CSE 478 Robot Autonomy
PID Control

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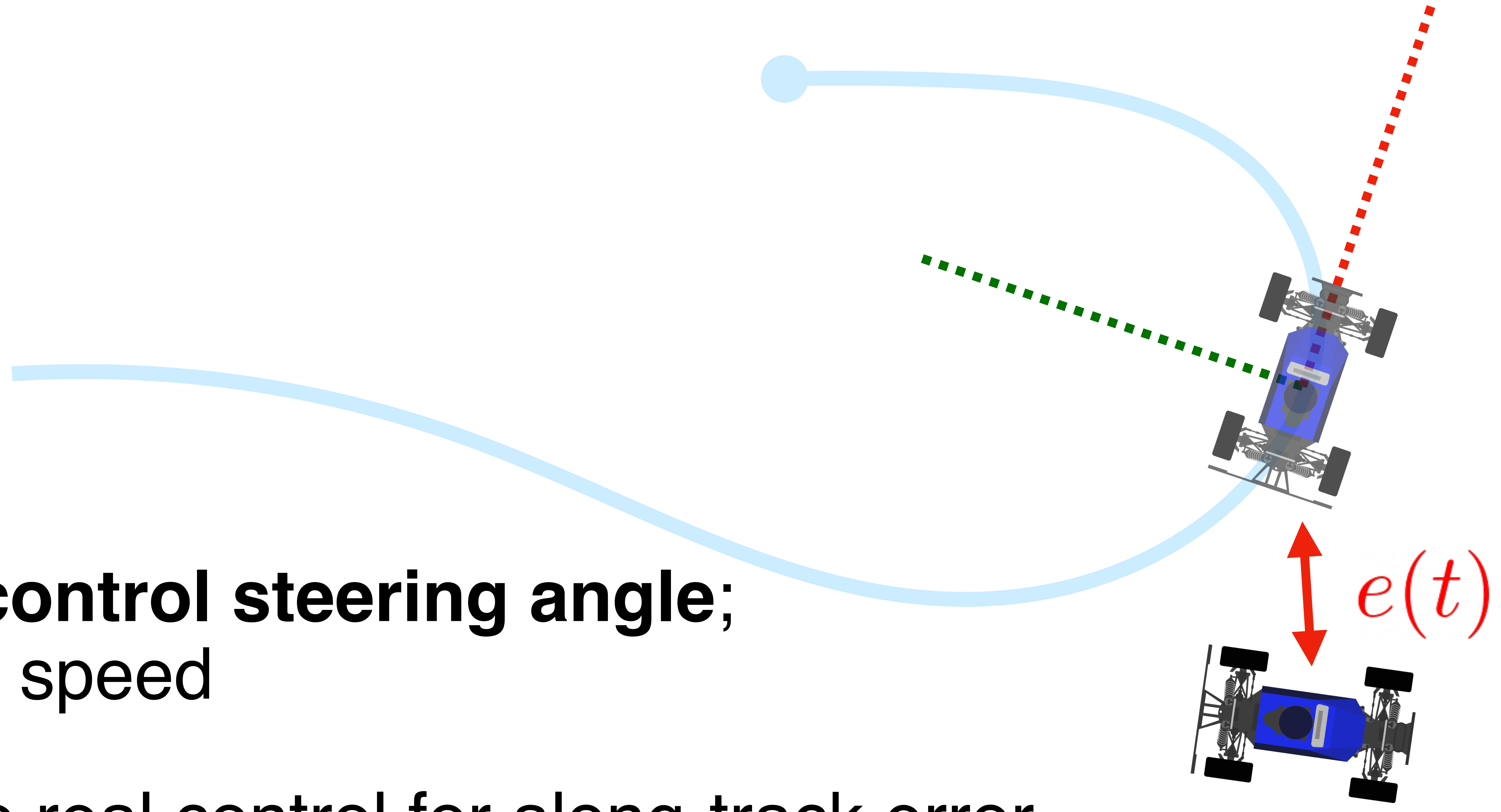
Step 4: Compute control law



- We will **only control steering angle**; fixed constant speed
- As a result, no real control for along-track error
- Some control laws will only minimize cross-track error, others will also minimize heading

$$u = K(e)$$

Step 4: Compute control law

Compute control action based on instantaneous error

$$u = K(\mathbf{x}, e)$$

control state error

(steering angle, speed)

Apply control action, robot moves a bit, compute new error, repeat

Different laws have different trade-offs

Kinematic Car Model

RECALL

$$\dot{x} = f(x, u)$$

$$x = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

X-COORDINATE

Y-COORDINATE

HEADING

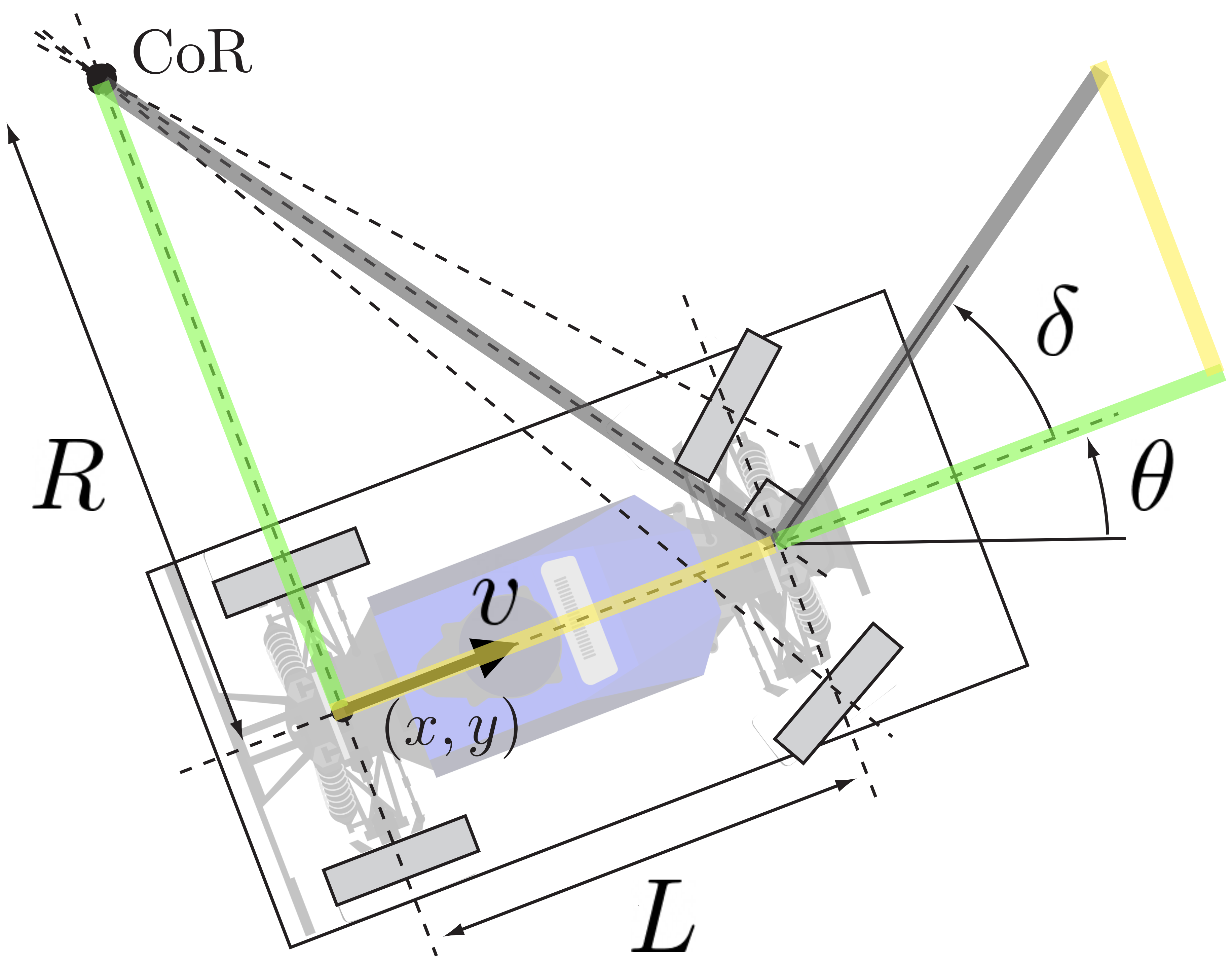
$$u = \begin{bmatrix} v \\ \delta \end{bmatrix}$$

SPEED

STEERING ANGLE

Equations of Motion

RECALL



$$\dot{x} = v\cos\theta$$

$$\dot{y} = v\sin\theta$$

$$\dot{\theta} = \omega = \frac{v}{R} = \frac{v\tan\delta}{L}$$

$$\tan\delta = \frac{L}{R} \rightarrow R = \frac{L}{\tan\delta}$$

RECALL

Kinematic Car Model

$$\dot{x} = f(x, u)$$

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

X-COORDINATE

Y-COORDINATE

HEADING

$$\begin{bmatrix} v \\ \delta \end{bmatrix}$$

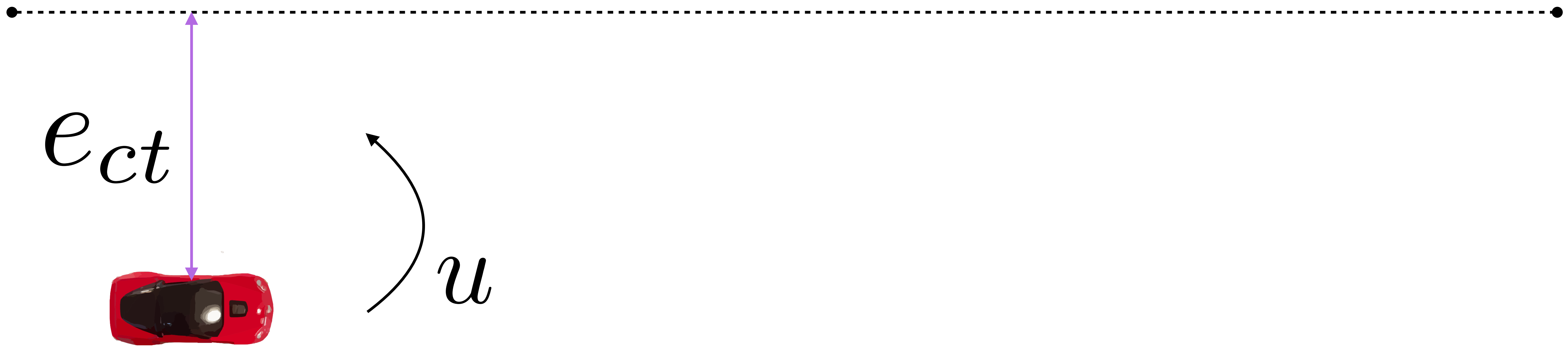
SPEED

STEERING ANGLE

$$\begin{bmatrix} 1 \\ u \end{bmatrix}$$

Bang-bang control

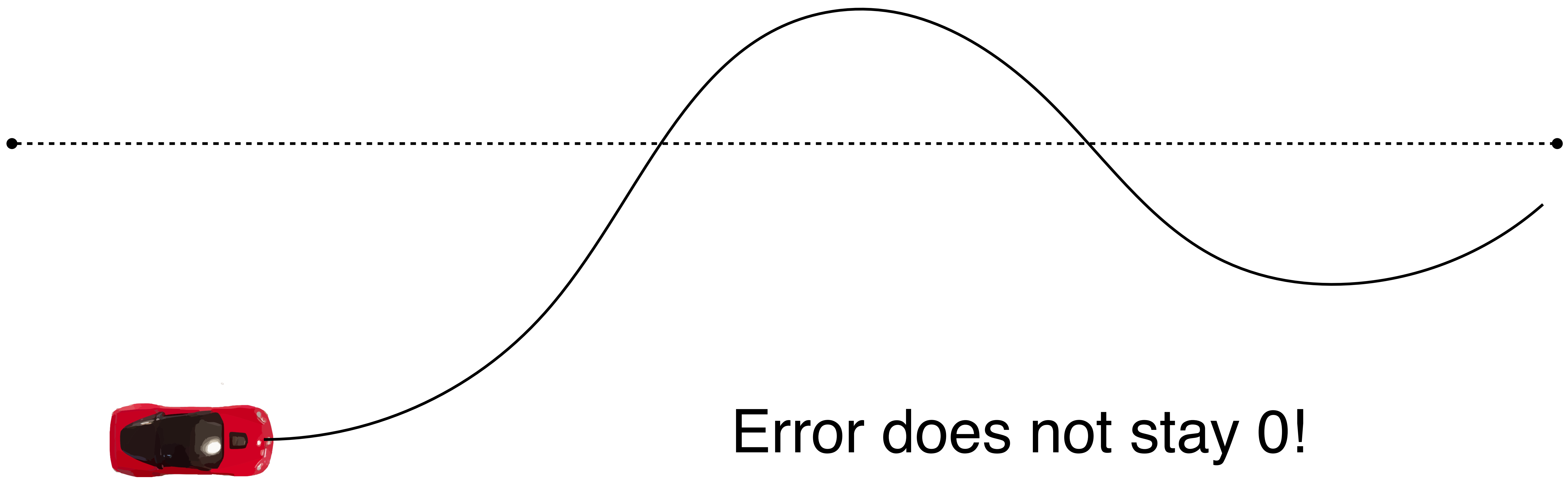
Simple control law - choose between hard left and hard right



$$u = \begin{cases} u_{max} & \text{if } e_{ct} < 0 \\ -u_{max} & \text{otherwise} \end{cases}$$

Bang-bang control

What happens when we run this control?



Error does not stay 0!

Need to adapt the magnitude of control proportional to the error …

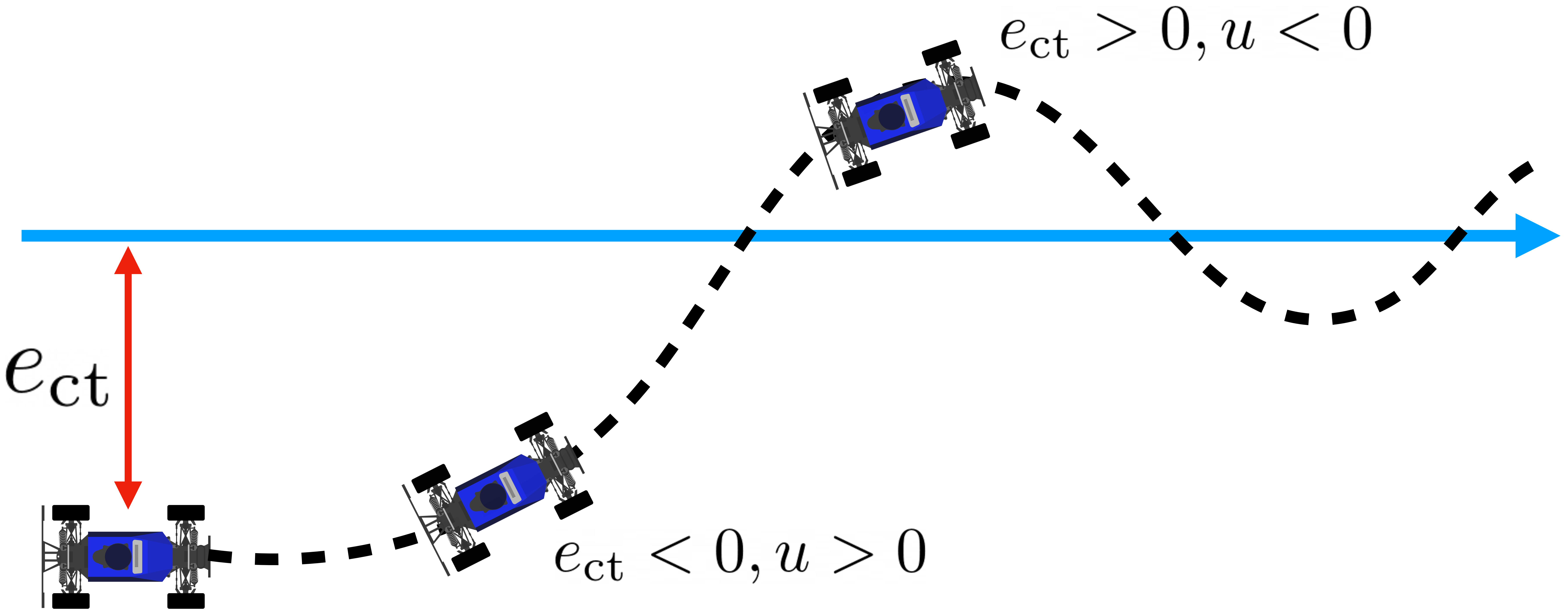
Proportional-Integral-Derivative (PID) Controller

- One of the most popular controllers in practice!
- Used widely in industrial control since 1900s (regulating temp., speed, etc.)

$$u = -\left(K_p e_{\text{ct}} + K_i \int e_{\text{ct}} dt + K_d \dot{e}_{\text{ct}} \right)$$

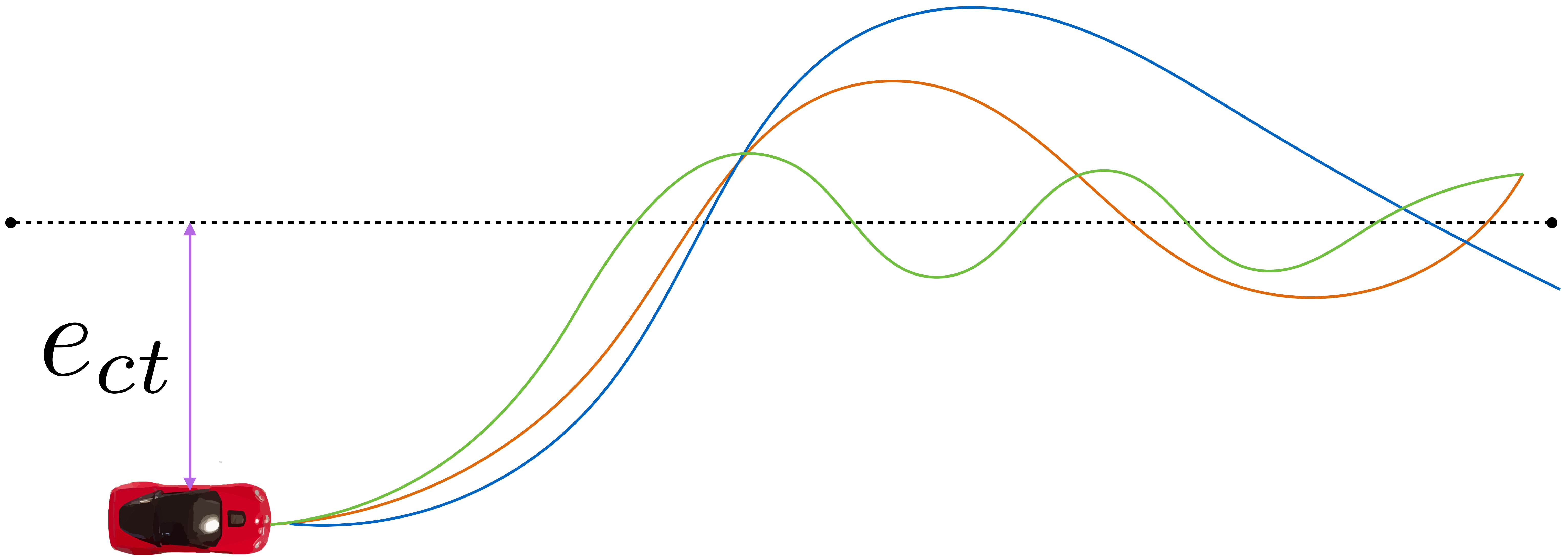
PROPORTIONAL (PRESENT) | **INTEGRAL** (PAST) | **DERIVATIVE** (FUTURE)

Proportional Control



$$u = -K_p e_{ct}$$

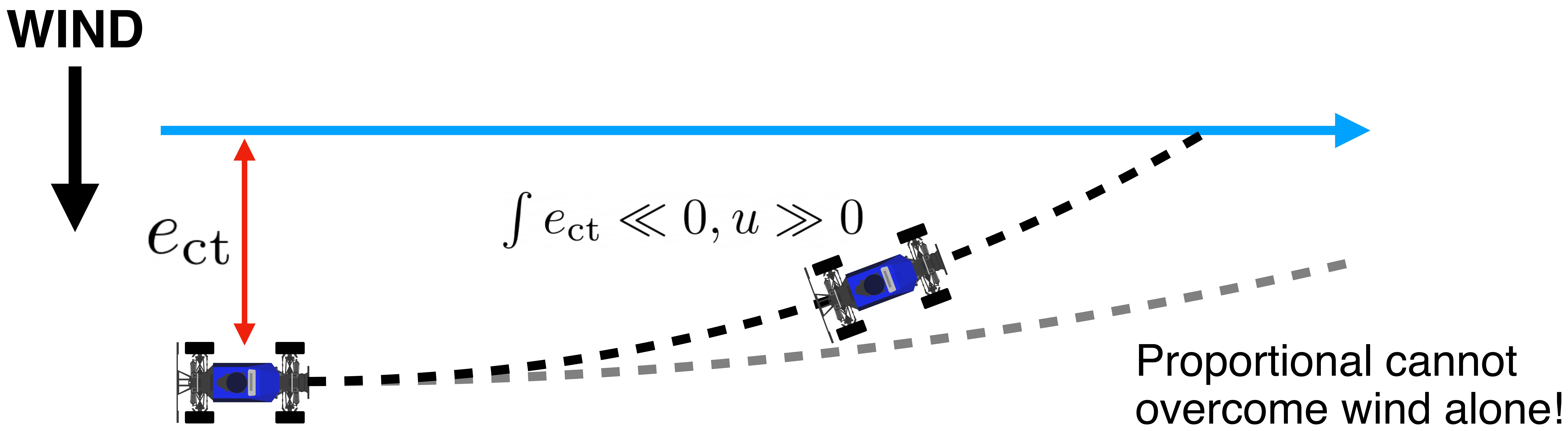
The proportional gain matters!



What happens when gain is low?

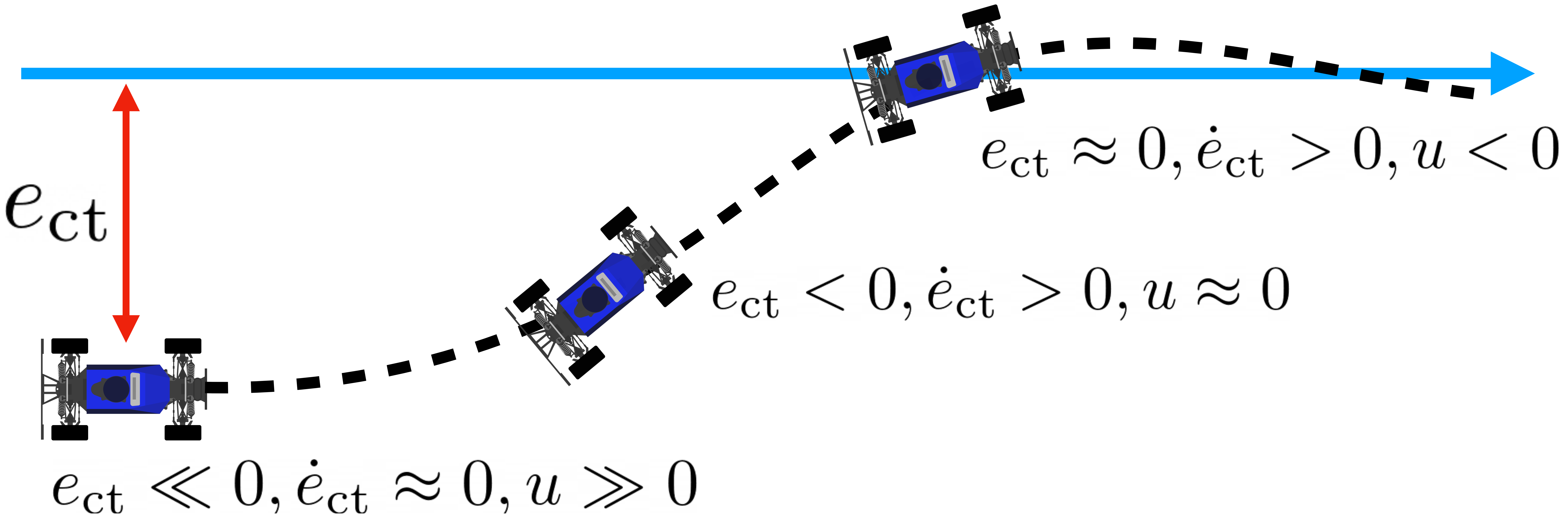
What happens when gain is high?

Proportional Integral (PI) Control



$$u = -\left(K_p e_{\text{ct}} + K_i \int e_{\text{ct}} dt\right)$$

Proportional Derivative (PD) Control



$$u = -(K_p e_{\text{ct}} + K_d \dot{e}_{\text{ct}})$$

How do you evaluate the derivative term?

Terrible way: Numerically differentiate error. Why is this a bad idea?

Smart way: Analytically compute the derivative of the cross track error

$$e_{ct} = -\sin(\theta_{ref})(x - x_{ref}) + \cos(\theta_{ref})(y - y_{ref})$$

$$\begin{aligned} \dot{e}_{ct} &= -\sin(\theta_{ref})\dot{x} + \cos(\theta_{ref})\dot{y} \\ &= -\sin(\theta_{ref})V\cos(\theta) + \cos(\theta_{ref})V\sin(\theta) \\ &= V\sin(\theta - \theta_{ref}) = V\sin(\theta_e) \end{aligned}$$

New control law! Penalize error in cross track and in heading

$$u = -\left(K_p e_{ct} + K_d V \sin\theta_e\right)$$

PID Intuition

$$u = -\left(K_p e_{\text{ct}} + K_i \int e_{\text{ct}}\,dt + K_d \dot{e}_{\text{ct}}\right)$$

PROPORTIONAL	INTEGRAL	DERIVATIVE
(PRESENT)	(PAST)	(FUTURE)

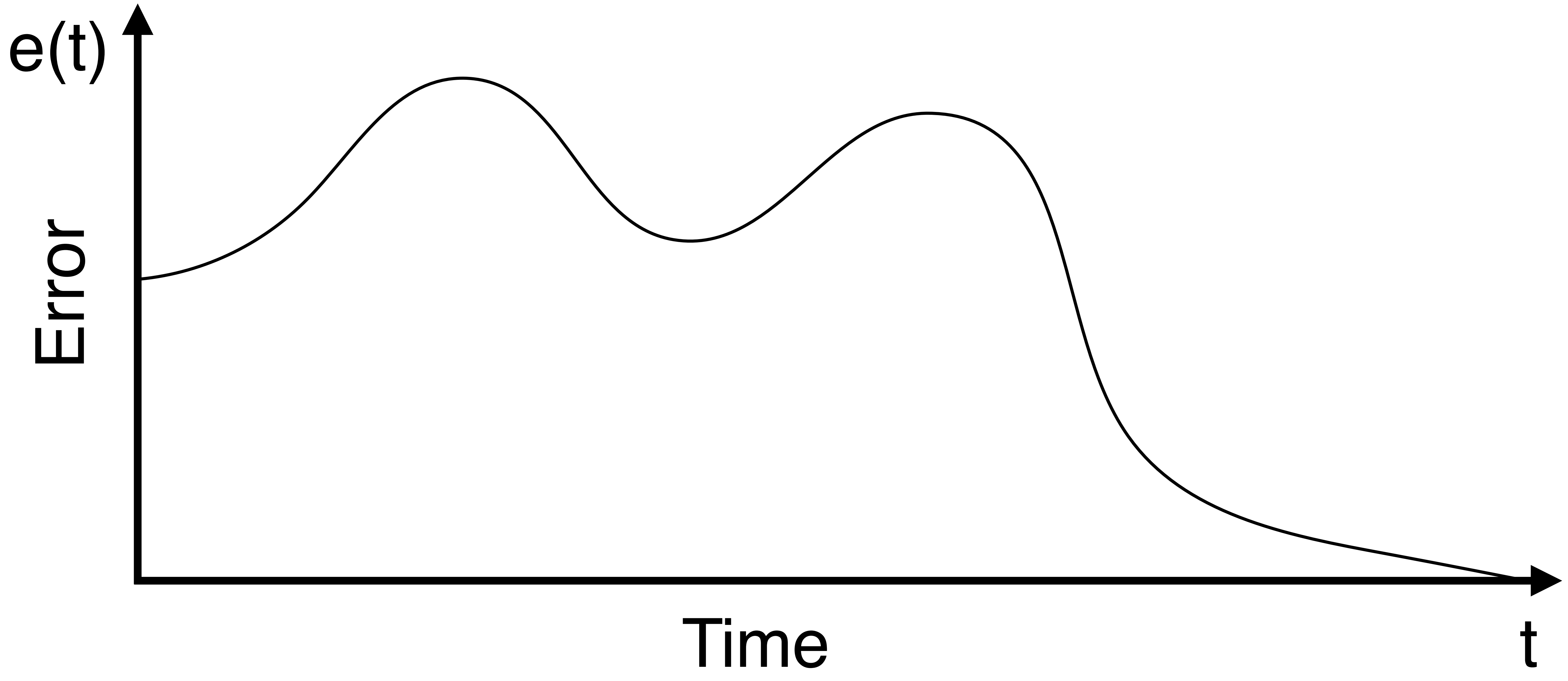
- Proportional: minimize the current error!
- Integral: if I'm accumulating error, try harder!
- Derivative: if I'm going to overshoot, slow down!

How can we prove
that a controller is stable?

Lyapunov Stability

What is stability?

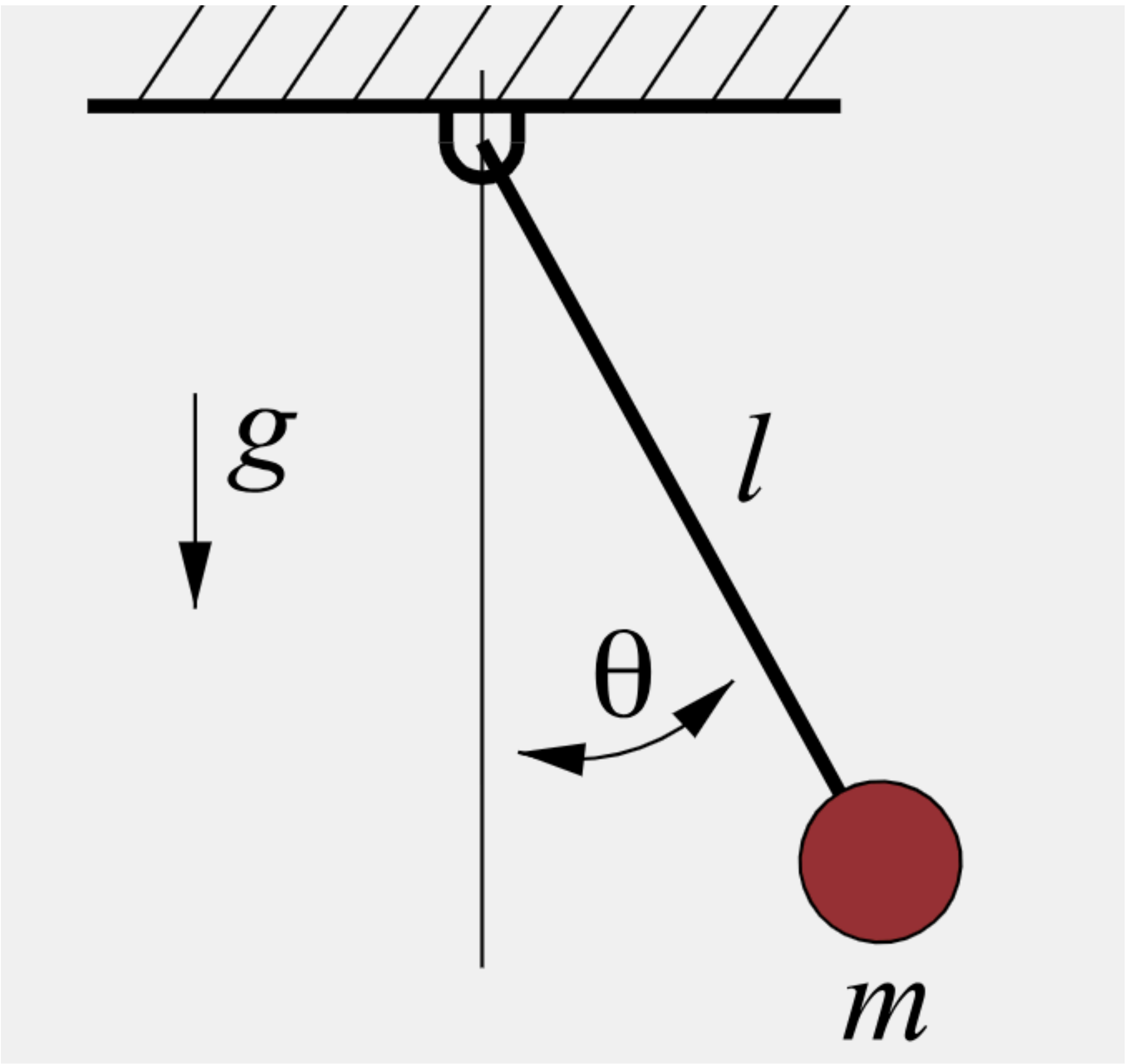
$$\lim_{t \to \infty} e(t) = 0$$



So we want both $e(t) \to 0$ and $\dot{e}(t) \to 0$

Question: Why does the error oscillate?

Detour: How do we make a pendulum stable?



$$ml^2\ddot{\theta} + mgl\sin\theta = u$$

What control law should we use to stabilize the pendulum, i.e.

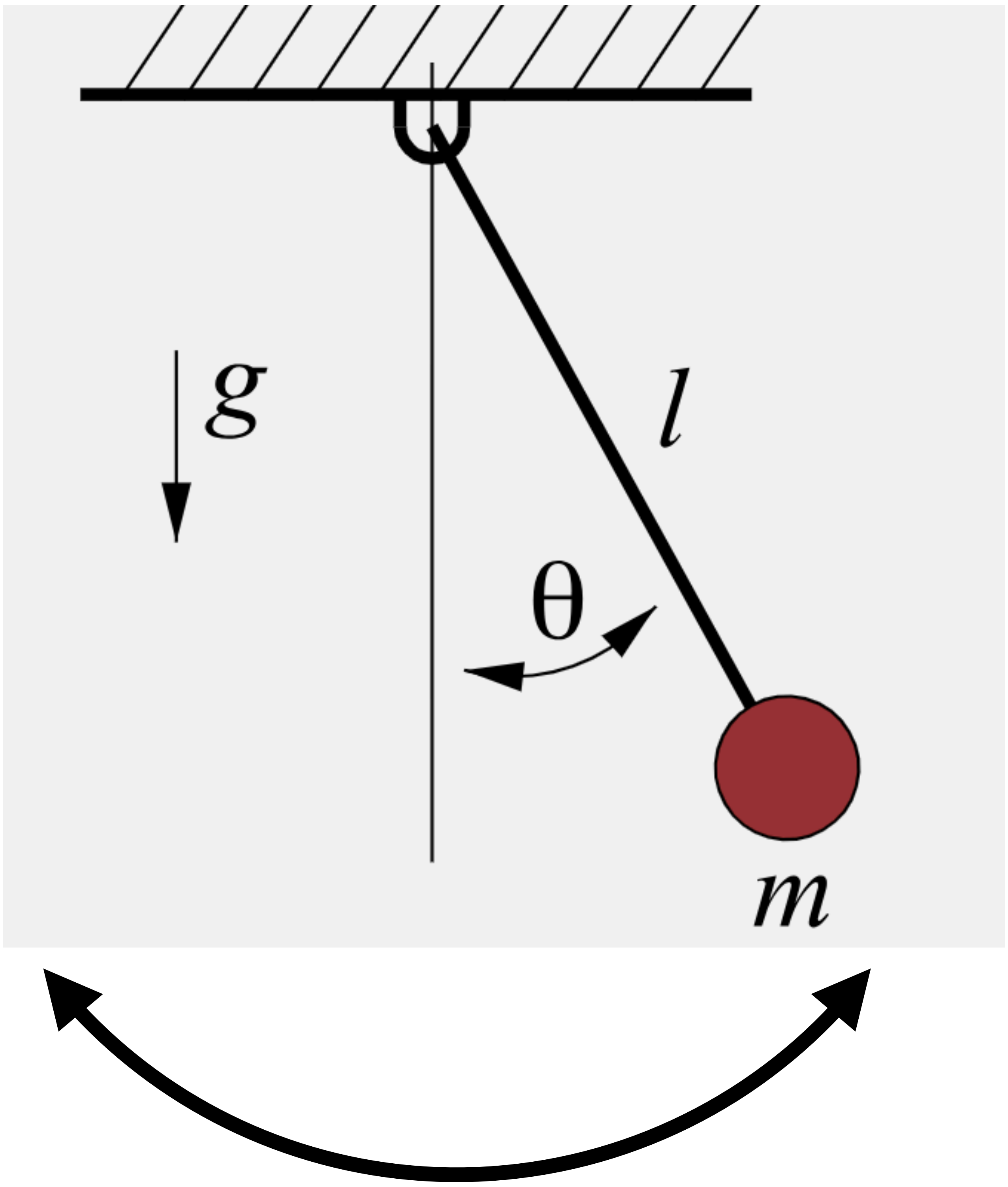
Choose $u = \pi(\theta, \dot{\theta})$ such that $\theta \to 0$

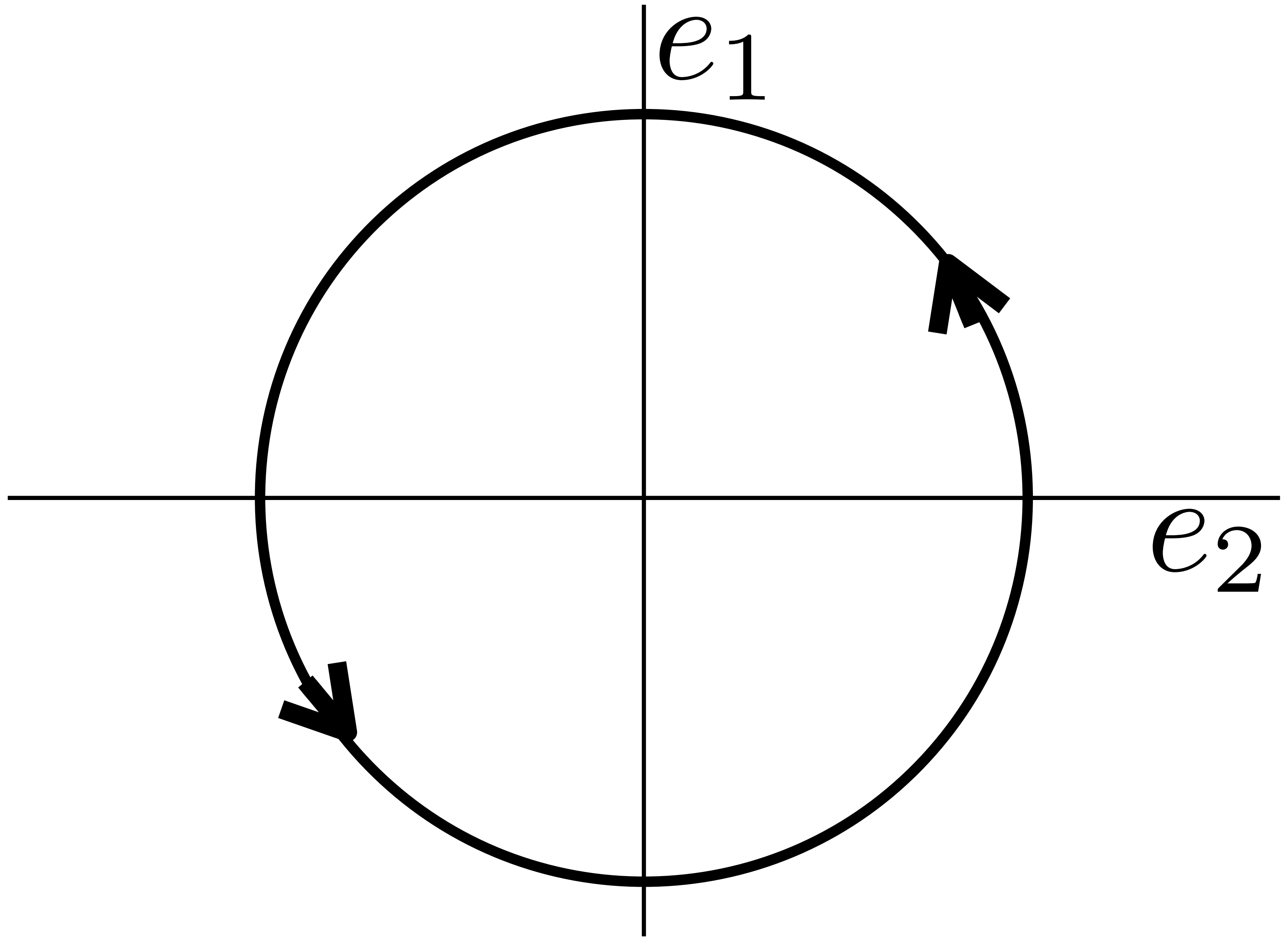
$$\dot{\theta} \to 0$$

How does the passive error dynamics behave?

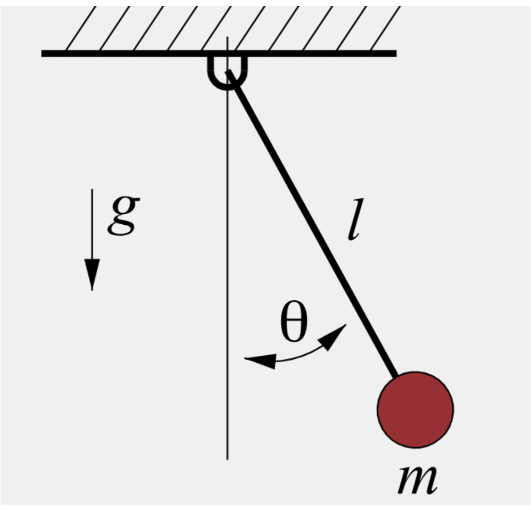
$$e_1 = \theta - 0 = \theta \qquad e_2 = \dot{\theta} - 0 = \dot{\theta}$$

Set u=0. Dynamics is not stable.





How do we verify if a controller is stable?



$$ml^2\ddot{\theta} + mgl\sin\theta = u$$

Lets pick the following law:

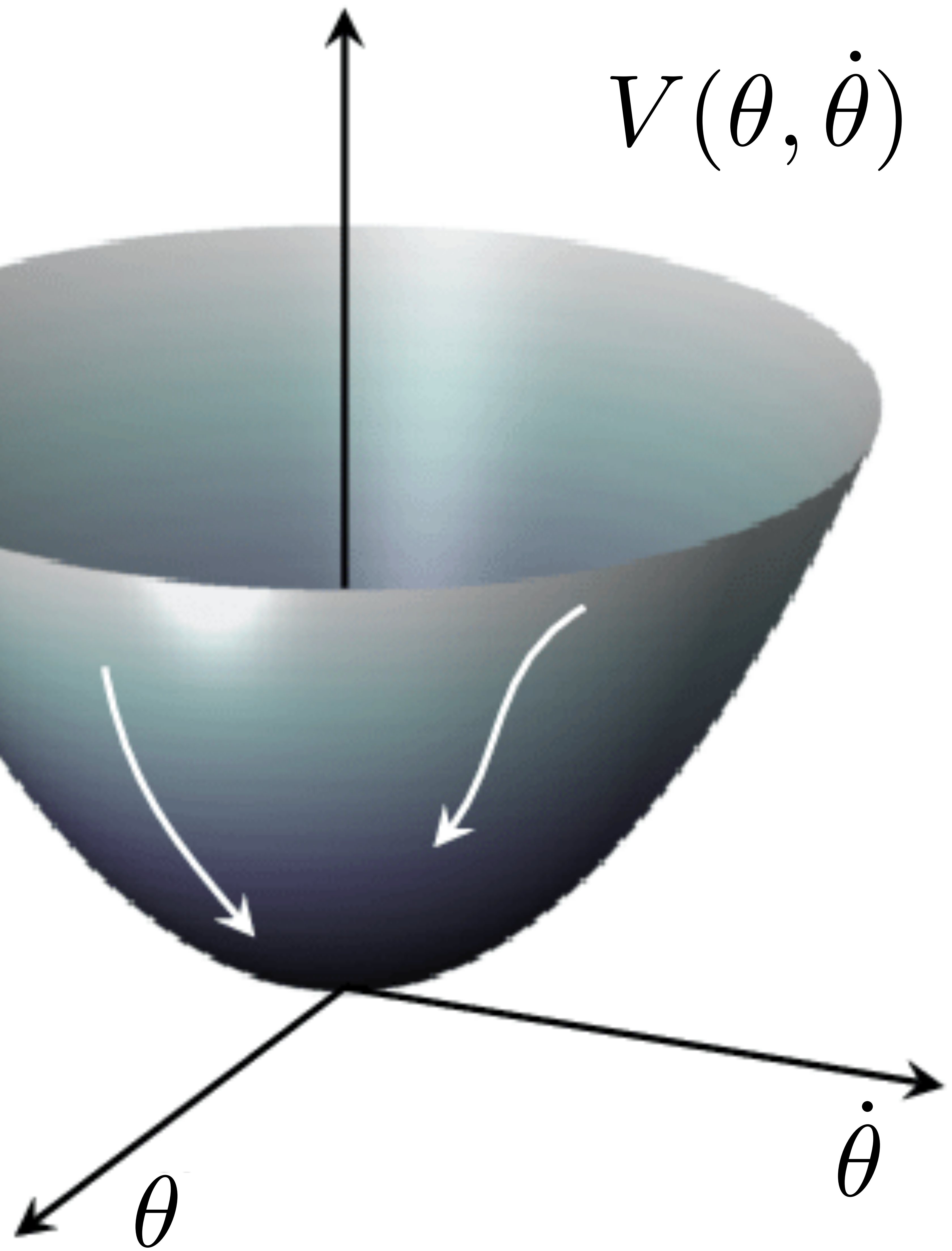
$$u = -K\dot{\theta}$$

Is this stable? How do we know?

We can simulate the dynamics from different start point and check….

but how many points do we check? what if we miss some points?

Key Idea: Think about energy!



Make energy decay to 0 and stay there

$$V(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$

$$> 0$$

$$\dot{V}(\theta, \dot{\theta}) = m l^2 \dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta}$$

$$= \dot{\theta}(u - mgl\sin\theta) + mgl(\sin\theta)\dot{\theta}$$

$$= \dot{\theta}u$$

Choose a control law $u = -k\dot{\theta}$

$$\dot{V}(\theta, \dot{\theta}) = -k\dot{\theta}^2 < 0$$

Lyapunov function:
A generalization of energy

Lyapunov function for a closed-loop system

1. Construct an energy function that is always positive

$$V(x) > 0, \forall x$$

Energy is only 0 at the origin, i.e. $V(0) = 0$

2. Choose a control law such that this energy always decreases

$$\dot{V}(x) < 0, \forall x$$

Energy rate is 0 at origin, i.e. $\dot{V}(0) = 0$

No matter where you start, energy will decay and you will reach 0!

Let's get provable control for our car!

Dynamics of the car

$$\dot{x} = V\cos\theta$$
$$\dot{y} = V\sin\theta$$
$$\dot{\theta} = \frac{V}{B}\tan u$$

Let's get provable control for our car!

Let's define the following Lyapunov function

$$V(e_{ct}, \theta_e) = \frac{1}{2} k_1 e_{ct}^2 + \frac{1}{2} \theta_e^2 \qquad > 0$$

Compute derivative

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} \dot{e_{ct}} + \theta_e \dot{\theta_e}$$

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin \theta_e + \theta_e \frac{V}{B} \tan u$$

Let's get provable control for our car!

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin\theta_e + \theta_e \frac{V}{B} \tan u$$

Trick: Set u intelligently to get this term to always be negative

$$\theta_e \frac{V}{B} \tan u = -k_1 e_{ct} V \sin\theta_e - k_2 \theta_e^2$$

$$\tan u = -\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e$$

$$u = \tan^{-1}\left(-\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e\right)$$

(Advanced Reading)

Bank-to-Turn Control for a Small UAV using Backstepping and Parameter Adaptation

Dongwon Jung and Panagiotis Tsiotras

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