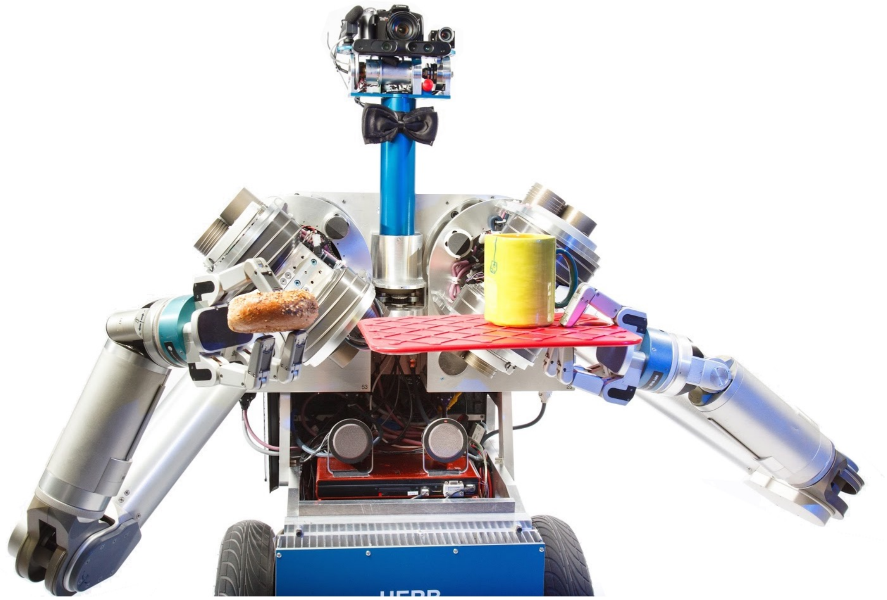
Autonomous Robotics

Winter 2026

Abhishek Gupta, Siddhartha Srinivasa

TAs: Carolina Higera, Entong Su, Rishabh Jain



Class Outline

State Estimation
- Robotic System Design
- Filtering
- Localization
- SLAM

Control
- Feedback Control
- PID Control
- MPC
- LQR

Planning
- Search
- Heuristic Search
- Motion Planning
- Lazy Search

Learning
- Imitation Learning
- Policy Gradient
- Actor-Critic
- Model-Based RL

Bayes Formula

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

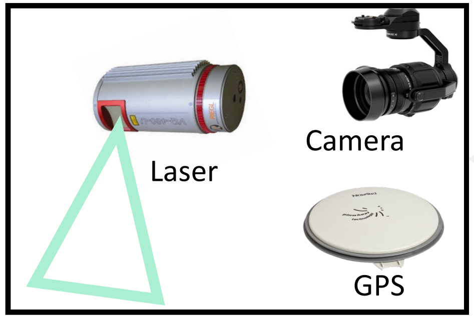
$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

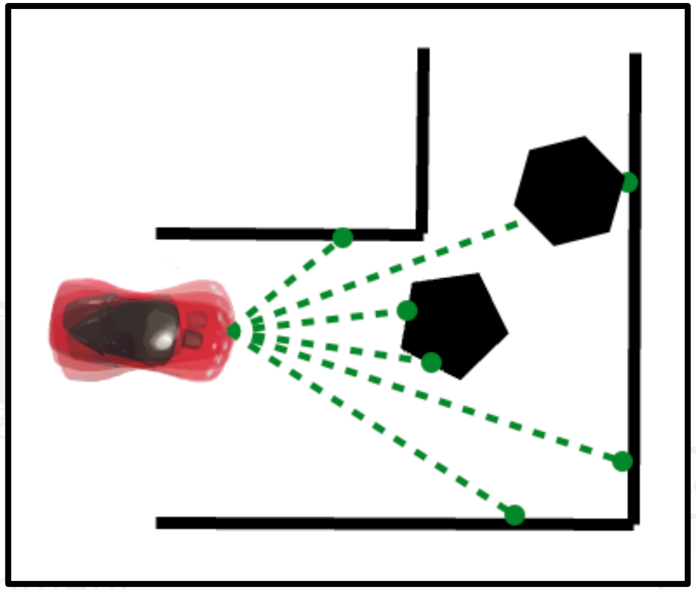
Today’s Objective: Understand how to formalize state estimation

What does it mean to **estimate** “state”?

State estimation

Sensors





Fundamental Problem: State is hidden

But all decision making depends on knowing state

Solution: Estimate belief over state

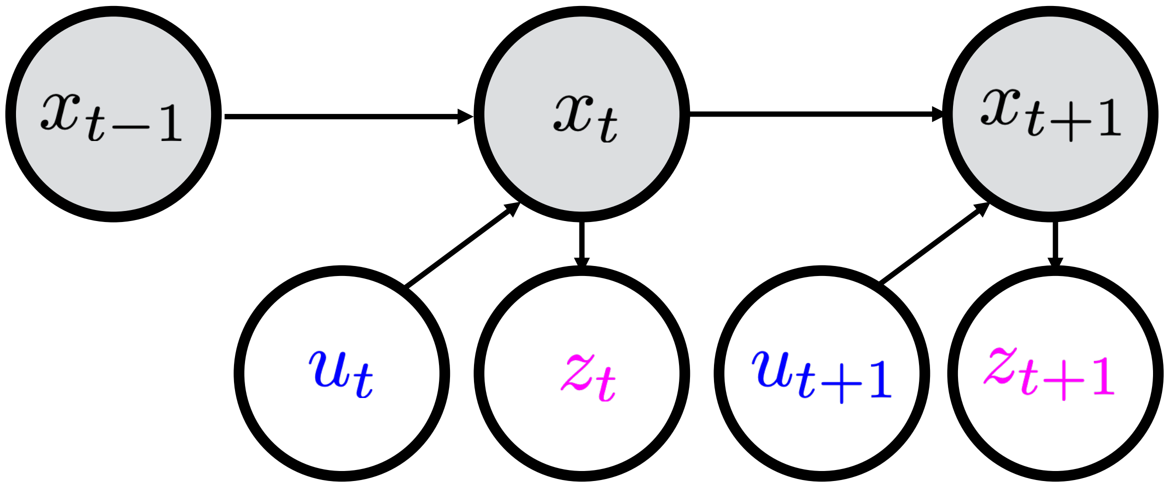
$$bel(x_t) = P(x_t | z_{1:t}, u_{1:t})$$

Belief is a probability of each possible state given history

Also called Posterior / Information state / State of knowledge

Represent belief? Parametric (Gaussian), Non-parametric (Histogram)

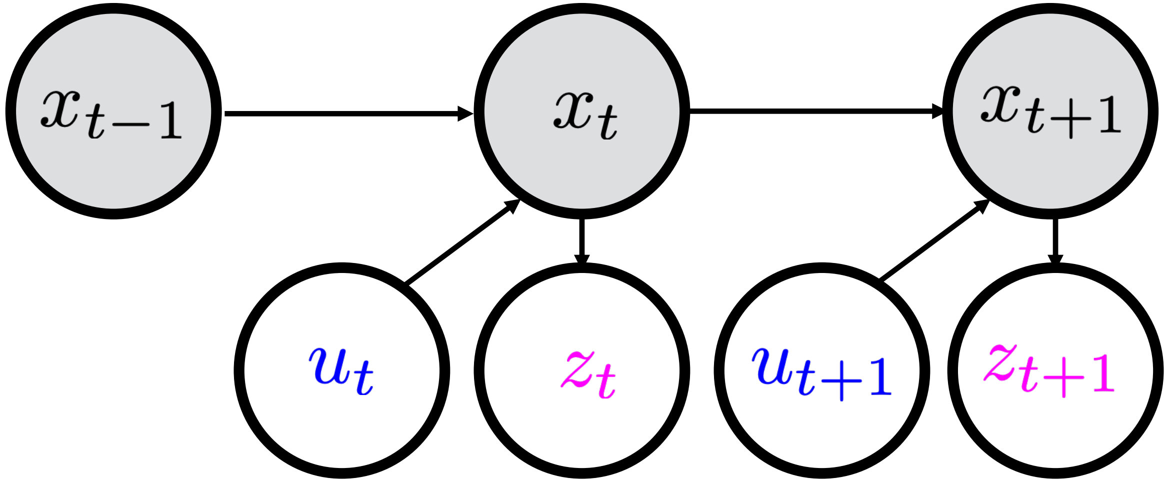
What is belief in this setting?



P(current state | all past information)

$$P(x_t | z_t, u_t, x_{t-1}, \ldots)$$

Solution: Markov Assumption



Markov assumption :

*Future state **conditionally independent** of past actions, measurements **given** present state.*

$$P(x_t | u_t, x_{t-1}, z_{t-1}, u_{t-1}, \ldots) = P(x_t | u_t, x_{t-1})$$

$$P(z_t | x_t, u_t, x_{t-1}, z_{t-1}, u_{t-1}, \ldots) = P(z_t | x_t)$$

Probabilistic models

State transition probability / dynamics / motion model

$$P(x_t | x_{t-1}, u_t)$$

Measurement probability / Observation model

$$P(z_t | x_t)$$

What does it mean to be Markov?

Lecture Outline

Recap

↓

Bayesian Filtering w/ Examples

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Motion Models

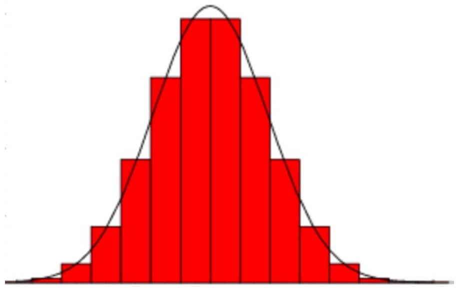
How do we tractably calculate belief?

Input data

Measurement $z_{1:t}$

Actions $u_{1:t}$

Belief



$$bel(x_t) = P(x_t | z_{1:t}, u_{1:t})$$

Ans: Bayes filter!

Bayes Filters

$$Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

We want to recursively express Bel(x_t) in terms of three entities

$$p(z_t|x_t)$$

Measurement

$$p(x_t|x_{t-1}, u_{t-1})$$

Dynamics

$$Bel(x_{t-1})$$

Previous Belief

Bayes filter in a nutshell

Key Idea: Apply Markov to get a recursive update!

Bayes filter in a nutshell

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

$bel(x_{t-1})$

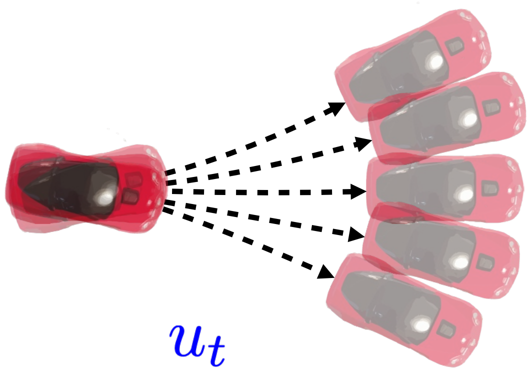
Bayes filter in a nutshell

Step 1: Prediction - push belief through dynamics given action

$$bel(x_{t-1}) = p(x_{t-1}|u_{1:t-1}, z_{1:t-1}) \longrightarrow \overline{bel}(x_t) = p(x_t|u_{1:t}, z_{1:t-1})$$

using

$$p(x_t|x_{t-1}, u_{t-1})$$



u_t

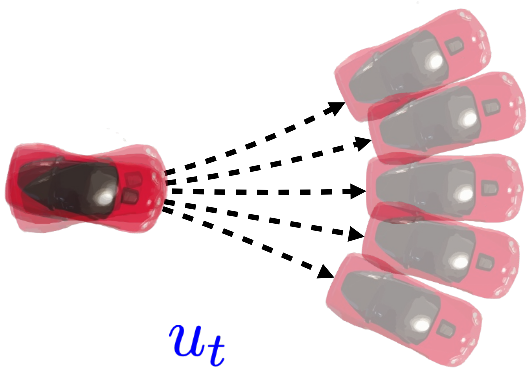
$bel(x_{t-1})$ $\overline{bel}(x_t)$

Bayes filter in a nutshell

Step 1: Prediction - push belief through dynamics given action

(discrete) $$\overline{bel}(x_t) = \sum P(x_t|u_t, x_{t-1})bel(x_{t-1})$$

(total probability)



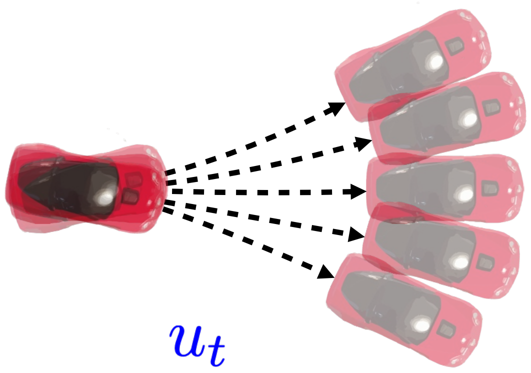
u_t

$bel(x_{t-1})$ $\overline{bel}(x_t)$

Derivation: Dynamics Update

Step 1: Prediction - push belief through dynamics given action

(discrete) $$\overline{bel}(x_t) = \sum P(x_t|u_t, x_{t-1})bel(x_{t-1})$$



(total probability)

$$p(x_t|u_{1:t}, z_{1:t-1}) = \sum_{x_{t-1}} p(x_t, x_{t-1}|u_{1:t}, z_{1:t-1})$$

$$p(x) = \sum_y p(x,y)$$

$$= \sum_{x_{t-1}} p(x_t|x_{t-1}, u_t, u_{1:t-1}, z_{1:t-1})p(x_{t-1}|u_{1:t-1}, z_{1:t-1})$$

$$p(A,B|C) = p(A|B,C)p(B|C)$$

$$= \sum_{x_{t-1}} p(x_t|x_{t-1}, u_t)p(x_{t-1}|u_{1:t-1}, z_{1:t-1})$$

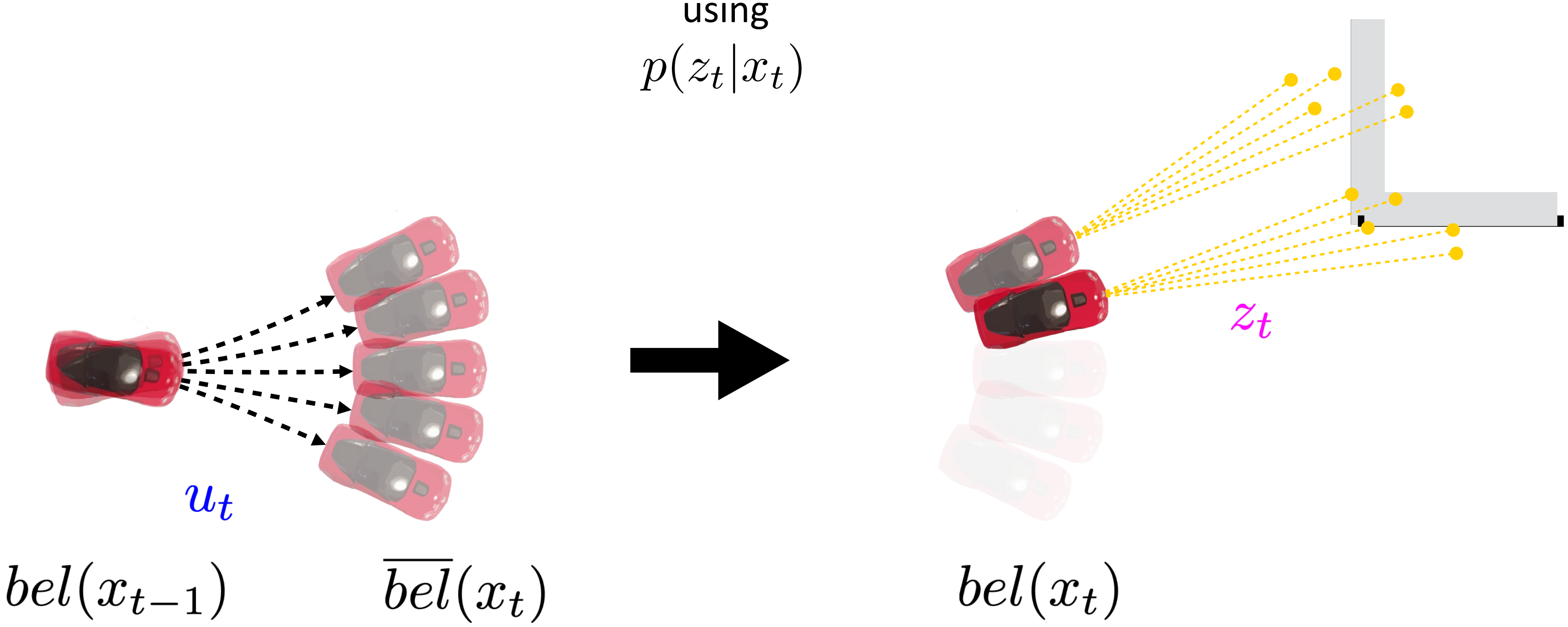
Motion model   Previous Belief

Bayes filter in a nutshell

Step 2: Correction - apply Bayes rule given measurement

$$\overline{bel}(x_t) = p(x_t|u_{1:t}, z_{1:t-1}) \longrightarrow bel(x_t) = p(x_t|u_{1:t}, z_{1:t})$$

using $p(z_t|x_t)$



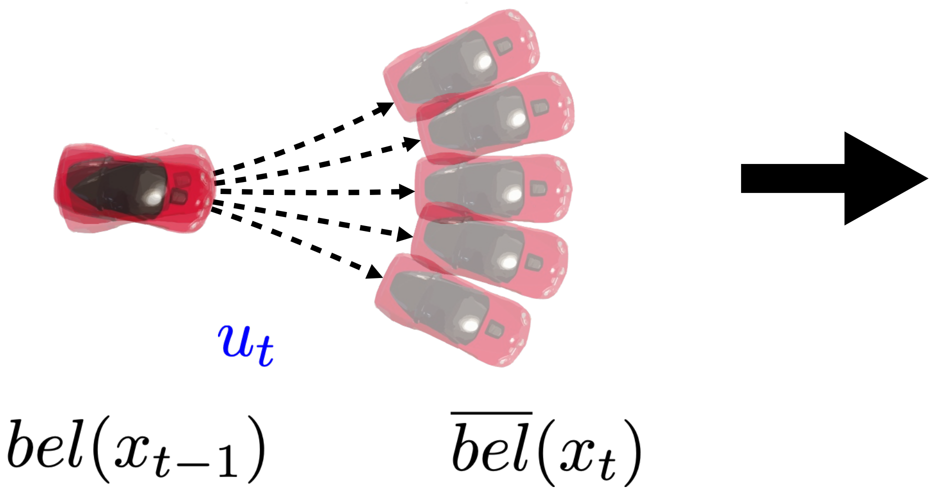
Bayes filter in a nutshell

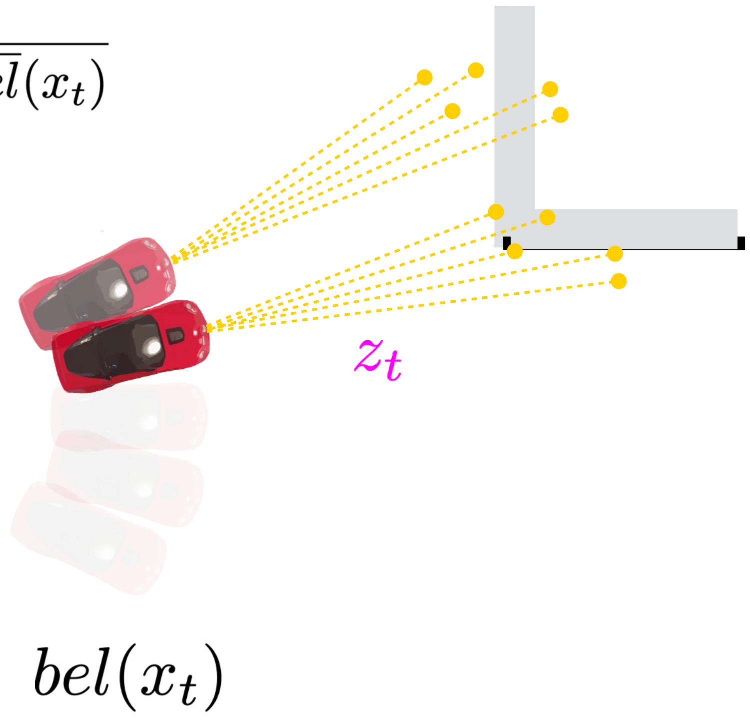
Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \frac{\overline{bel}(x_t)p(z_t|x_t)}{\sum_{x_t} \overline{bel}(x_t)p(z_t|x_t)} \Rightarrow$$

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

$$\eta = \frac{1}{\sum P(z_t|x_t)\overline{bel}(x_t)}$$



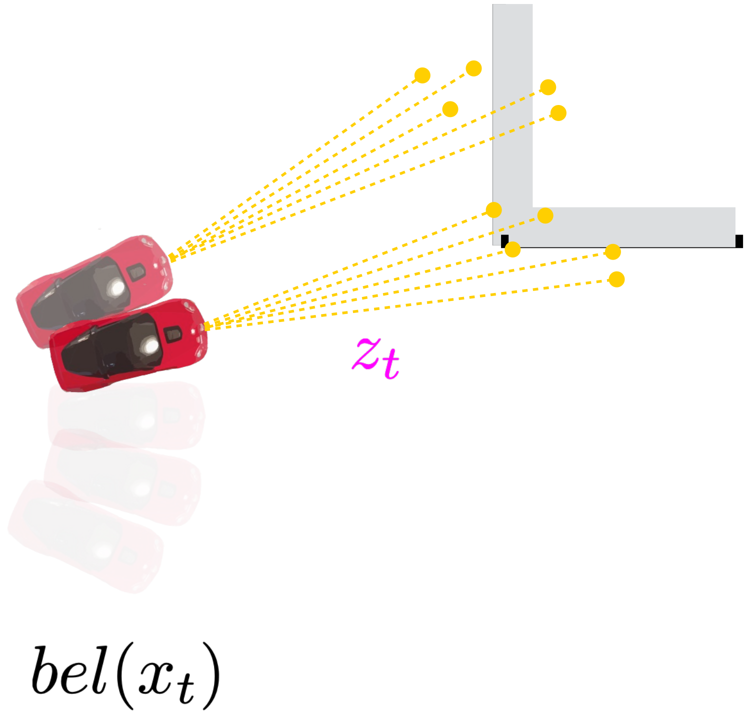


Derivation: Measurement Update

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \frac{\overline{bel}(x_t)p(z_t|x_t)}{\sum_{x_t}\overline{bel}(x_t)p(z_t|x_t)} \Rightarrow \quad bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

$$\eta = \frac{1}{\sum P(z_t|x_t)\overline{bel}(x_t)}$$



$$bel(x_t) = p(x_t|u_{1:t}, z_{1:t})$$

$$= p(x_t|u_{1:t}, z_{1:t-1}, z_t)$$

$$P(Y|X,Z) = \frac{P(X|Y,Z)P(Y|Z)}{\sum_Y P(X|Y,Z)P(Y|Z)} \quad \text{(Bayes)}$$

$$= \frac{p(z_t|u_{1:t}, z_{1:t-1}, x_t)p(x_t|u_{1:t}, z_{1:t-1})}{\sum_{x_t} p(z_t|u_{1:t}, z_{1:t-1}, x_t)p(x_t|u_{1:t}, z_{1:t-1})}$$

$$= \frac{p(z_t|x_t)p(x_t|u_{1:t}, z_{1:t-1})}{\sum_{x_t} p(z_t|x_t)p(x_t|u_{1:t}, z_{1:t-1})} \quad \text{(Markov)}$$

Bayes filter in a nutshell

Key Idea: Apply Markov to get a recursive update!

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \sum P(x_t|u_t, x_{t-1})bel(x_{t-1})$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

Bayes filter is a powerful tool

Localization Mapping SLAM POMDP

Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE $P(x_t | x_{t-1}, u_t)$

$$P(O|C,P) = 0.7$$
$$P(C|C,P) = 0.3$$

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Example: Opening a Door

$$\begin{bmatrix} P(x_t = \mathbf{O} | x_{t-1} = \mathbf{O}, u_t) & P(x_t = \mathbf{O} | x_{t-1} = \mathbf{C}, u_t) \\ P(x_t = \mathbf{C} | x_{t-1} = \mathbf{O}, u_t) & P(x_t = \mathbf{C} | x_{t-1} = \mathbf{C}, u_t) \end{bmatrix}$$

$$P(.|., \mathbf{P}) = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \qquad P(.|., \mathbf{L}) = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$$

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Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

$\mathcal{Z} =$ **O**PEN, **C**LOSED

$$P(z_t|x_t)$$

$$\begin{bmatrix} P(z_t|\mathbf{O}) \\ P(z_t|\mathbf{C}) \end{bmatrix} \qquad P(\mathbf{O}|.) = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix} \quad P(\mathbf{C}|.) = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix}$$

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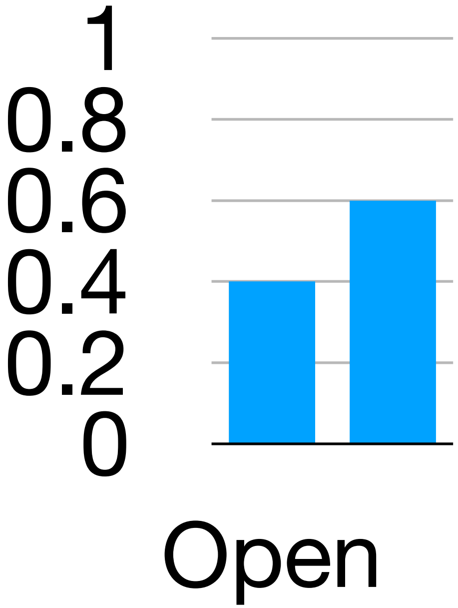
Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

$\mathcal{Z} =$ **O**PEN, **C**LOSED

$$Bel(x_0) = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$



PULL

Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

$\mathcal{Z} =$ **O**PEN, **C**LOSED

Prediction: Given action, propagate belief through dynamics

$$\overline{Bel}(x_t) = \sum_{x_{t-1}} P(x_t | u_t, x_{t-1}) Bel(x_{t-1})$$

$$\underbrace{\begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix}}_{\overline{Bel}(x_t)} = \begin{bmatrix} P(x_t = \mathbf{O} | x_{t-1} = \mathbf{O}, u_t) & P(x_t = \mathbf{O} | x_{t-1} = \mathbf{C}, u_t) \\ P(x_t = \mathbf{C} | x_{t-1} = \mathbf{O}, u_t) & P(x_t = \mathbf{C} | x_{t-1} = \mathbf{C}, u_t) \end{bmatrix} \underbrace{\begin{bmatrix} P(x_{t-1} = \mathbf{O}) \\ P(x_{t-1} = \mathbf{C}) \end{bmatrix}}_{Bel(x_{t-1})}$$

Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

$\mathcal{Z} =$ **O**PEN, **C**LOSED

Prediction: Given action, propagate belief through dynamics

$$\overline{Bel}(x_t) = \sum_{x_{t-1}} P(x_t | u_t, x_{t-1}) Bel(x_{t-1})$$

$$\underbrace{\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}}_{\overline{Bel}(x_t)} = \underbrace{\begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix}}_{P(.|.,\mathbf{P})} \underbrace{\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}}_{Bel(x_{t-1})}$$

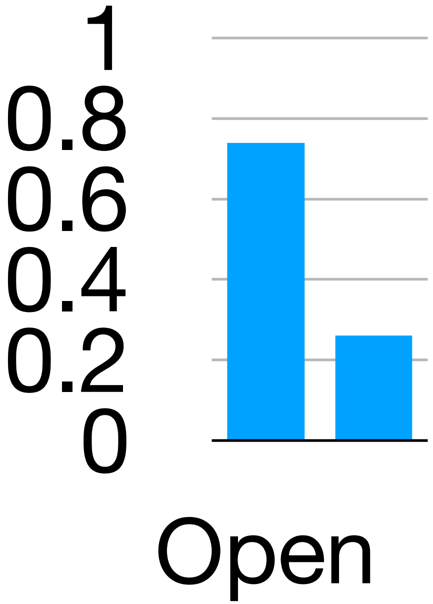
Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

$\mathcal{Z} =$ **O**PEN, **C**LOSED

$$\overline{Bel}(x_t) = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$$



CLOSED

Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

$\mathcal{Z} =$ **O**PEN, **C**LOSED

Correction: Given measurement, apply Bayes' rule

$$Bel(x_t) = \eta P(z_t | x_t) \overline{Bel}(x_t)$$

$$\begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix} = \eta \begin{bmatrix} P(z_t | \mathbf{O}) \\ P(z_t | \mathbf{C}) \end{bmatrix} * \begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix}$$

$Bel(x_t)$ $\quad P(\mathbf{C}|.)$ $\quad \overline{Bel}(x_t)$

Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

$\mathcal{Z} =$ **O**PEN, **C**LOSED

Correction: Given measurement, apply Bayes' rule

$$Bel(x_t) = \eta P(z_t|x_t)\overline{Bel}(x_t)$$

$$\underbrace{\begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix}}_{Bel(x_t)} = \eta \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} * \underbrace{\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}}_{\overline{Bel}(x_t)} = \eta \begin{bmatrix} 0.296 \\ 0.208 \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$$

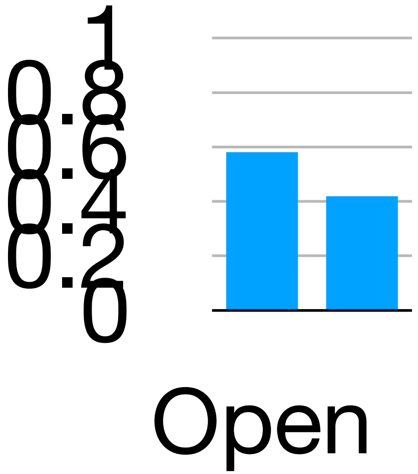
Example: Opening a Door

$\mathcal{X} =$ **O**PEN, **C**LOSED

$\mathcal{A} =$ **P**ULL, **L**EAVE

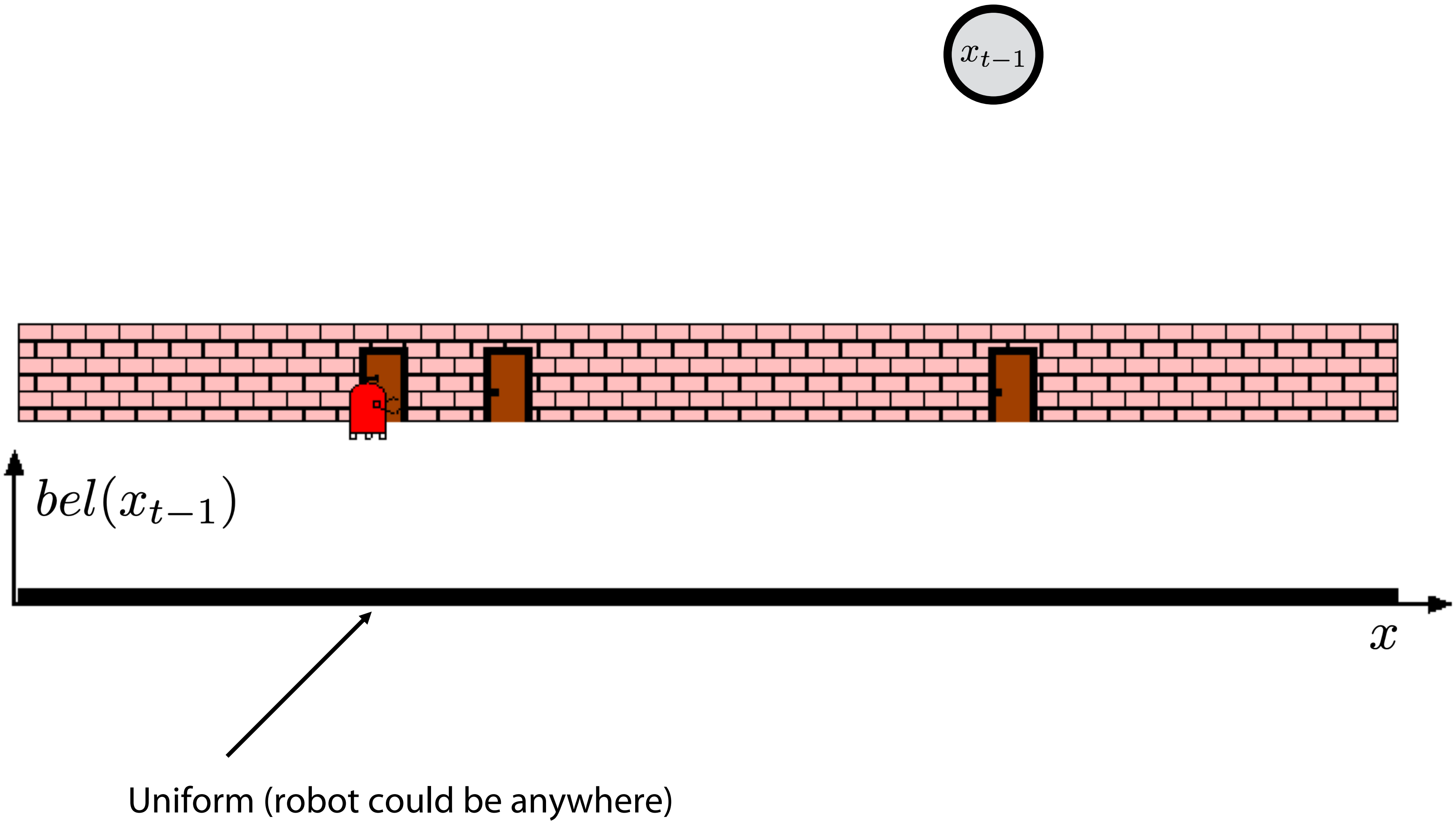
$\mathcal{Z} =$ **O**PEN, **C**LOSED

$$Bel(x_t) = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$$



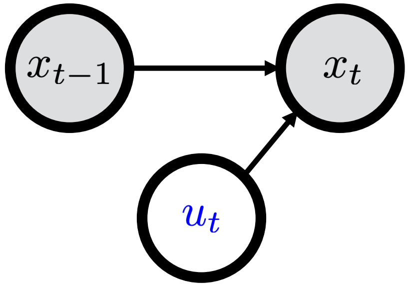
- Robot initially thought the door was open with 0.4 prob
- Robot took the PULL action, then thought the door was open with 0.74 prob
- Robot received a CLOSED measurement, now thinks open with 0.58 prob

Robot lost in a 1-D hallway

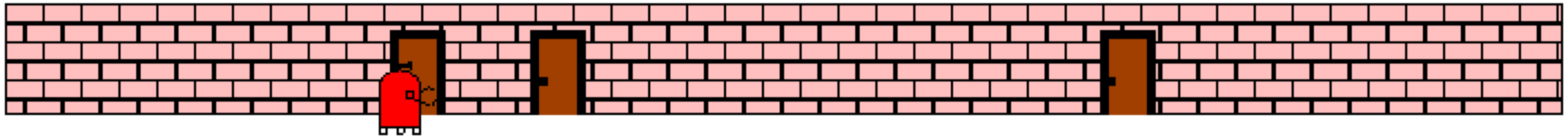


Action at time t: NOP

$$u_t = \text{NOP}$$



$$P(x_t | u_t, x_{t-1}) = \delta(x_t = x_{t-1})$$



$$\overline{bel}(x_t) = \int P(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1} = bel(x_t)$$

x

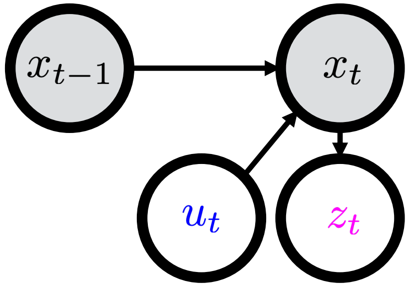
NOP action implies belief remains the same!

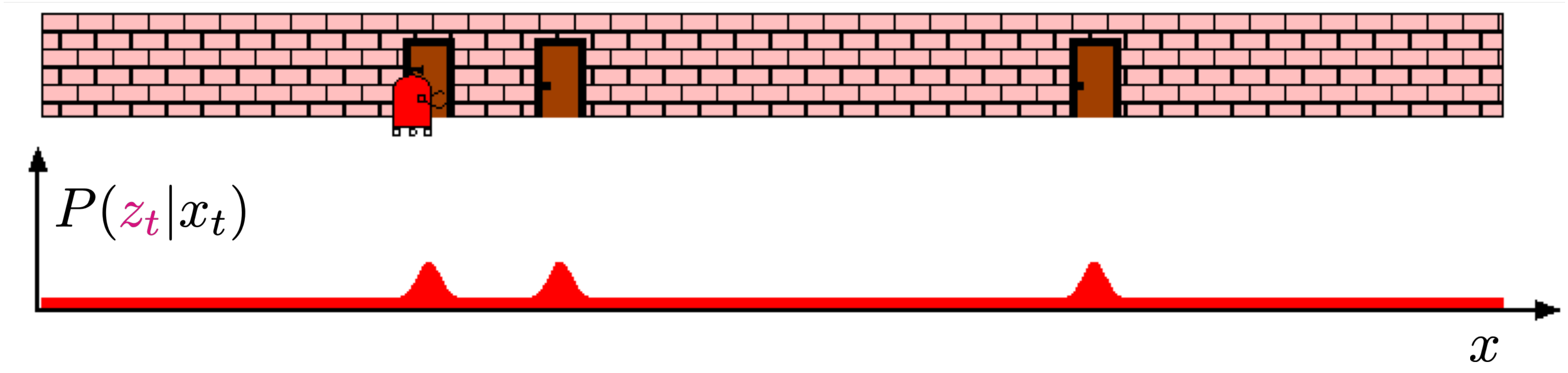
(still uniform — no idea where I am)

Measurement at time t: “Door”

$z_t = \text{Door}$

$P(z_t|x_t) = \mathcal{N}(\text{door centre}, 0.75m)$

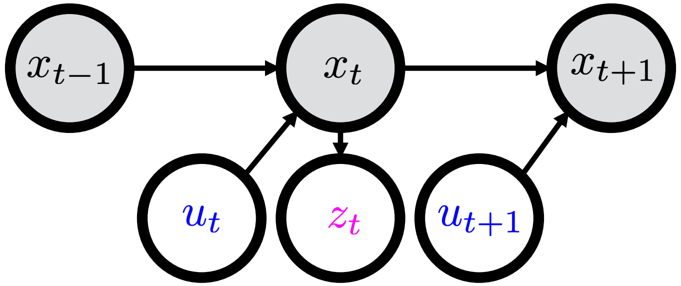


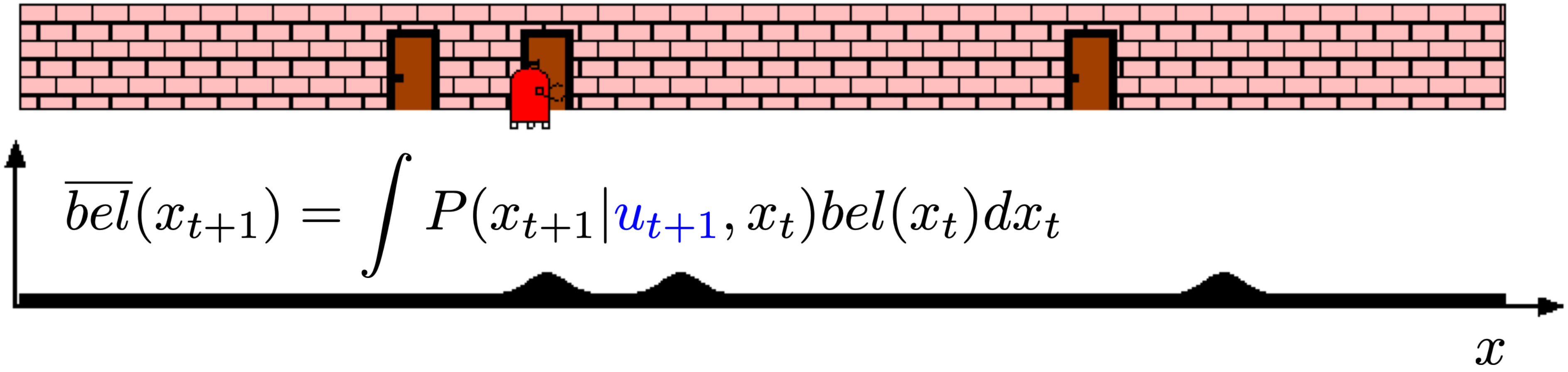


Action at time t+1: Move 3m right

$u_{t+1} = \text{3m right}$

$$P(x_{t+1}|u_{t+1}, x_t) = \mathcal{N}(x_t + u_{t+1}, 0.25m)$$



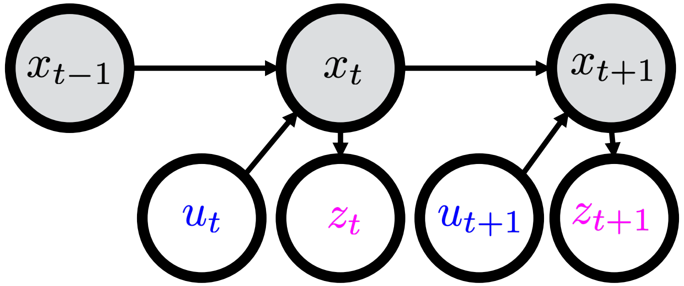


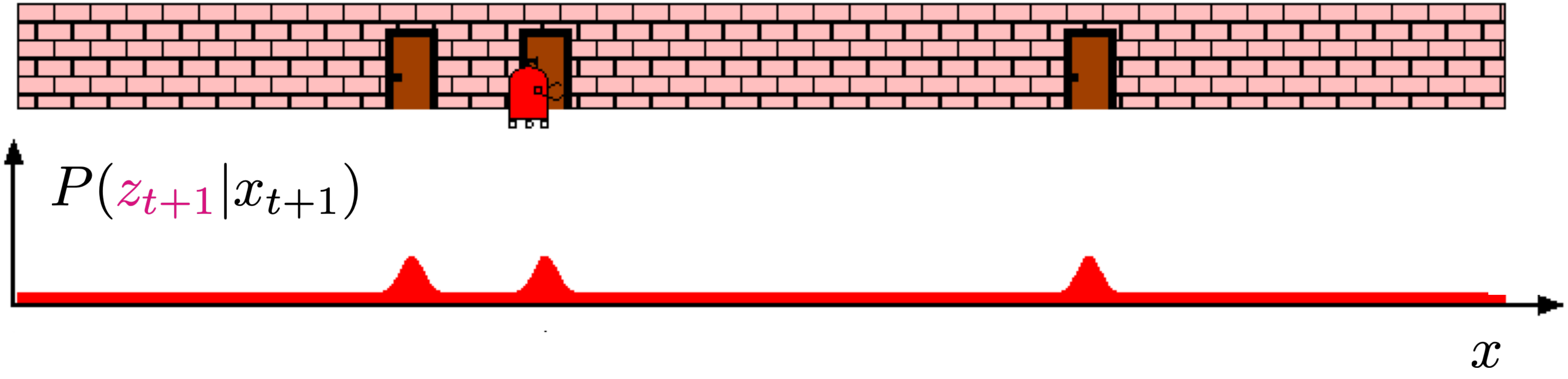
$$\overline{bel}(x_{t+1}) = \int P(x_{t+1}|u_{t+1}, x_t) bel(x_t) dx_t$$

Measurement at time t+1: “Door”

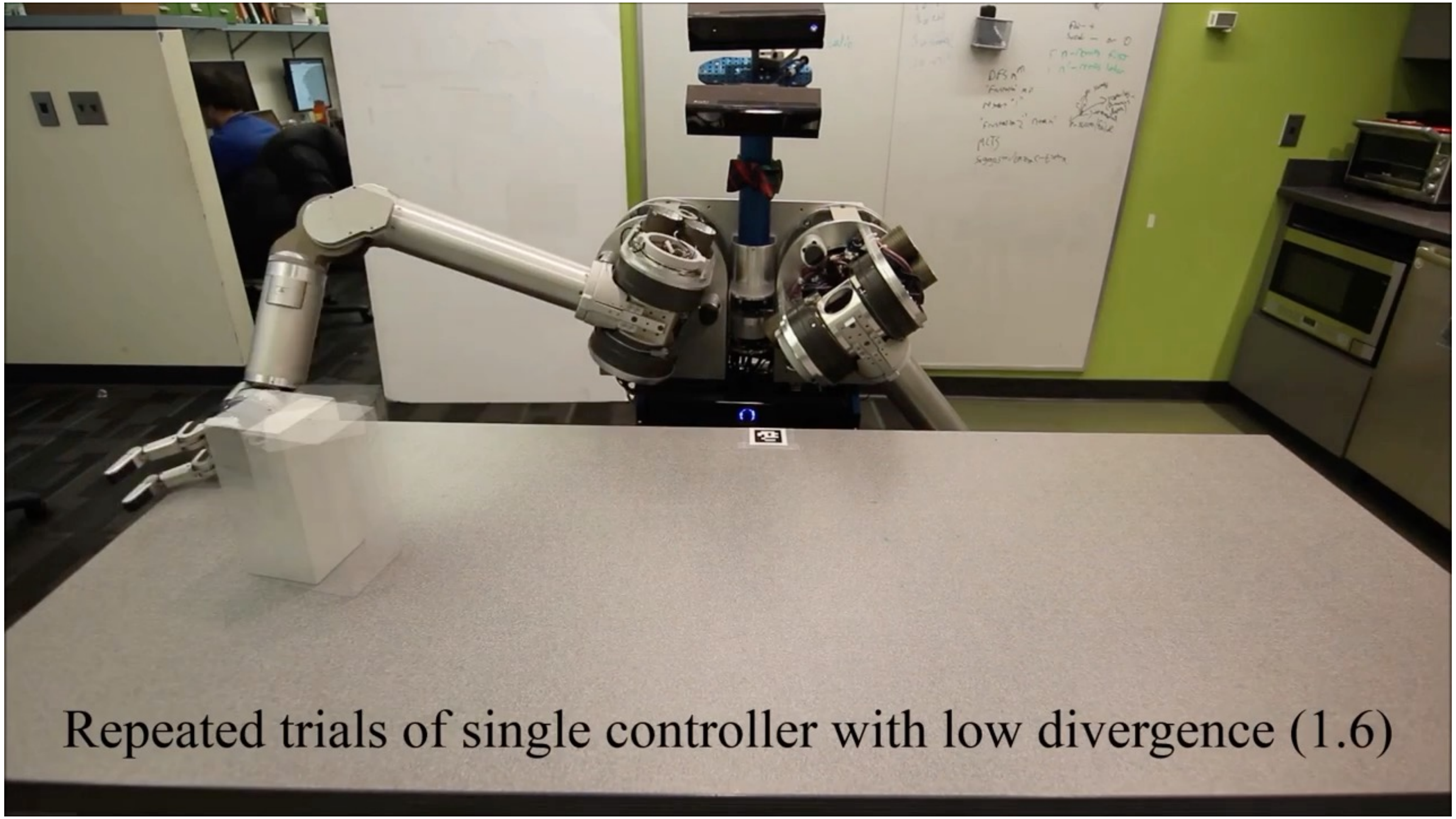
$z_{t+1} = \text{Door}$

$$P(z_{t+1}|x_{t+1}) = \mathcal{N}(\text{door centre}, 0.75m)$$





Do actions always increase uncertainty?



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Do measurements always reduce uncertainty?

- Level of uncertainty can be formalized as **entropy**
  - Low entropy if belief is tightly concentrated (e.g., concentrated on one state)
  - High entropy if belief is very spread out (e.g., uniform distribution)
- What if you reach into your pocket and can't find your keys?
  - Initially: low entropy (belief concentrated around pocket, some probability in other states around the house)
  - After: high entropy (very little probability in pocket, other states around the house have increased probability)

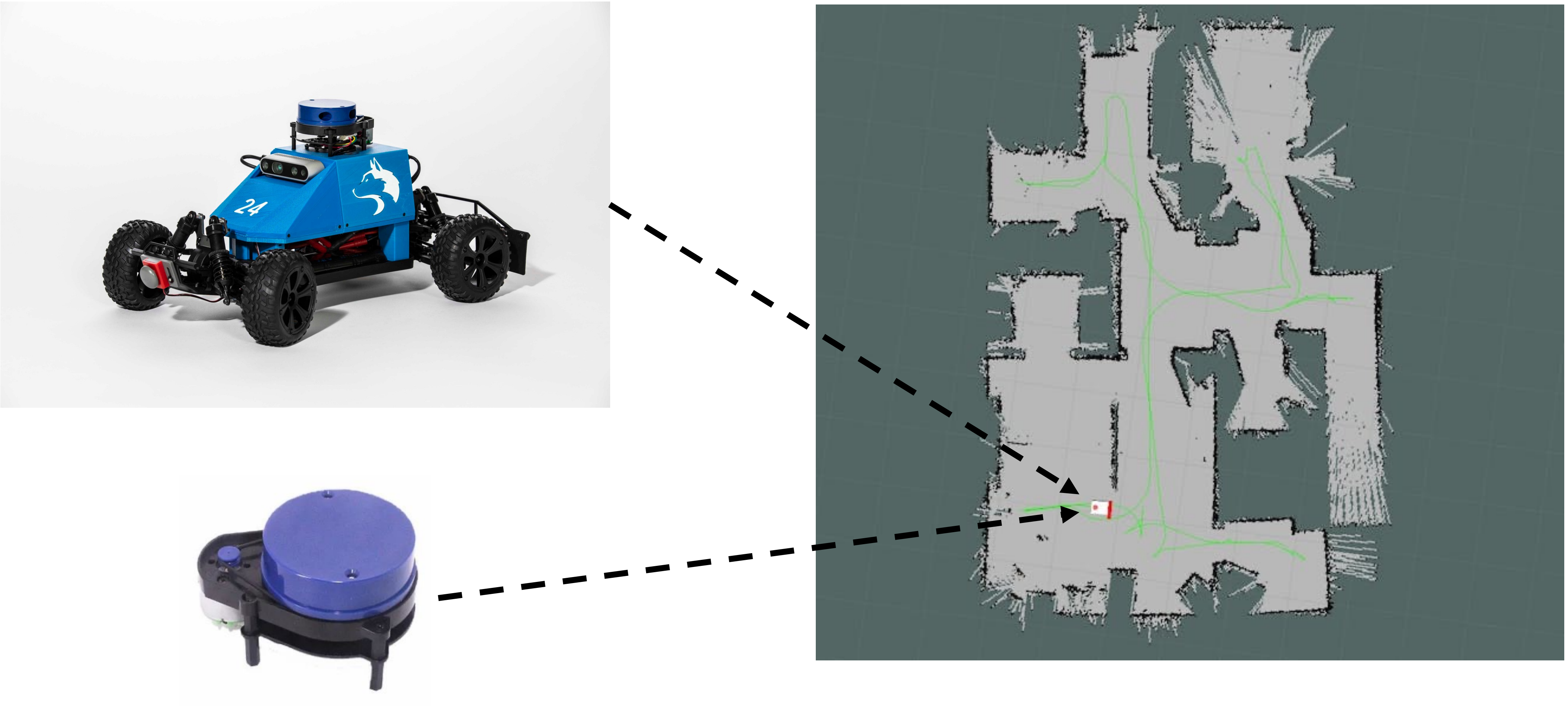
Ok this seems simple? What makes this hard!

$$Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}$$

Tractable Bayesian inference is challenging in the general case

We will work out the conjugate prior and discrete case, leaving the MCMC/VI cases as an exercise

How does this connect back to our racecar?



Where am I in the world?

Lecture Outline

Probability Recap

↓

Bayesian Filtering w/ Examples

↓

Motion Models

So what do we need to define to instantiate this?

Key Idea: Apply Markov to get a recursive update!

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

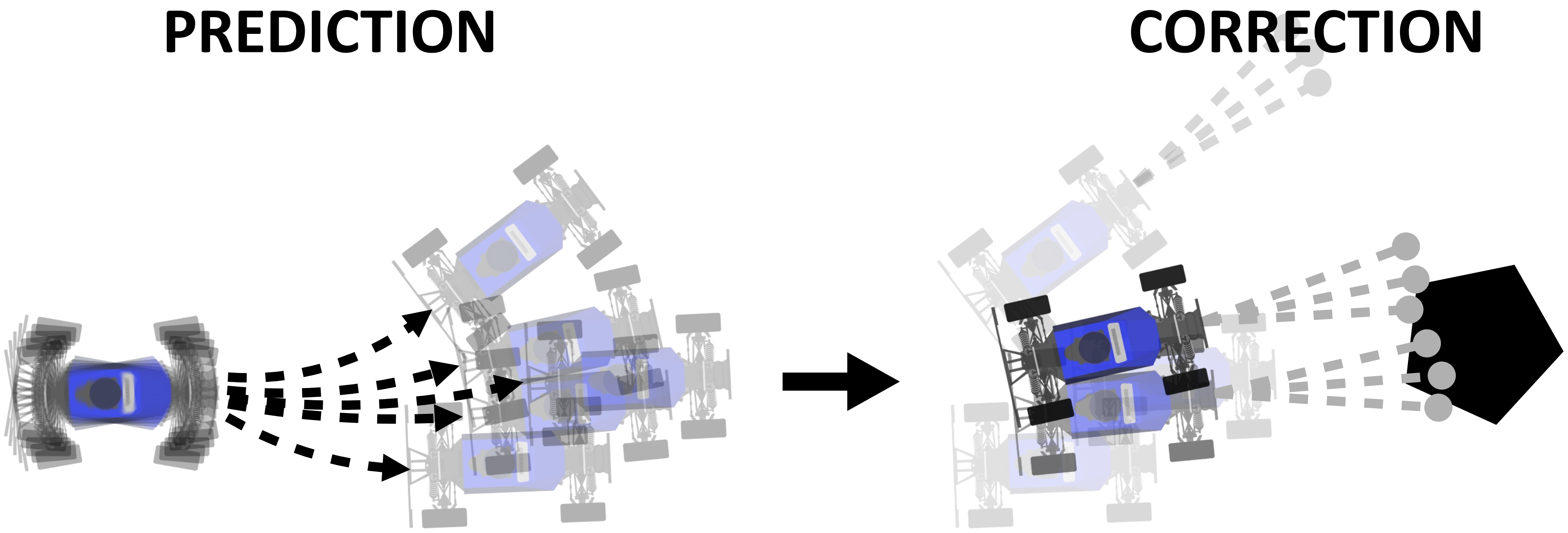
Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \sum P(x_t|u_t, x_{t-1})bel(x_{t-1})$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

Let's ground this in the context of the car



PREDICTION

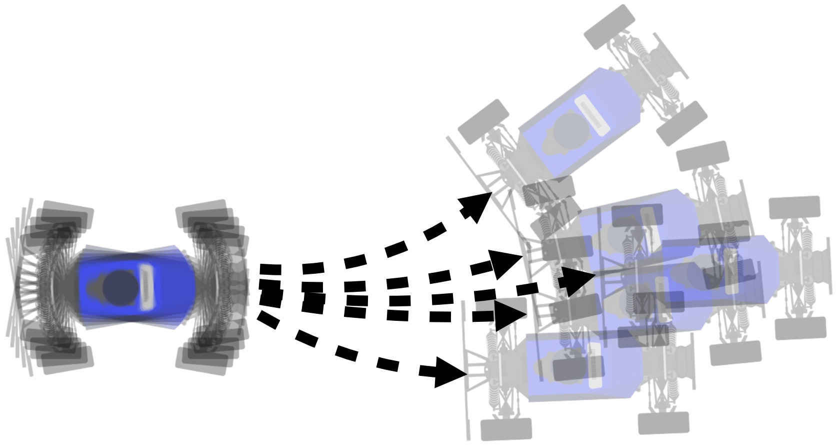
$$P(x_t | u_t, x_{t-1})$$

CORRECTION

$$P(z_t | x_t)$$

Motion Model

How do we know this?
→ it's just physics!



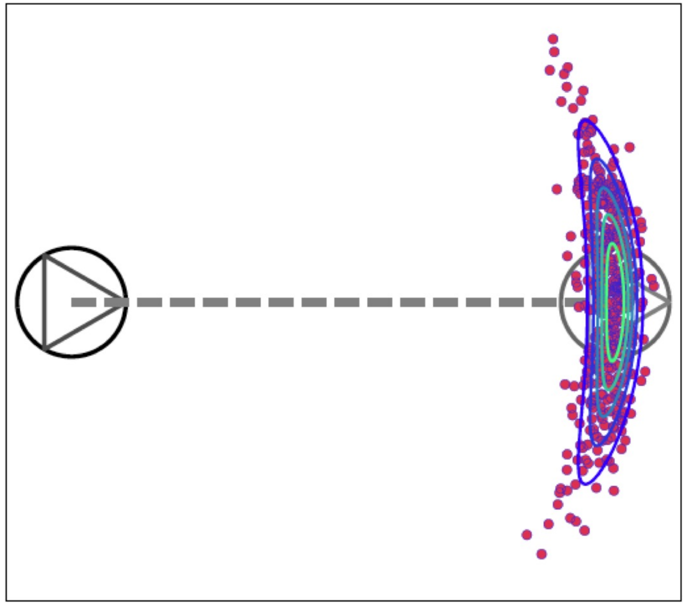
$$P(x_t | u_t, x_{t-1})$$

A Spectrum of Motion Models

Highest-fidelity models capturing everything we know

(Red Bull F1 Simulator)

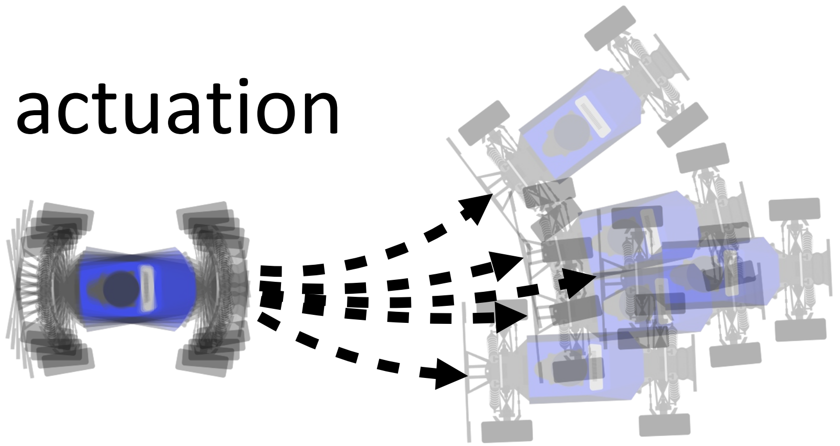
VS



Simple model
with lots of noise

Why is the motion model probabilistic?

- If we know how to write out equations of motion, shouldn’t we be able to predict exactly where an object ends up?
- “All models are wrong, but some are useful” — George Box
  - Examples: ideal gas law, Coulomb friction
- Stochasticity is a catch-all for model error, actuation error, …



What defines a good motion model?

- In theory: try to accurately model the uncertainty (e.g., actuation errors)
- In practice…
  - We need just enough stochasticity to **explain any measurements** we'll see (Bayes filter uses measurements to hone in on the right state)
  - We need a model that can deal with **unknown unknowns** (No matter the model, we need to overestimate uncertainty)
  - We would like a model that is **computationally cheap** (Bayes filter repeatedly invokes this model to predict state after actions)
- Key idea: simple model + stochasticity

What motion model should I use for MuSHR?

- A **kinematic model** governs how wheel speeds map to robot velocities and change in position
- A **dynamic model** governs how wheel torques map to robot accelerations and change in velocity
- For MuSHR, we'll ignore dynamics and focus on kinematics (assuming the wheel actuators can set speed directly)
- Other assumptions: wheels roll on hard, flat, horizontal ground without slipping

Kinematic Car Model

$$\dot{x} = f(x, u)$$

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

X-COORDINATE

Y-COORDINATE

HEADING

$$\begin{bmatrix} v \\ \delta \end{bmatrix}$$

SPEED

STEERING ANGLE

Kinematic Car Model

$$\dot{x} = f(x, u)$$

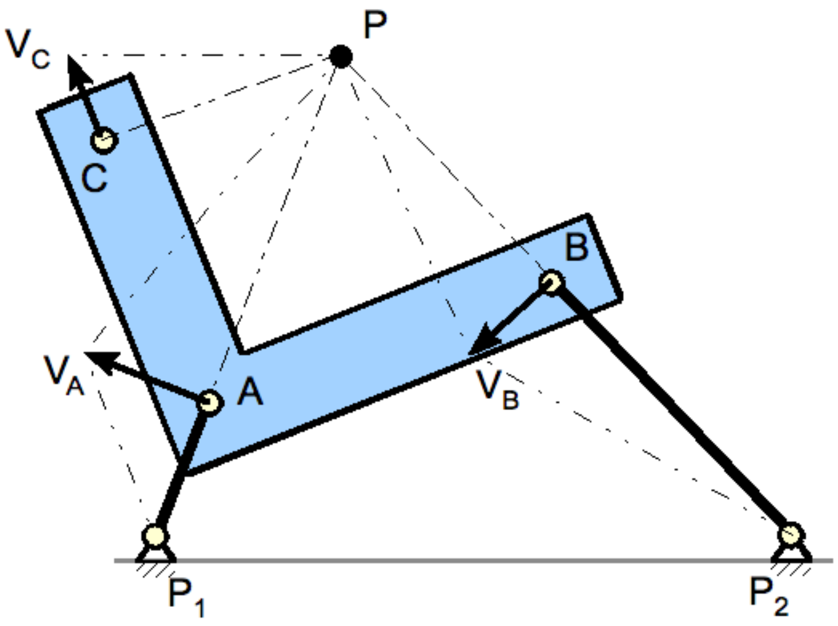
→ INTEGRATE

$$\begin{bmatrix} x_{t-1} + \Delta x \\ y_{t-1} + \Delta y \\ \theta_{t-1} + \Delta\theta \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}$$

→ ADD NOISE

$$P(x_t | u_t, x_{t-1})$$

Definition: Instant Center of Rotation (CoR)



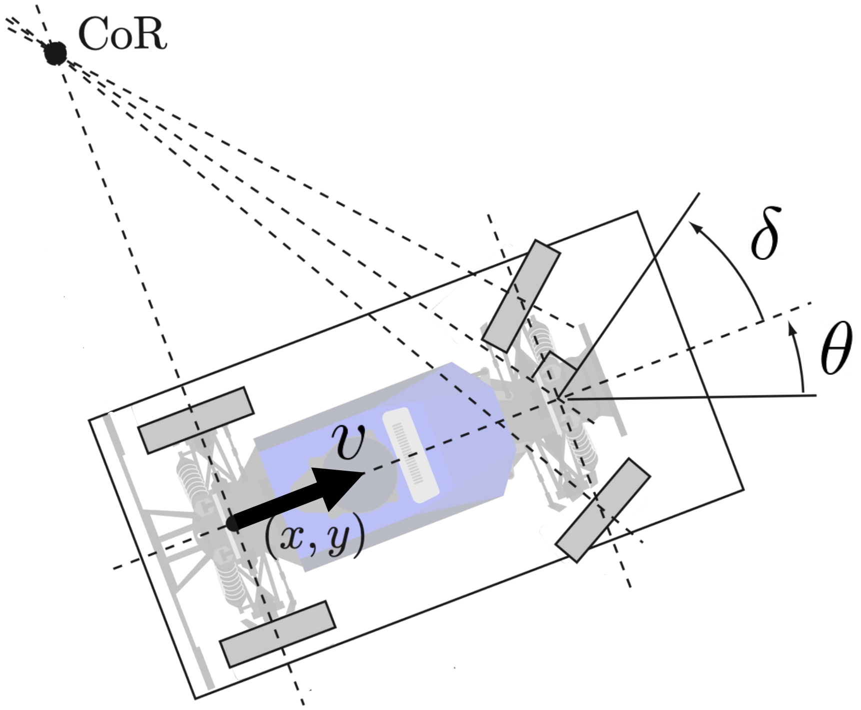
A planar **rigid body** undergoing a **rigid transformation** can be viewed as undergoing a **pure rotation** about an instant center of rotation.

rigid body: a non-deformable object

rigid transformation: a combined rotation and translation

HTTPS://EN.WIKIPEDIA.ORG/WIKI/INSTANT_CENTRE_OF_ROTATION

Equations of Motion

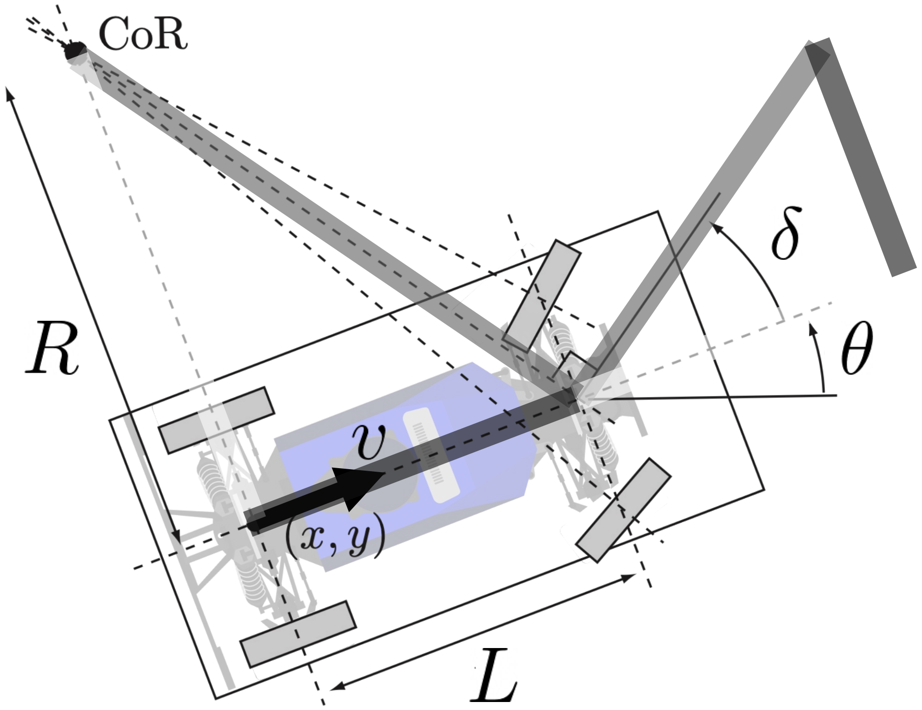


$$\dot{x} = v \cos\theta$$

$$\dot{y} = v \sin\theta$$

$$\dot{\theta} = \textbf{?}$$

Equations of Motion



$$\dot{x} = v\cos\theta$$

$$\dot{y} = v\sin\theta$$

$$\dot{\theta} = \omega = \frac{v}{R} = \frac{v\tan\delta}{L}$$

$$\tan\delta = \frac{L}{R} \rightarrow R = \frac{L}{\tan\delta}$$

Kinematic Car Model

$$\dot{x} = f(x, u) \quad \xrightarrow{\text{INTEGRATE}} \quad \begin{bmatrix} x_{t-1} + \Delta x \\ y_{t-1} + \Delta y \\ \theta_{t-1} + \Delta\theta \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}$$

Integrate the Kinematics Numerically

$$\dot{x} = v \cos\theta$$
$$\dot{y} = v \sin\theta$$
$$\dot{\theta} = \frac{v}{L} \tan\delta$$

Assume that steering angle is **piecewise constant** between t and t'

Integrate the Kinematics Numerically

$$\Delta x = \int_t^{t'} v\cos\theta(t)dt = \int_t^{t'} \frac{v\cos\theta}{\dot\theta}\frac{d\theta}{dt}dt = \frac{v}{\dot\theta}\int_\theta^{\theta'}\cos\theta d\theta$$

$$= \frac{L}{\tan\delta}(\sin\theta' - \sin\theta)$$

$$\Delta y = \frac{L}{\tan\delta}(\cos\theta - \cos\theta')$$

$$\Delta\theta = \int_t^{t'} \dot\theta dt = \frac{v}{L}\tan\delta\Delta t$$

$$\dot x = v\cos\theta$$
$$\dot y = v\sin\theta$$
$$\dot\theta = \frac{v}{L}\tan\delta$$

Assume that steering angle is **piecewise constant** between t and t’

Kinematic Car Update

$$\theta_t = \theta_{t-1} + \Delta\theta = \theta_{t-1} + \frac{v}{L}\tan\delta\Delta t$$

$$x_t = x_{t-1} + \Delta x = x_{t-1} + \frac{L}{\tan\delta}(\sin\theta_t - \sin\theta_{t-1})$$

$$y_t = y_{t-1} + \Delta y = y_{t-1} + \frac{L}{\tan\delta}(\cos\theta_{t-1} - \cos\theta_t)$$

Kinematic Car Model

$$\dot{x} = f(x, u)$$

→ INTEGRATE

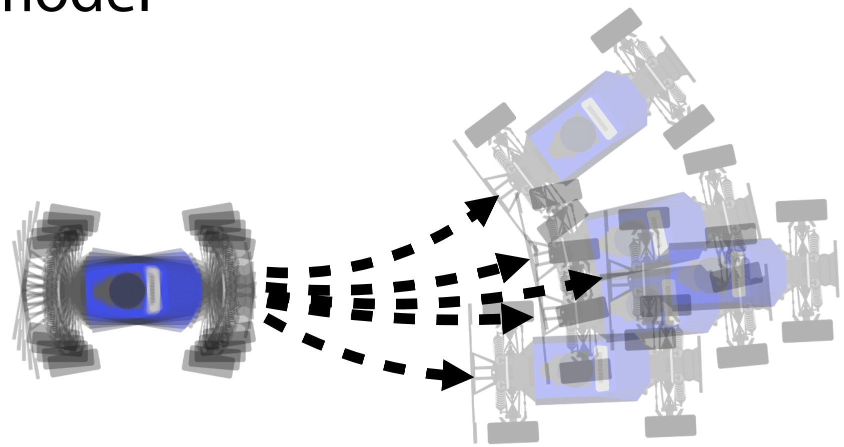
$$\begin{bmatrix} x_{t-1} + \Delta x \\ y_{t-1} + \Delta y \\ \theta_{t-1} + \Delta\theta \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}$$

→ ADD NOISE

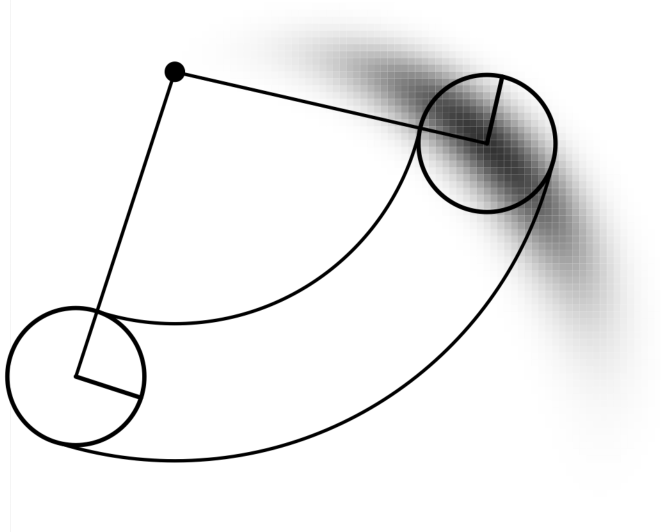
$$P(x_t | u_t, x_{t-1})$$

Why is the kinematic car model probabilistic?

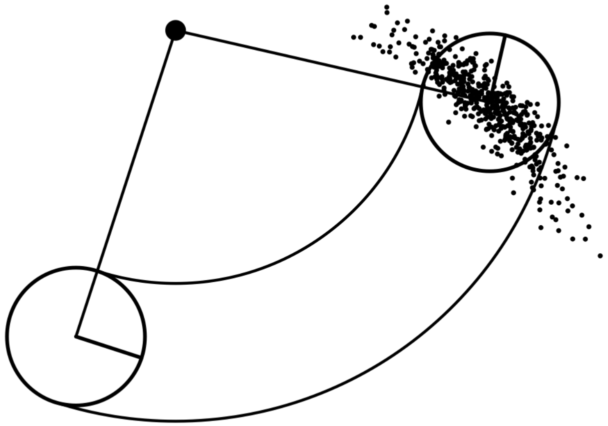
- Control signal error: voltage discretization, communication lag
- Unmodeled physics parameters: friction of carpet, tire pressure
- Incorrect physics: ignoring tire deformation, ignoring wheel slippage
- Our probabilistic motion model
  - Add noise to control before propagating through model
  - Add noise to state after propagating through model



Motion Model Summary



MOTION MODEL
PROB. DENSITY FUNCTION



MOTION MODEL
SAMPLES

- Write down the deterministic equations of motion (kinematic car model)
- Introduce stochasticity to account against various factors

Class Outline

State Estimation

- Robotic System Design
- Filtering
- Localization
- SLAM

Control

- Feedback Control
- PID Control
- MPC
- LQR

Planning

- Search
- Heuristic Search
- Motion Planning
- Lazy Search

Learning

- Imitation Learning
- Policy Gradient
- Actor-Critic
- Model-Based RL