CSE 478 Robot Autonomy
Linear Quadratic Regulator

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Control as an Optimization Problem

- For a sequence of H control actions
    1. Use model to predict consequence of actions (i.e., H future states)
    2. Evaluate the cost function
- Compute optimal sequence of H control actions (minimizes cost)

Linear Quadratic Regulator

- **Linear** system (model)
- **Quadratic** cost function to minimize

$$x_{t+1} = Ax_t + Bu_t$$

$$\sum_t x_t^\top Q x_t + u_t^\top R u_t$$

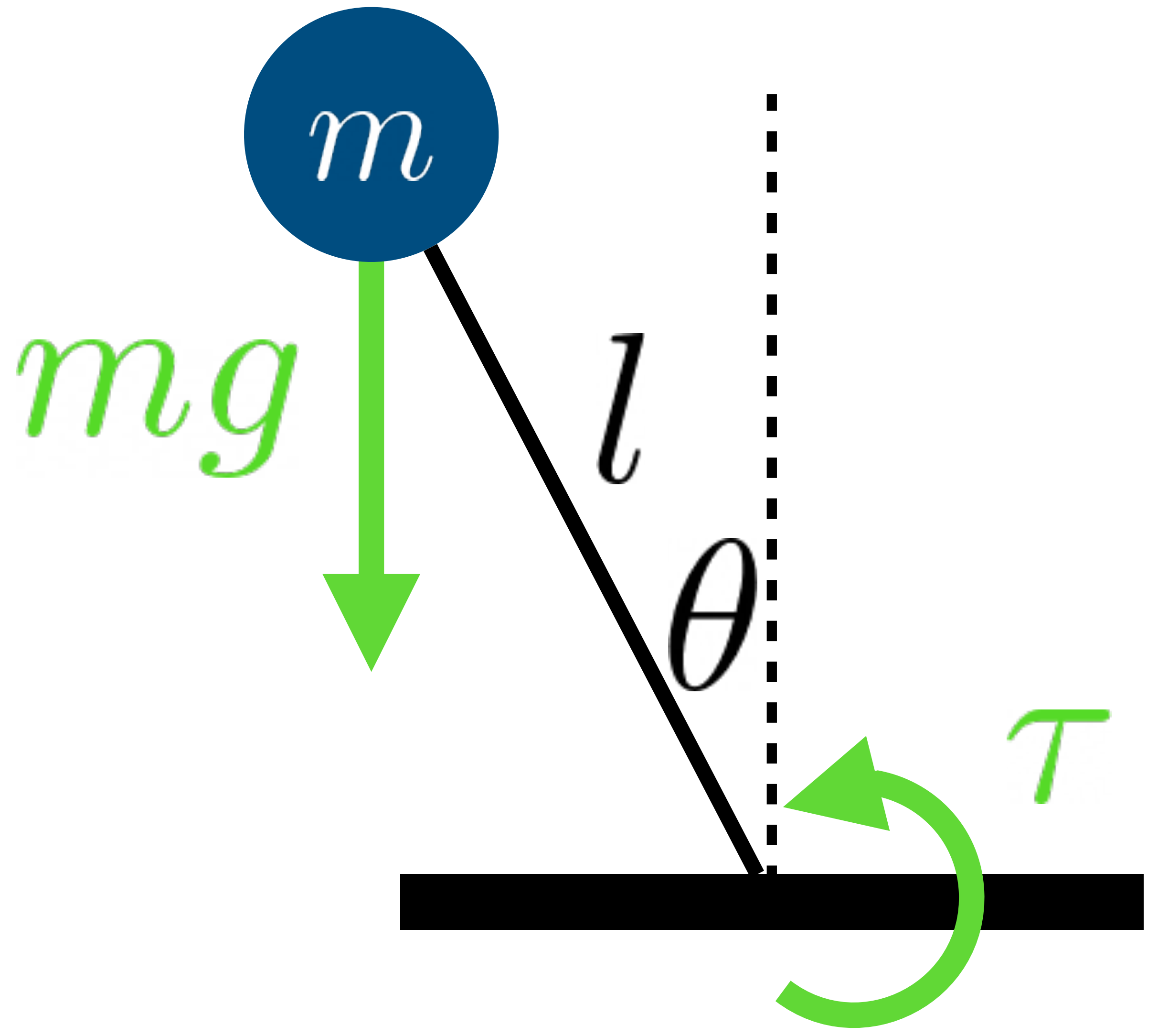
Linear System

- **Linear** system (model)
- **Quadratic** cost function to minimize

$$x_{t+1} = Ax_t + Bu_t$$
$$\sum_t x_t^\top Q x_t + u_t^\top R u_t$$

x_{t+1}	$=$	A	x_t	$+$	B	u_t
(N x 1)		(N x N)	(N x 1)		(N x M)	(M x 1)
		STATE → NEXT STATE			**CONTROL → NEXT STATE**	

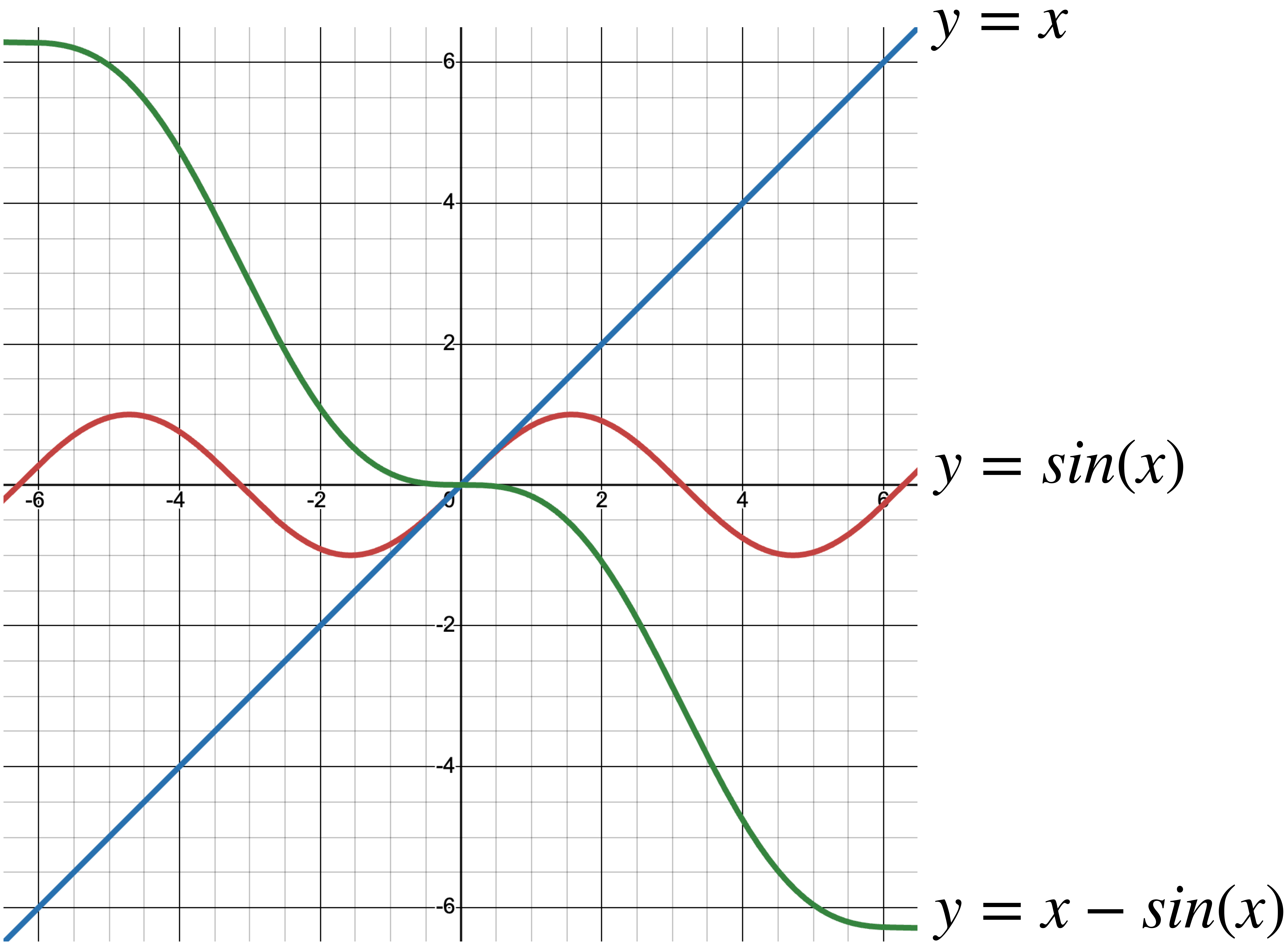
Example: Inverted Pendulum (Linear System)



$$mgl\sin\theta + \tau = ml^2\ddot{\theta}$$

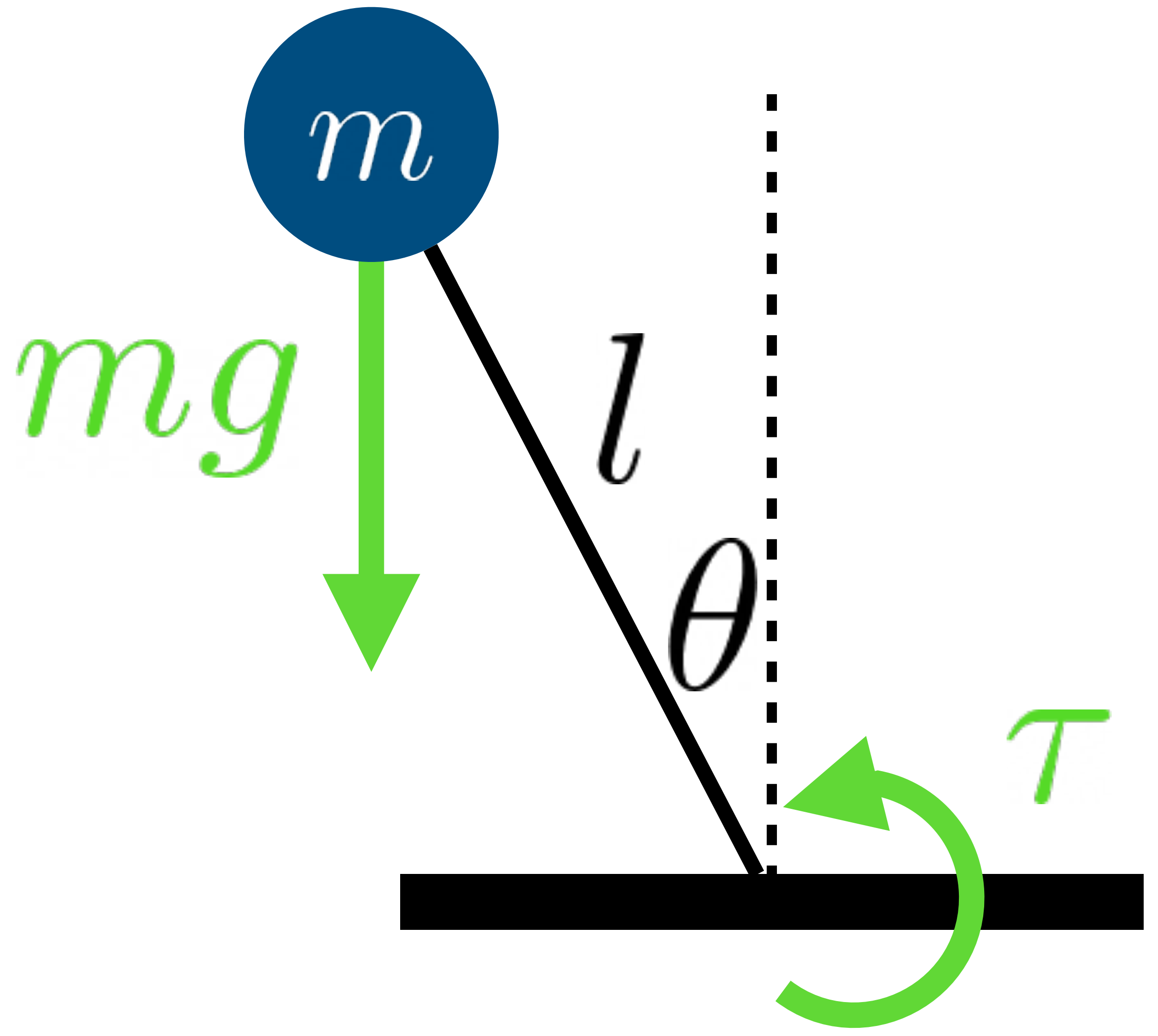
$$\ddot{\theta} = \frac{g}{l}\sin\theta + \frac{\tau}{ml^2} \approx \frac{g}{l}\theta + \frac{\tau}{ml^2}$$

Small angle assumption



6
4
2
-2
-4
-6
-6
-4
-2
0
2
4
6
y = x
y = sin(x)
y = x − sin(x)

Example: Inverted Pendulum (Linear System)



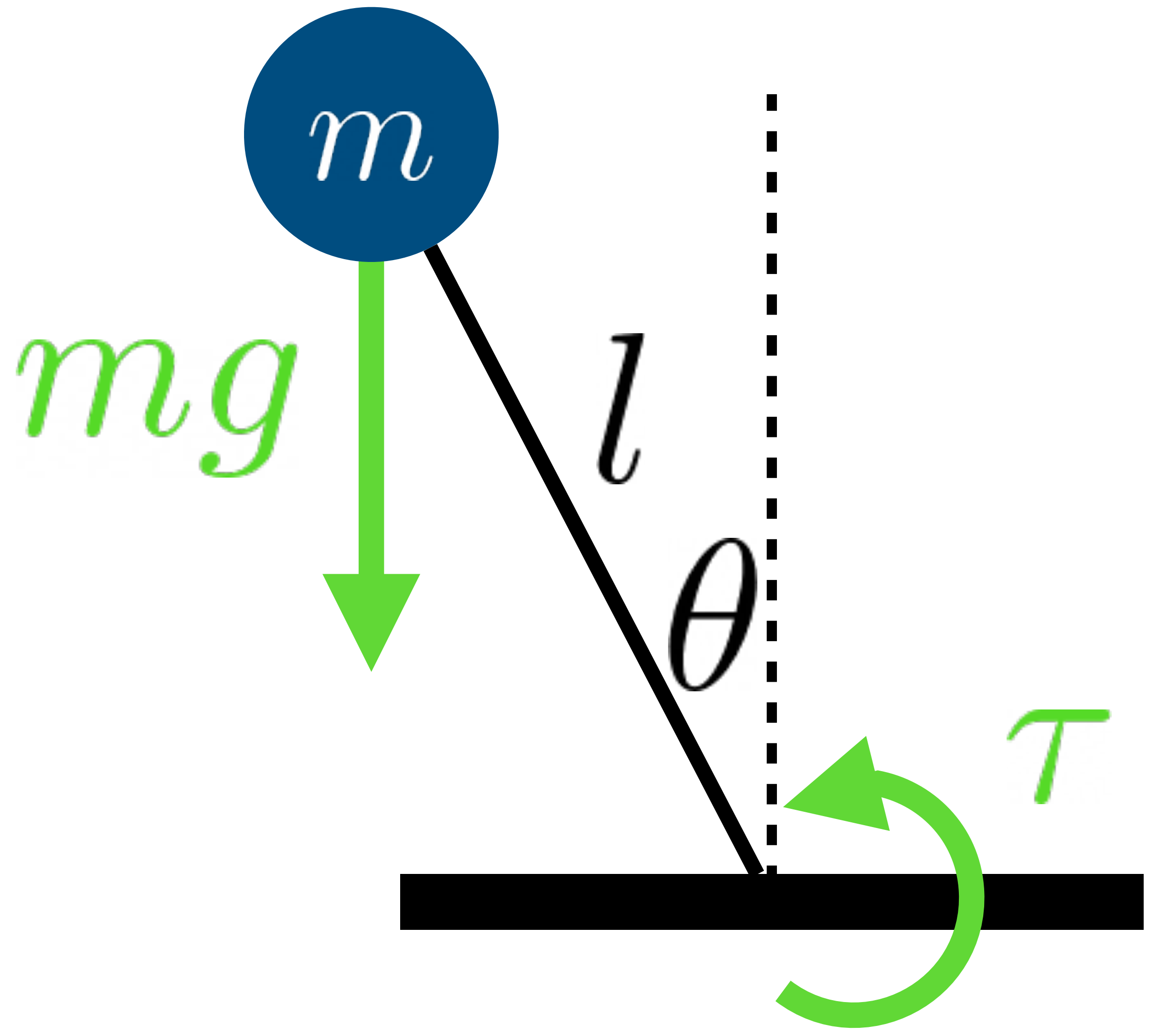
$$mgl\sin\theta + \tau = ml^2\ddot{\theta}$$

$$\ddot{\theta} = \frac{g}{l}\sin\theta + \frac{\tau}{ml^2} \approx \frac{g}{l}\theta + \frac{\tau}{ml^2}$$

$$\dot{\theta}_{t+1} = \frac{\theta_{t+1} - \theta_t}{\Delta t} \qquad \ddot{\theta}_{t+1} = \frac{\dot{\theta}_{t+1} - \dot{\theta}_t}{\Delta t}$$

Finite Differences

Example: Inverted Pendulum (Linear System)



$$mgl\sin\theta + \tau = ml^2\ddot{\theta}$$

$$\ddot{\theta} = \frac{g}{l}\sin\theta + \frac{\tau}{ml^2} \approx \frac{g}{l}\theta + \frac{\tau}{ml^2}$$

$$\underbrace{\begin{bmatrix}\theta_{t+1} \\ \dot{\theta}_{t+1}\end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix}1 & \Delta t \\ \frac{g}{l}\Delta t & 1\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}\theta_t \\ \dot{\theta}_t\end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix}0 \\ \Delta t\end{bmatrix}}_{B} \underbrace{\frac{\tau}{ml^2}}_{u_t}$$

Quadratic Cost Function

- **Linear** system (model)
- **Quadratic** cost function to minimize

$$x_{t+1} = Ax_t + Bu_t$$

$$\sum_t x_t^\top Q x_t + u_t^\top R u_t$$

What state errors do we care about?

$$x_t^\top Q x_t$$

(1 x N) (N x N) (N x 1)

STATE COST

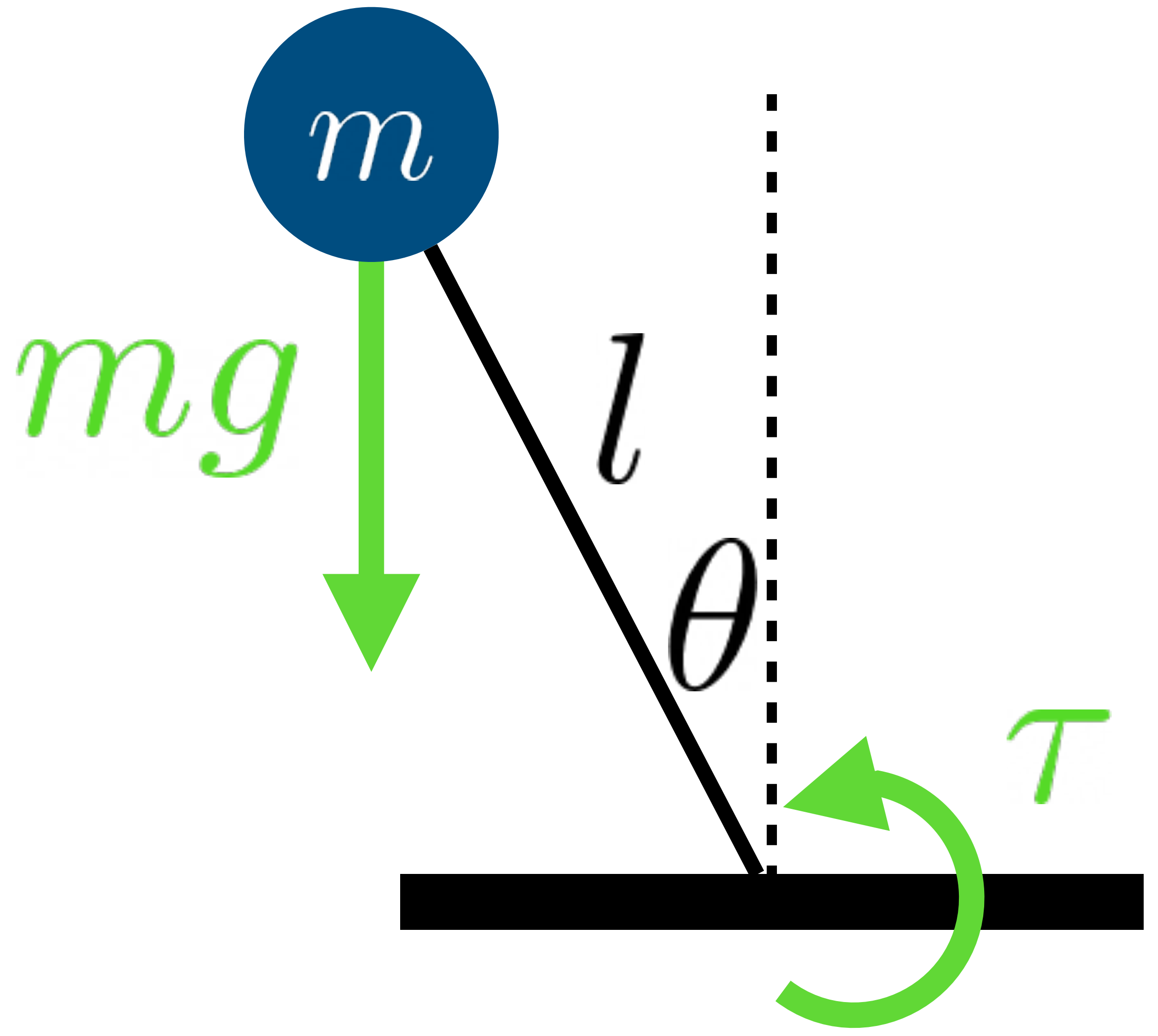
How much does the effort hurt?

$$u_t^\top R u_t$$

(1 x M)(M x M)(M x 1)

CONTROL COST

Example: Inverted Pendulum (State Cost)



$$x_t^\top Q x_t$$

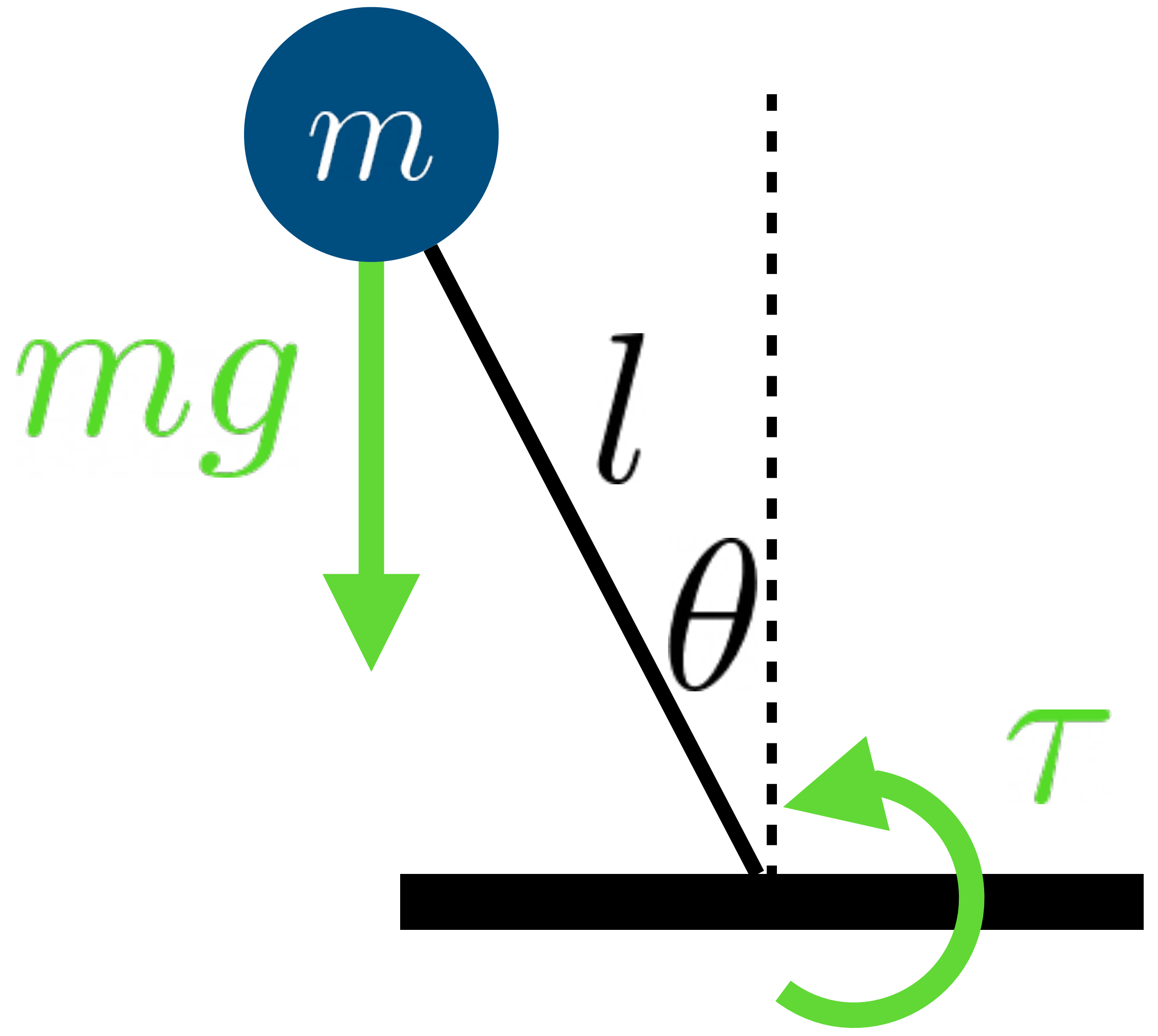
(QUADRATIC FORM)

$$= \begin{bmatrix} \theta_t \\ \dot{\theta}_t \end{bmatrix}^\top \begin{bmatrix} Q_{\theta\theta} & Q_{\theta\dot{\theta}} \\ Q_{\dot{\theta}\theta} & Q_{\dot{\theta}\dot{\theta}} \end{bmatrix} \begin{bmatrix} \theta_t \\ \dot{\theta}_t \end{bmatrix}$$

$$= Q_{\theta\theta}\theta_t^2 + 2Q_{\theta\dot{\theta}}\theta_t\dot{\theta}_t + Q_{\dot{\theta}\dot{\theta}}\dot{\theta}_t^2$$

$$Q \succ 0 \leftrightarrow z^\top Q z > 0,\ \forall z \neq 0$$

Example: Inverted Pendulum (Control Cost)



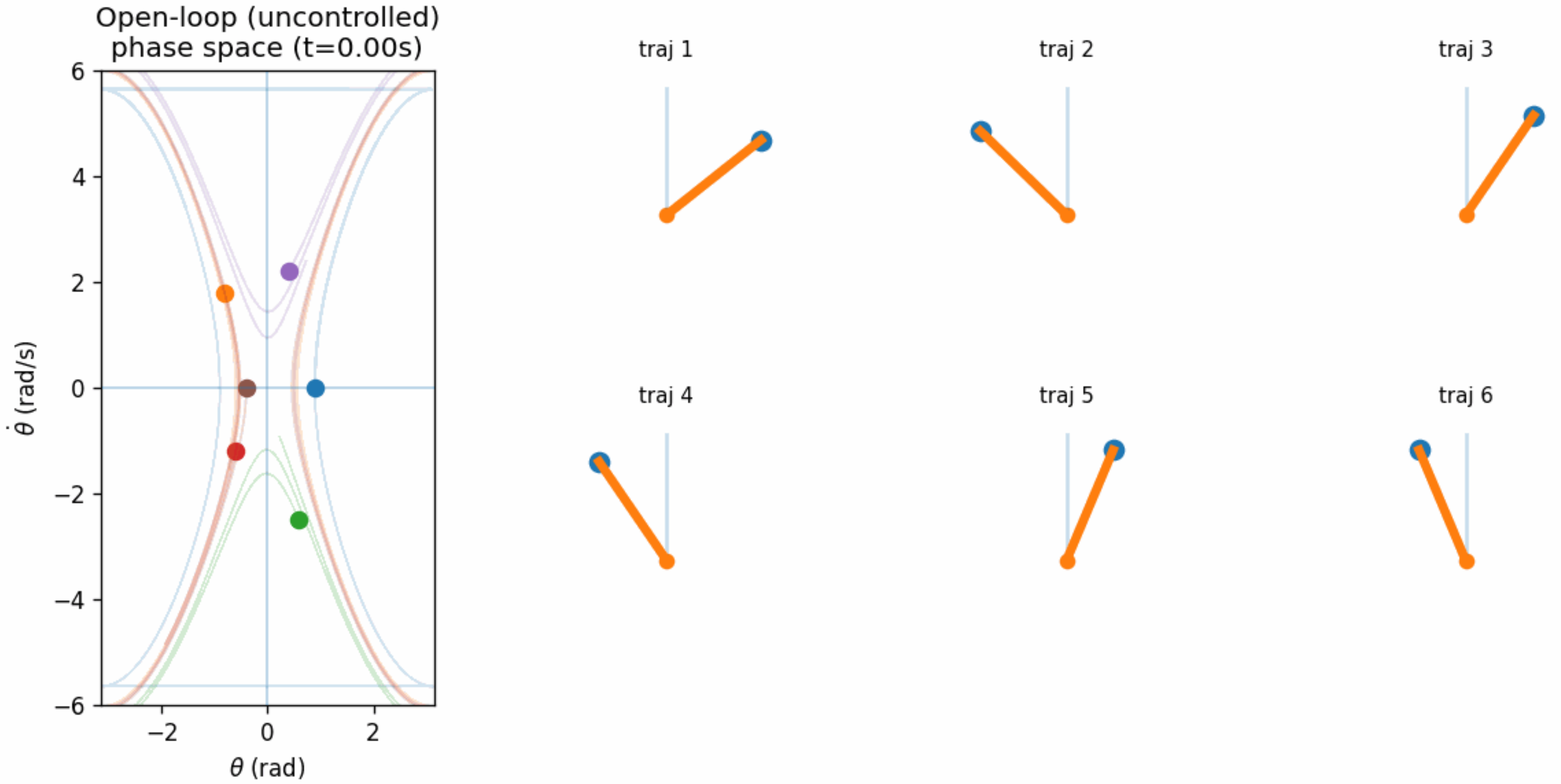
$$u_t^\top R u_t$$

(QUADRATIC FORM)

$$= \frac{\tau_t}{ml^2} \left[R_{\tau\tau} \right] \frac{\tau_t}{ml^2}$$

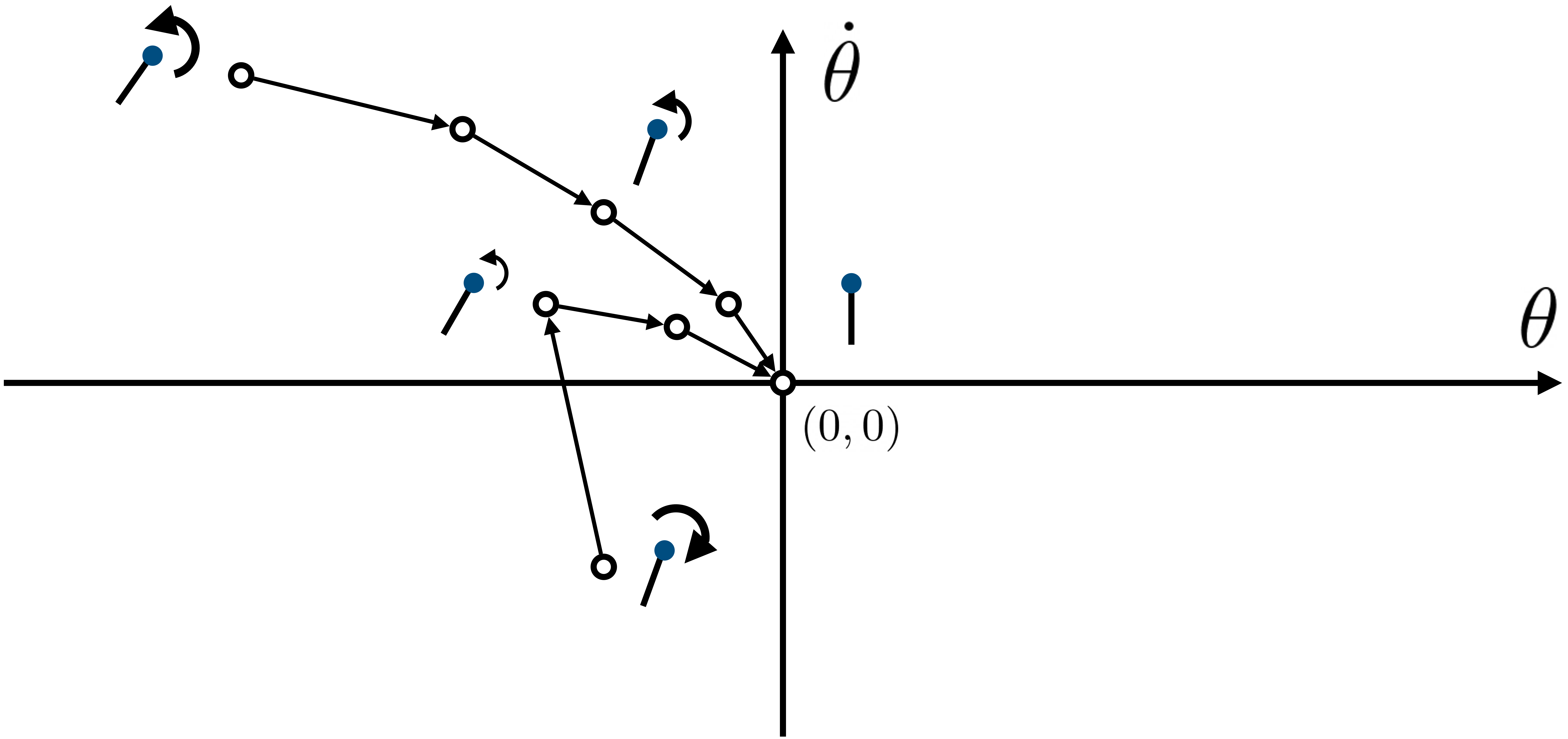
$$= R_{\tau\tau} \left(\frac{\tau_t}{ml^2} \right)^2$$

$$R \succ 0 \leftrightarrow z^\top R z > 0, \ \forall z \neq 0$$



Open-loop (uncontrolled)
phase space (t=0.00s)
$\dot{\theta}$ (rad/s)
θ (rad)
6
4
2
0
−2
−4
−6
−2
0
2
traj 1
traj 2
traj 3
traj 4
traj 5
traj 6

Example: Inverted Pendulum



Bellman Equation for Dynamic Programming

- **Linear** system (model) $$x_{t+1} = Ax_t + Bu_t$$
- **Quadratic** cost function to minimize $$\sum_t x_t^\top Q x_t + u_t^\top R u_t$$

Whatever you do now, you must finish optimally from wherever you end up.

$$J^*(x_t) = \min_{u_t} x_t^\top Q x_t + u_t^\top R u_t + J^*(x_{t+1})$$

MINIMUM COST, STARTING FROM x_t

IMMEDIATE COST

MINIMUM FUTURE COST, STARTING FROM x_{t+1}

Value Iteration (Horizon = 0)

$$J_0(x) = \min_u x^\top Q x + u^\top R u = x^\top Q x = x^\top P_0 x$$

Value Iteration (Horizon = 1)

$$J_0(x) = \min_u x^\top Q x + u^\top R u = x^\top Q x = x^\top P_0 x$$

$$J_1(x) = \min_u x^\top Q x + u^\top R u + J_0(Ax + Bu)$$

Value Iteration (Horizon = 1)

$$J_1(x) = \min_u \left[x^\top Q x + u^\top R u + (Ax + Bu)^\top P_0 (Ax + Bu) \right]$$

$$\nabla_u [\cdot] = 2Ru + 2B^\top P_0 (Ax + Bu) = 0$$

$$u = -(R + B^\top P_0 B)^{-1} B^\top P_0 A x$$

$$J_1(x) = x^\top P_1 x$$

$$P_1 = Q + K_1^\top R K_1 + (A + BK_1)^\top P_0 (A + BK_1)$$

$$K_1 = -(R + B^\top P_0 B)^{-1} B^\top P_0 A$$

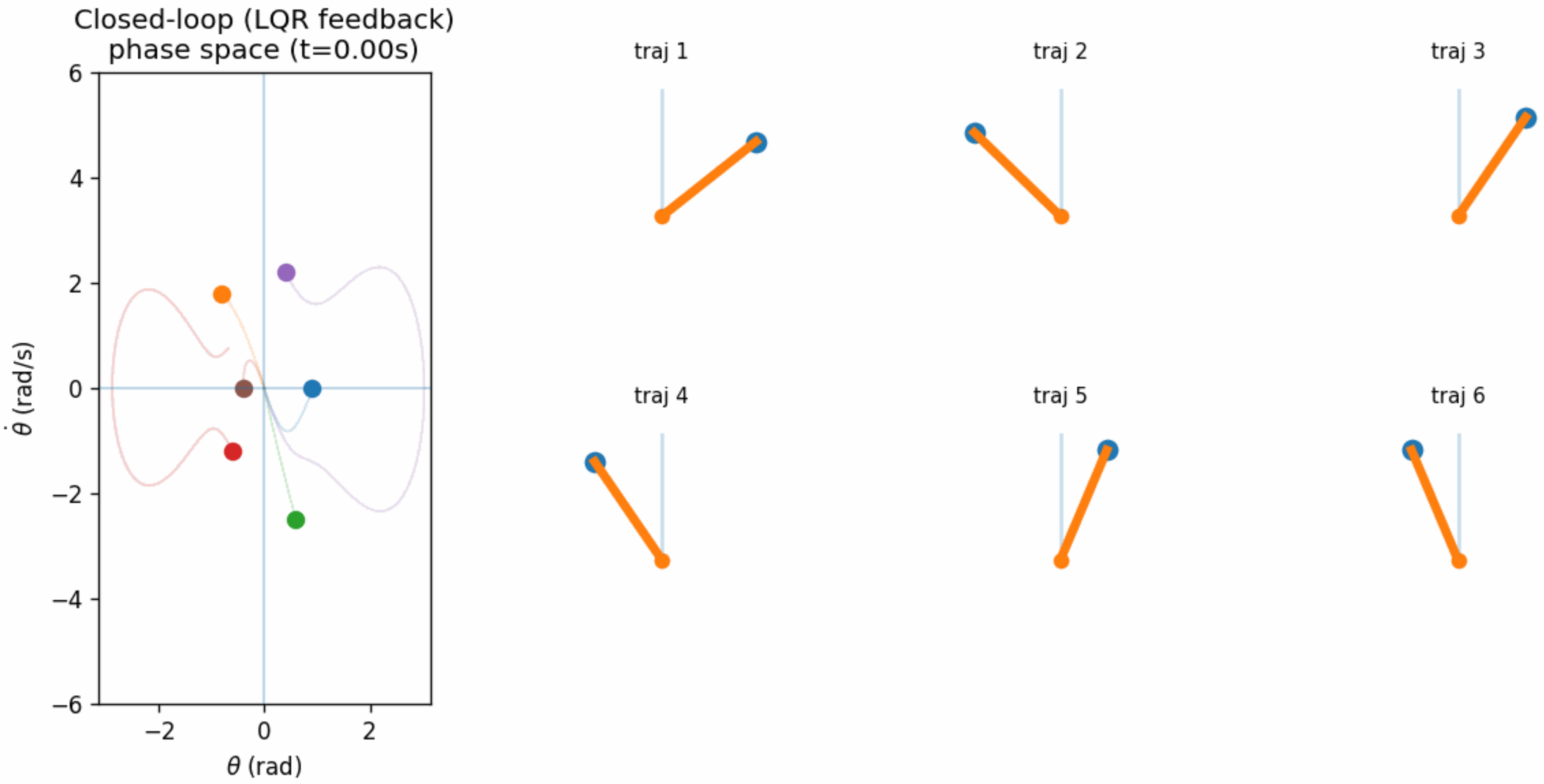
Value Iteration (Horizon = i)

$$K_i = -(R + B^\top P_{i-1} B)^{-1} B^\top P_{i-1} A$$

$$P_i = Q + K_i^\top R K_i + (A + BK_i)^\top P_{i-1} (A + BK_i)$$

$$u = K_i x, \;\; J_i(x) = x^\top P_i x$$

RUNTIME: $O(H(n^3 + m^3))$


Closed-loop (LQR feedback)
phase space (t=0.00s)
6
4
2
0
−2
−4
−6
$\dot{\theta}$ (rad/s)
−2
0
2
θ (rad)
traj 1
traj 2
traj 3
traj 4
traj 5
traj 6

LQR in Action: Stanford Helicopter

ABBEEL ET AL., 2006

HTTPS://YOUTU.BE/0JL04JJJOCC

Linear Quadratic Regulator

- For **linear** systems with **quadratic** costs, we can write down very efficient algorithms that return the optimal sequence of actions!
  - Special case where dynamic programming can be applied to continuous states and actions (typically only discrete states and actions)
- Many LQR extensions: non-linear systems, linear time-varying systems, trajectory following for non-linear systems, arbitrary costs, etc.

Control as an Optimization Problem

- For a sequence of H control actions
  1. Use model to predict consequence of actions (i.e., H future states)
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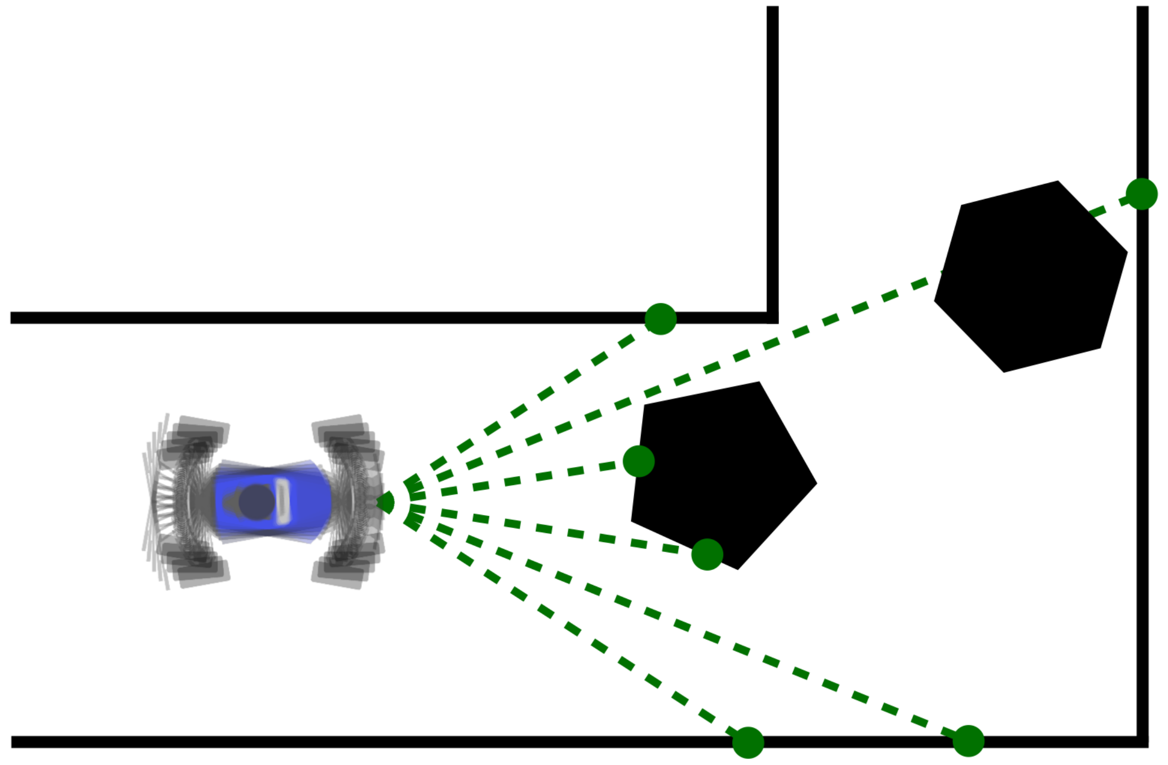
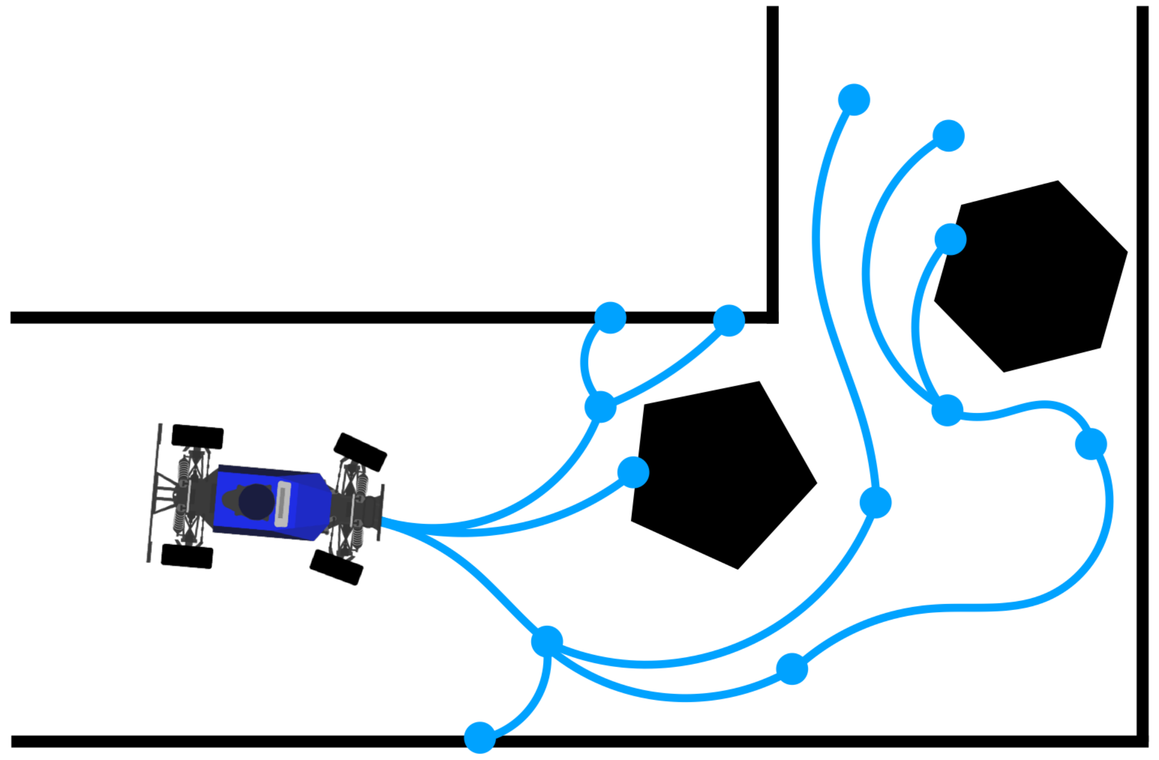
Different Control Laws

- Proportional-integral-derivative (PID) control
- Pure-pursuit control
- Model-predictive control (MPC)
- Linear-quadratic regulator (LQR)
- And many many more!

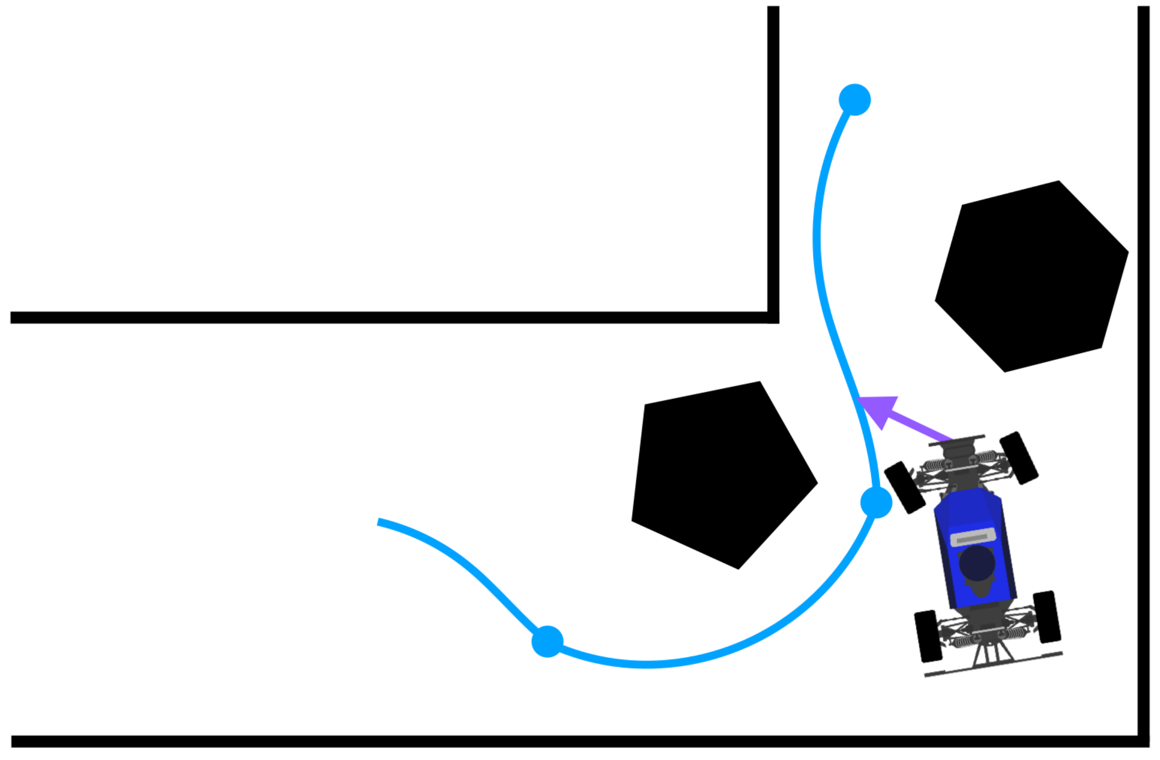
The Sense-Plan-Act Paradigm



Estimate
robot state



Plan sequence of
motions



Control robot to
follow plan

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