Autonomous Robotics

Winter 2024

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Class Outline

State Estimation

Robotic System Design
Filtering
Localization
SLAM

Control

Feedback Control
PID Control
MPC
LQR

Planning

Search
Heuristic Search
Motion Planning
Lazy Search

Learning

Imitation Learning
Policy Gradient
Actor-Critic
Model-Based RL

Logistics

- HW 1 due today!
- Come talk to us if any extenuating circumstances!
- Everyone should have received a team by now and gotten access to their car and workstation!

Lecture Outline

Recap

↓

Gaussian Properties

↓

Kalman Filtering

Recap: Bayes Filters

Key Idea: Apply Markov to get a recursive update!

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \sum P(x_t|u_t, x_{t-1})bel(x_{t-1})$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

Recap: Motion and Sensor Models

Motion Model

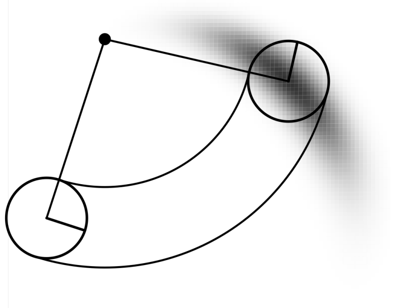
$$\theta_t = \theta_{t-1} + \Delta\theta = \theta_{t-1} + \frac{v}{L}\tan\delta\Delta t$$

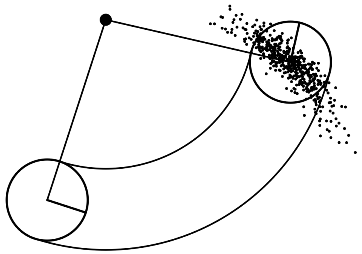
$$x_t = x_{t-1} + \Delta x = x_{t-1} + \frac{L}{\tan\delta}(\sin\theta_t - \sin\theta_{t-1})$$

$$y_t = y_{t-1} + \Delta y = y_{t-1} + \frac{L}{\tan\delta}(\cos\theta_{t-1} - \cos\theta_t)$$

Sensor Model

$$p(z_t^k \mid x_t, m) \quad = \quad \begin{pmatrix} z_{\text{hit}} \\ z_{\text{short}} \\ z_{\text{max}} \\ z_{\text{rand}} \end{pmatrix}^T \cdot \begin{pmatrix} p_{\text{hit}}(z_t^k \mid x_t, m) \\ p_{\text{short}}(z_t^k \mid x_t, m) \\ p_{\text{max}}(z_t^k \mid x_t, m) \\ p_{\text{rand}}(z_t^k \mid x_t, m) \end{pmatrix}$$





Can we do tractable filtering without huge memory requirements?

Need to choose form of probability distributions

- Dynamics (Prediction)

$$\overline{Bel}(x_t) = \int p(x_t|u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Measurement (Correction)

$$Bel(x_t) = \eta P(z_t|x_t) \overline{Bel}(x_t)$$

Tractable computation of Bayesian posteriors

Solution: Linear Gaussian Models

- Dynamics (Prediction)

$$\overline{Bel}(x_t) = \int p(x_t | u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Measurement (Correction)

$$Bel(x_t) = \eta P(z_t | x_t) \overline{Bel}(x_t)$$

Model as Linear Gaussian

Lecture Outline

Recap

↓

Gaussian Properties

↓

Kalman Filtering

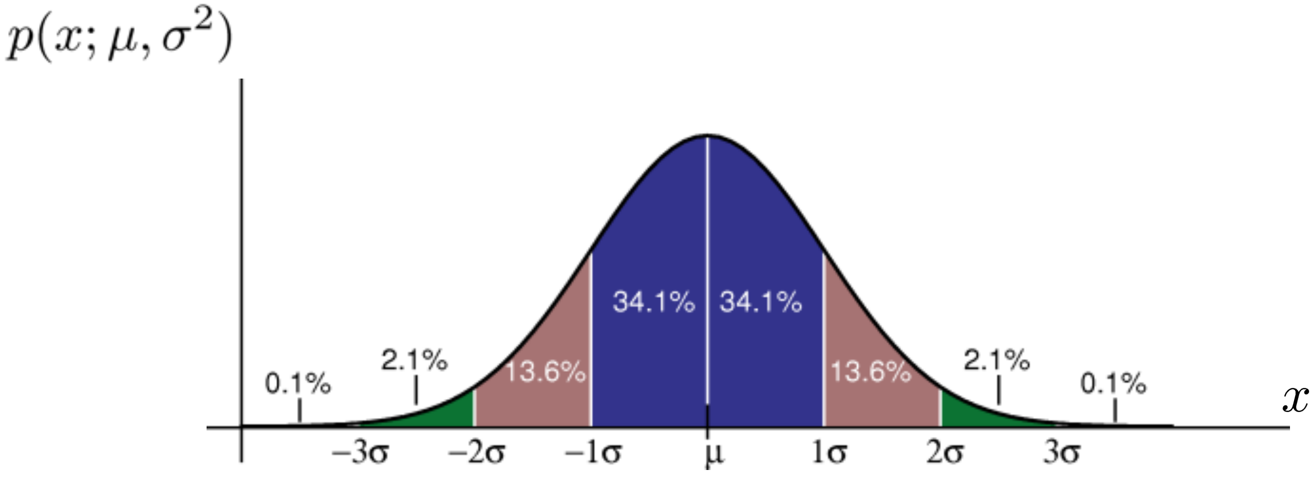
Let’s take a little Gaussian detour

Gaussians (1D)

- Gaussian with mean (μ) and standard deviation (σ)

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$p(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$



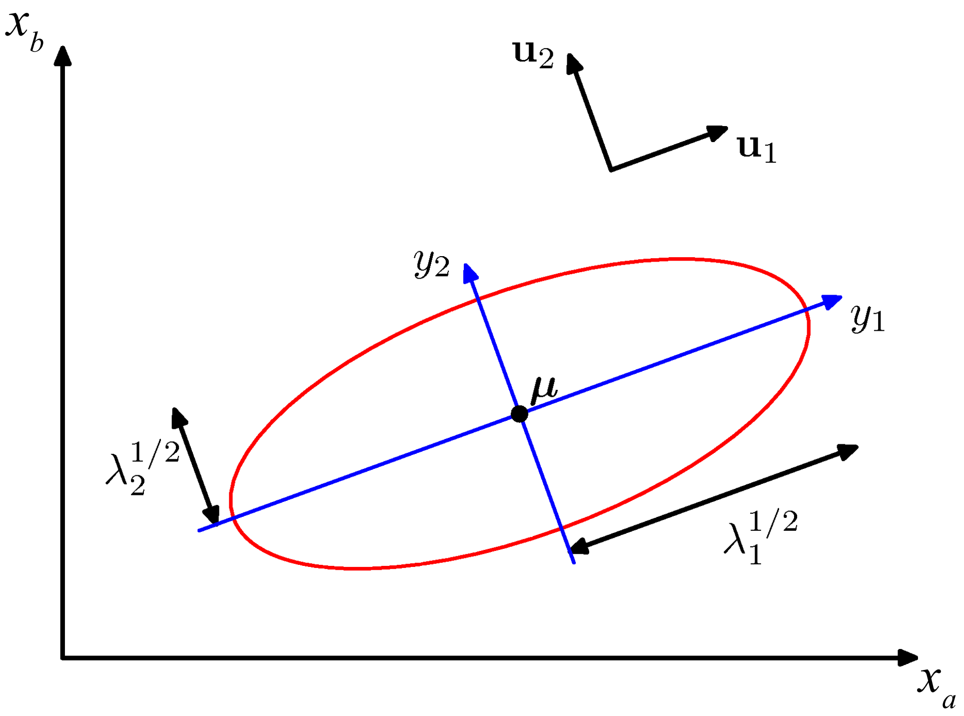
Gaussians (2D) – we won’t get too deep into this!

$$p(\mathbf{x}) = \mathrm{N}(\mu, \Sigma)$$

$$\mathbf{x} = \begin{pmatrix} x_a \\ x_b \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$$

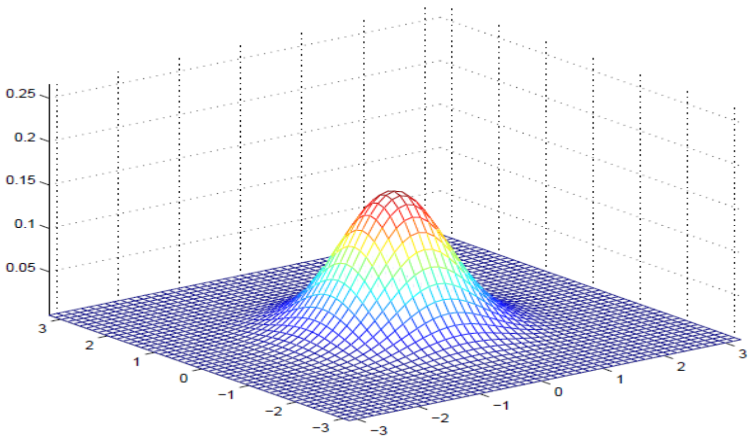
$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

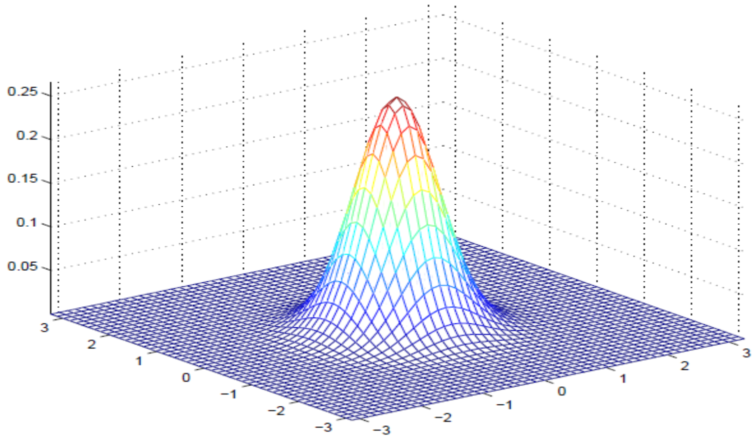
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)}$$

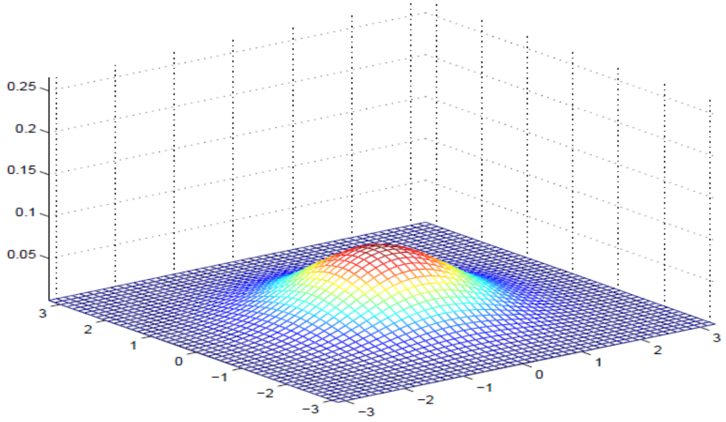


2D examples

Slide from Pieter Abbeel







- μ = [0; 0]
- Σ = [1 0 ; 0 1]

- μ = [0; 0]
- Σ = [.6 0 ; 0 .6]

- μ = [0; 0]
- Σ = [2 0 ; 0 2]

Important Identities: Gaussians

Forward propagation
$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = AX + B + \epsilon \\ \epsilon \sim \mathcal{N}(0, C) \end{cases} \implies Y \sim \mathcal{N}(A\mu + B, A\Sigma A^T + Q)$$

Conditioning
$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = CX + B + \delta \\ \delta \sim \mathcal{N}(0, R) \end{cases} \implies X|Y = y_0 \sim \mathcal{N}(\mu + K(y_0 - C\mu), (I - KC)\Sigma)$$

- Marginalization and conditioning in Gaussians results in Gaussians
- We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations.

Lecture Outline

Recap

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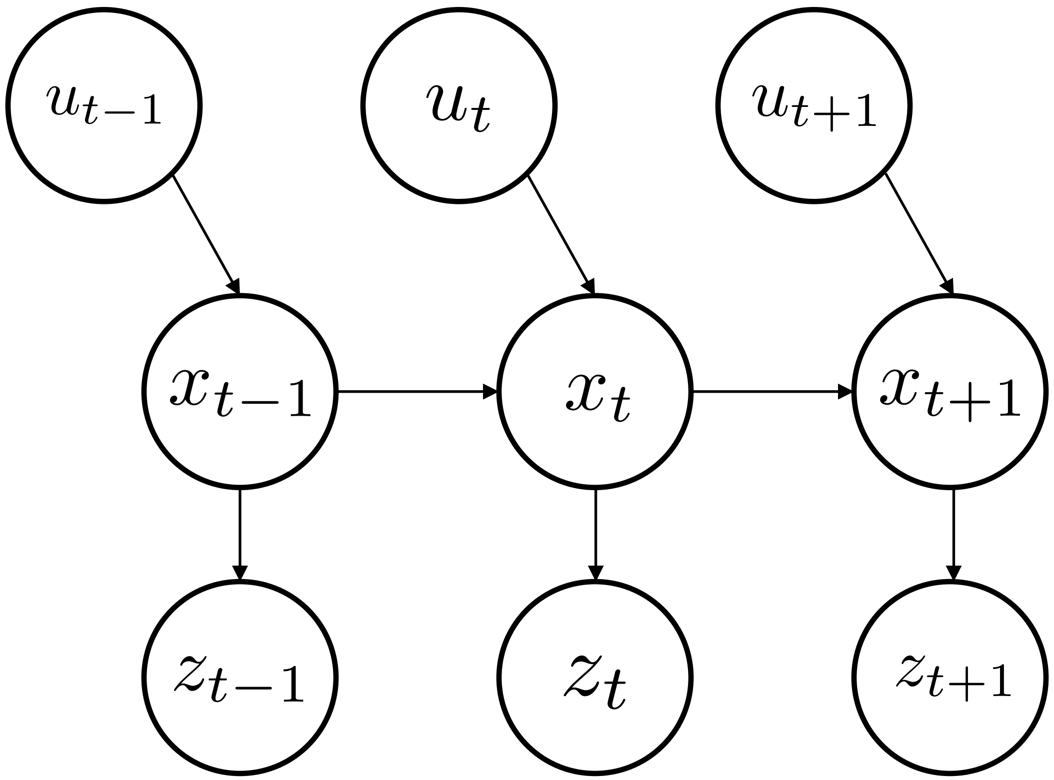
Gaussian Properties

↓

Kalman Filtering

Discrete Kalman Filter

Kalman filter = Bayes filter with Linear Gaussian dynamics and sensor models



Discrete Kalman Filter: Scalar Version

Estimates the state *x* of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = a x_{t-1} + b u_t + \epsilon_t$$

$$\epsilon_t \sim \mathcal{N}(0, q)$$

with a measurement

$$z_t = c x_t + \delta_t$$

$$\delta_t \sim \mathcal{N}(0, r)$$

Linear Gaussian

Discrete Kalman Filter: Matrix Version

Estimates the state *x* of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A x_{t-1} + B u_t + \epsilon_t$$

$$\epsilon_t \sim \mathcal{N}(0, Q)$$

Linear Gaussian

with a measurement

$$z_t = C x_t + \delta_t$$

$$\delta_t \sim \mathcal{N}(0, R)$$

Components of a Kalman Filter

A Matrix (n x n) that describes how the state evolves from *t-1* to *t* without controls or noise.

B Matrix (n x l) that describes how the control u_{t-1} changes the state from *t-1* to *t*

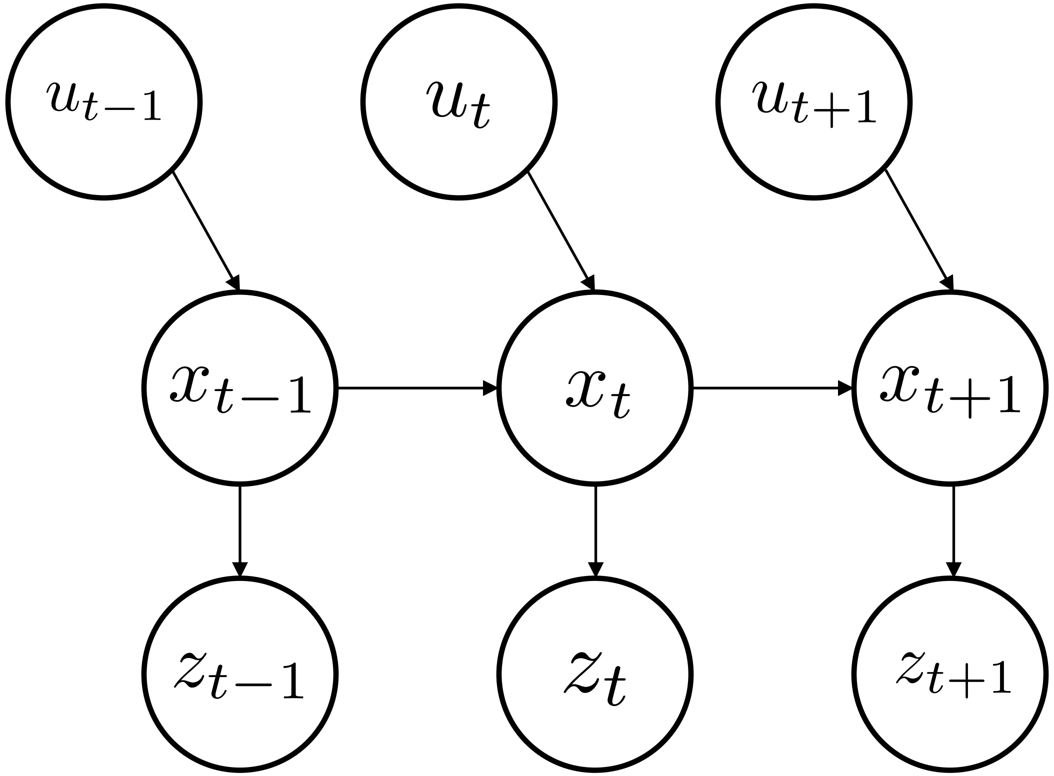
C Matrix (k x n) that describes how to map the state x_t to an observation z_t.

ϵ_t

δ_t

Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance *R* and *Q* respectively.

Goal of the Kalman Filter: Same as Bayes Filter



Belief

$$p(x_t | z_{0:t}, u_{0:t})$$

Idea: recursive update

$$\propto p(z_t|x_t) \int p(x_t|x_{t-1}, u_t) p(x_{t-1}|z_{0:t-1}, u_{0:t-1})$$

Measurement

Dynamics

Recursive Belief

2 step process:

- Dynamics update (incorporate action)
- Measurement update (incorporate sensor reading)

Bayes Filters

Key Idea: Apply Markov to get a recursive update!

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

Step 1: Prediction - push belief through dynamics given action

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Linear Gaussian

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$Bel(x_0) = \mathcal{N}(\mu_0, \Sigma_0)$$

- $Bel(x_t)$ at any step t is: $\mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$
- $\overline{Bel}(x_t)$ at any step t is: $\mathcal{N}(\mu_{t|0:t-1}, \Sigma_{t|0:t-1})$

Linear Gaussian Systems: Prediction

- Integrate the effect of one action under the dynamics, before measurement comes in

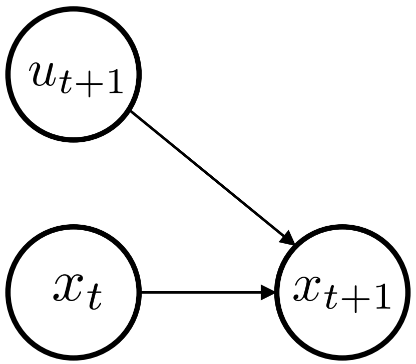
$$x_{t+1} = Ax_t + Bu_{t+1} + \epsilon_{t+1} \qquad \epsilon_{t+1} \sim \mathcal{N}(0, Q_{t+1})$$

$$p(x_{t+1}|x_t, u_{t+1}) = \mathcal{N}(Ax_t + Bu_{t+1}, Q_{t+1})$$

$\overline{Bel}(x_{t+1})$ $\qquad\qquad\qquad$ $Bel(x_t)$

$$p(x_{t+1}|z_{0:t}, u_{0:t+1}) = \int p(x_t|u_{0:t}, z_{0:t})p(x_{t+1}|u_{t+1}, x_t)dx_t$$

Gaussian, easy!



Linear Gaussian Systems: Prediction

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$$x_{t+1} = Ax_t + Bu_{t+1} + \epsilon_{t+1} \qquad \epsilon_{t+1} \sim \mathcal{N}(0, Q_{t+1})$$

$$p(x_{t+1}|x_t, u_{t+1}) = \mathcal{N}(Ax_t + Bu_{t+1}, Q_{t+1})$$

$$\underbrace{p(x_{t+1}|z_{0:t}, u_{0:t+1})}_{\overline{Bel}(x_{t+1})} = \int \underbrace{p(x_t|u_{0:t}, z_{0:t})}_{Bel(x_t)} p(x_{t+1}|u_{t+1}, x_t) dx_t$$

Gaussian, easy!

$$\boxed{\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = AX + B + \epsilon \\ \epsilon \sim \mathcal{N}(0, C) \end{cases} \implies Y \sim \mathcal{N}(A\mu + B, A\Sigma A^T + Q)}$$

Linear Gaussian Systems: Prediction

- Integrate the effect of one action under the dynamics, before measurement comes in

$$p(x_t|u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$$

$$x_{t+1} = Ax_t + Bu_{t+1} + \epsilon_{t+1}$$

$$\epsilon_{t+1} \sim \mathcal{N}(0, Q_{t+1})$$

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = AX + B + \epsilon \implies Y \sim \mathcal{N}(A\mu + B, A\Sigma A^T + Q) \\ \epsilon \sim \mathcal{N}(0, C) \end{cases}$$

Previous belief	$p(x_t|u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$
Belief Update	$p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(A\mu_{t|0:t} + Bu_{t+1}, A\Sigma_{t|0:t}A^T + Q_{t+1})$

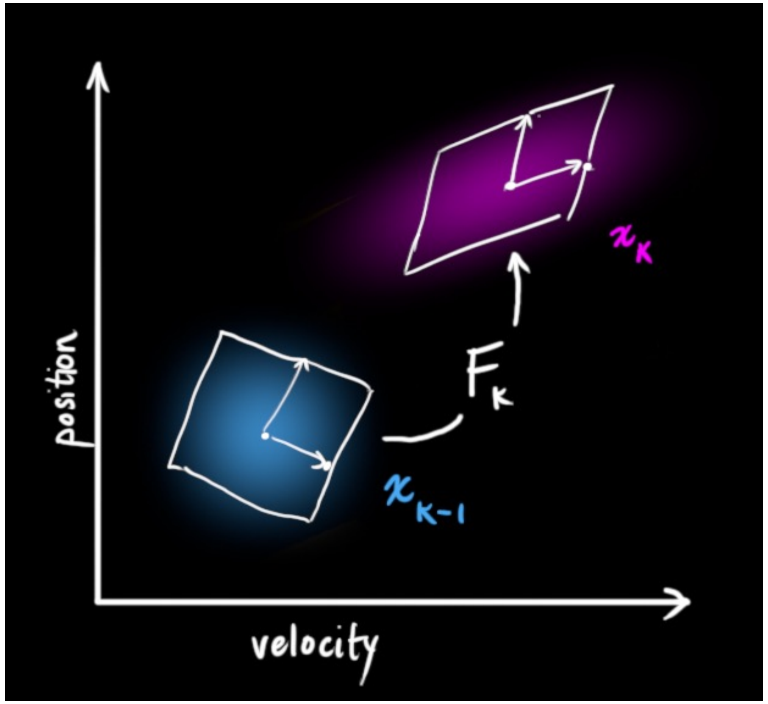
Intuition: Scale and shift the mean according to dynamics, uncertainty grows quadratically!

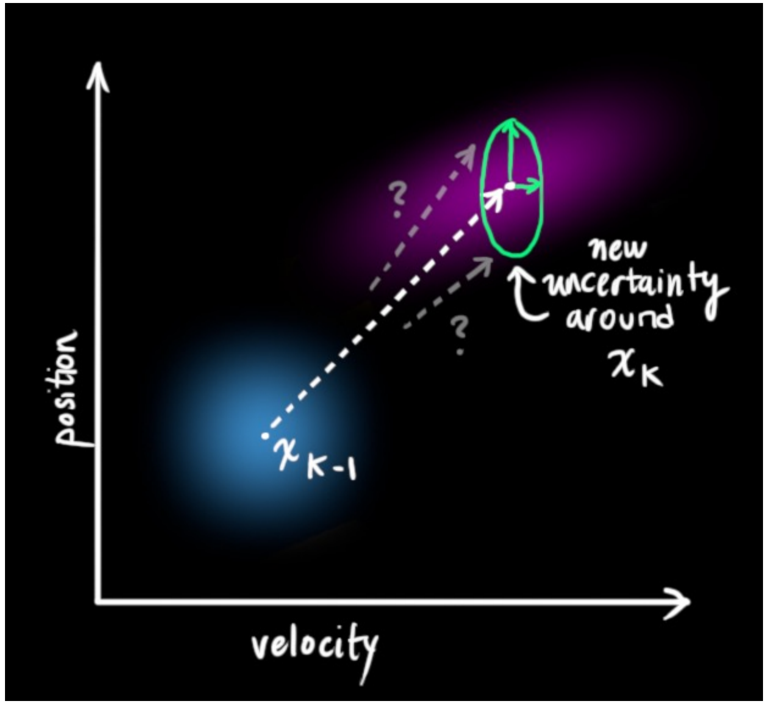
Linear Gaussian Systems: Prediction

Previous belief $$p(x_t|u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$$

Belief Update $$p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(A\mu_{t|0:t} + Bu_{t+1}, A\Sigma_{t|0:t}A^T + Q_{t+1})$$

Intuition: Scale and shift the mean according to dynamics, uncertainty grows!



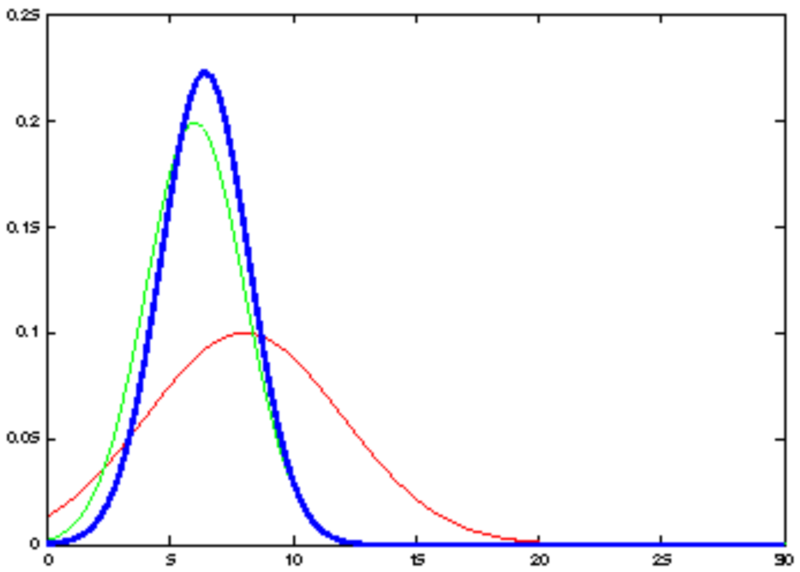


Intuition Behind Prediction Step

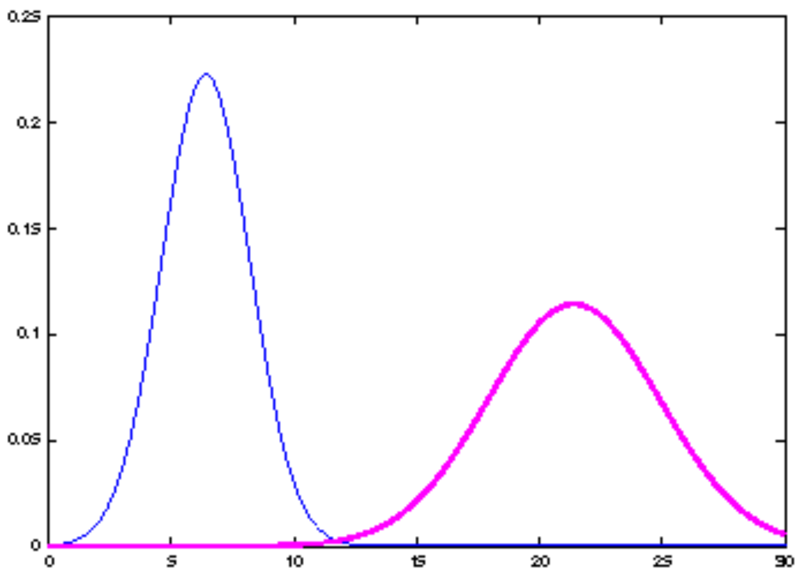
Previous belief $p(x_t | u_{0:t}, z_{0:t}) = \mathcal{N}(\mu_{t|0:t}, \Sigma_{t|0:t})$

Belief Update $p(x_{t+1} | u_{0:t+1}, z_{0:t}) = \mathcal{N}(A\mu_{t|0:t} + Bu_{t+1}, A\Sigma_{t|0:t}A^T + Q_{t+1})$

Intuition: Scale and shift the mean according to dynamics, uncertainty grows!



Belief at x_t



Belief post dynamics → shifted mean, scaled and shifted variance

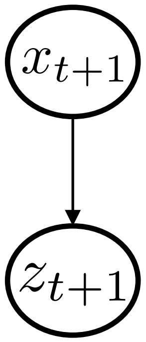
Linear Gaussian Systems: Observations

- Integrate the effect of an observation using sensor model, after dynamics

$$z_{t+1} = Cx_{t+1} + \delta_{t+1} \qquad\qquad \delta_{t+1} \sim \mathcal{N}(0, R_{t+1})$$

$$p(z_{t+1}|x_{t+1}) = \mathcal{N}(Cx_{t+1}, R_{t+1})$$

$$\overset{Bel(x_{t+1})}{p(x_{t+1}|u_{0:t+1}, z_{0:t+1})} \propto p(z_{t+1}|x_{t+1})\overset{\overline{Bel}(x_{t+1})}{p(x_{t+1}|u_{0:t+1}, z_{0:t})}$$



Gaussian, easy to normalize

Slightly harder than the dynamics step!

Linear Gaussian Systems: Observations

- Integrate the effect of an observation using sensor model, after dynamics

$$z_{t+1} = Cx_{t+1} + \delta_{t+1} \qquad \delta_{t+1} \sim \mathcal{N}(0, R_{t+1})$$

$$p(z_{t+1}|x_{t+1}) = \mathcal{N}(Cx_{t+1}, R_{t+1})$$

$$\underbrace{p(x_{t+1}|u_{0:t+1}, z_{0:t+1})}^{Bel(x_{t+1})} \propto p(z_{t+1}|x_{t+1})\overbrace{p(x_{t+1}|u_{0:t+1}, z_{0:t})}^{\overline{Bel}(x_{t+1})}$$

Conditioning

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = CX + B + \delta \\ \delta \sim \mathcal{N}(0, R) \end{cases} \implies \begin{array}{c} X|Y = y_0 \sim \mathcal{N}(\mu + K(y_0 - C\mu), (I - KC)\Sigma) \\ K = \Sigma C^T(C\Sigma C^T + R)^{-1} \end{array}$$

Linear Gaussian Systems: Observations

- Integrate the effect of an observation using sensor model, after dynamics

$$p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$$
$$z_{t+1} = Cx_{t+1} + \delta_{t+1}$$
$$\delta_{t+1} \sim \mathcal{N}(0, R_{t+1})$$

$$\begin{cases} X \sim \mathcal{N}(\mu, \Sigma) \\ Y = CX + B + \delta \\ \delta \sim \mathcal{N}(0, R) \end{cases} \implies \begin{array}{c} X|Y = y_0 \sim \mathcal{N}(\mu + K(y_0 - C\mu), (I - KC)\Sigma) \\ K = \Sigma C^T (C \Sigma C^T + R)^{-1} \end{array}$$

Previous belief $\quad p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$ Computed from dynamics step

Updated belief $\quad p(x_{t+1}|u_{0:t+1}, z_{0:t+1})$

$$= \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

$$K_{t+1} = \Sigma_{t+1|0:t} C^T (C \Sigma_{t+1|0:t} C^T + R)^{-1}$$

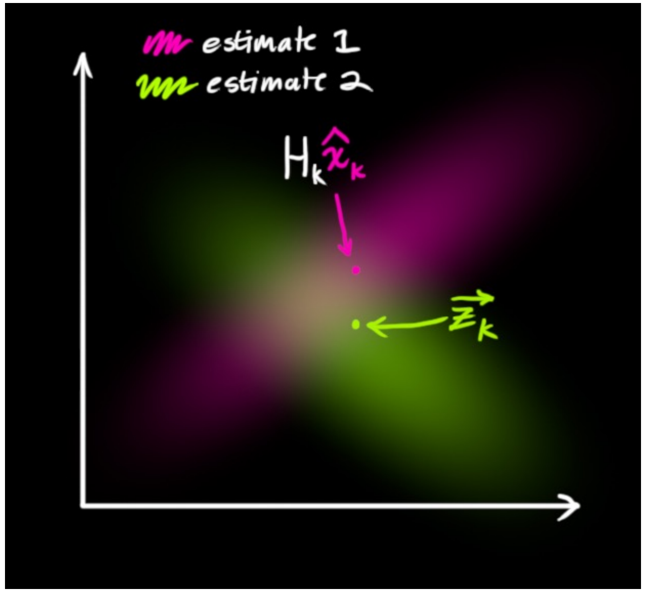
Linear Gaussian Systems: Observations

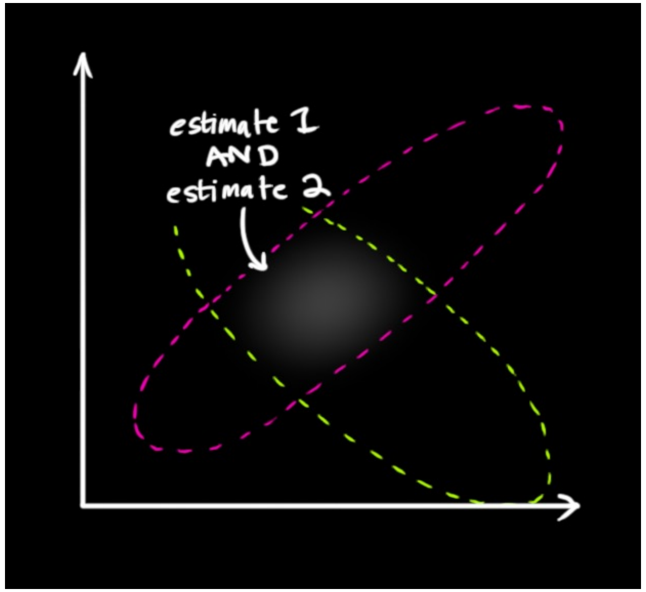
Previous belief $p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$ Computed from dynamics step

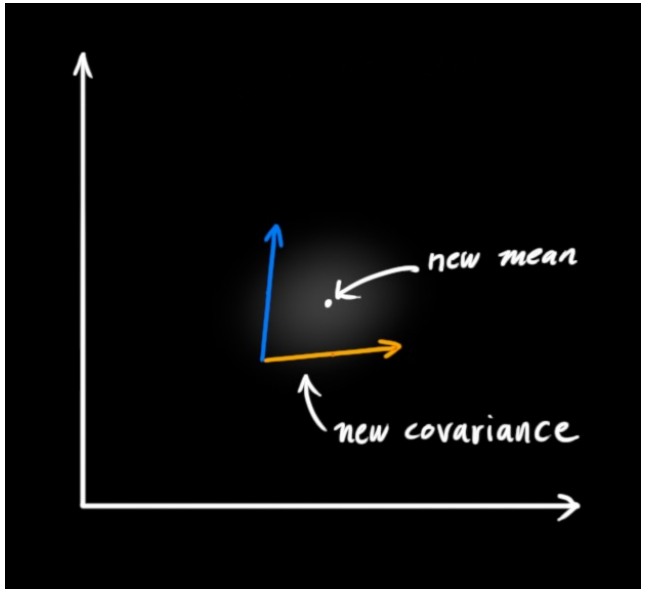
Updated belief $p(x_{t+1}|u_{0:t+1}, z_{0:t+1})$

$$= \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

Intuition: Correct the update linearly according to measurement error from expectation, shrink uncertainty accordingly



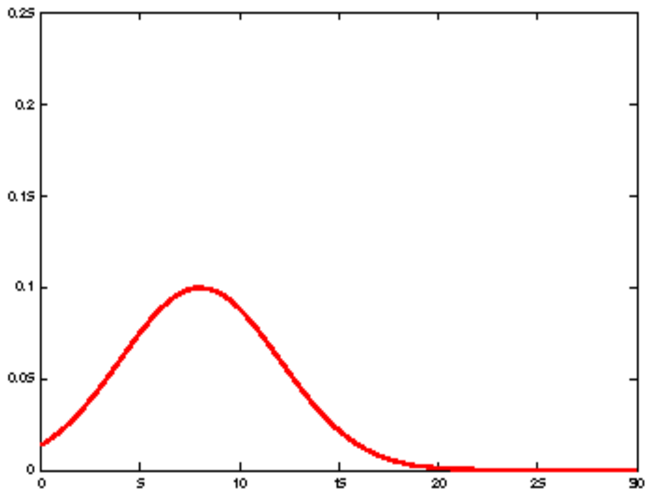


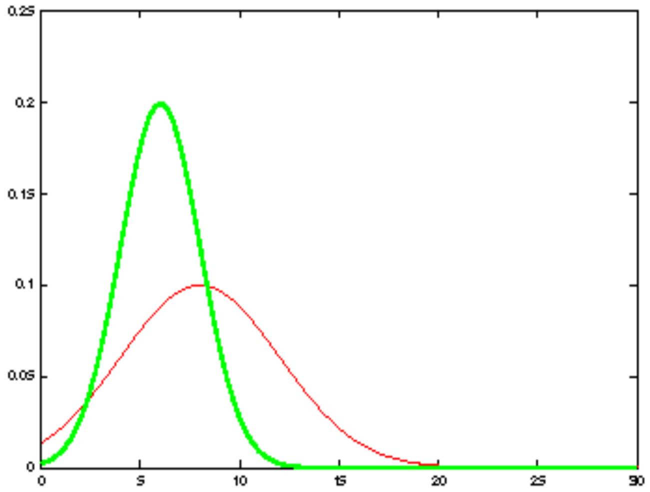


Intuition Behind Correction Step

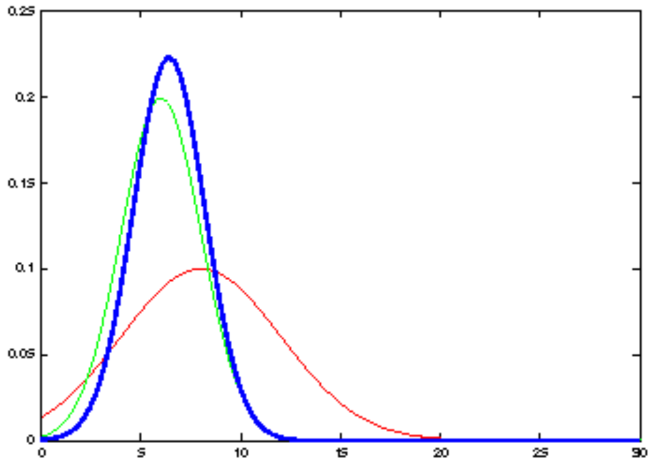
Previous belief

New Measurement





$$p(x_{t+1}|u_{0:t+1}, z_{0:t+1}) = \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$
$$K_{t+1} = \Sigma_{t+1|0:t}C^T(C\Sigma_{t+1|0:t}C^T + R)^{-1}$$



For the sake of simplicity, let's say C = I

$$K_{t+1} = \frac{\Sigma_{t+1|0:t}}{\Sigma_{t+1|0:t} + R}$$

Corrects belief based on measurement

→ Average between mean and measurement based on K

→ Scale down uncertainty based on K

Unpacking the Kalman Gain

Previous belief $p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(\mu_{t+1|0:t}, \Sigma_{t+1|0:t})$ Computed from dynamics step

Updated belief $p(x_{t+1}|u_{0:t+1}, z_{0:t+1})$

$$= \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

$$K_{t+1} = \Sigma_{t+1|0:t}C^T(C\Sigma_{t+1|0:t}C^T + R)^{-1}$$

For the sake of simplicity, let's say C = I

$$K_{t+1} = \frac{\Sigma_{t+1|0:t}}{\Sigma_{t+1|0:t} + R}$$

Case 1: Very noisy sensor, R>>Σ

Case 2: Deterministic sensor, R = 0

Kalman Filter Algorithm

Initial Prior

$$p(x_0)$$

Dynamics/Prediction
(given some u)

Estimate $\overline{Bel}(x_{t+1})$

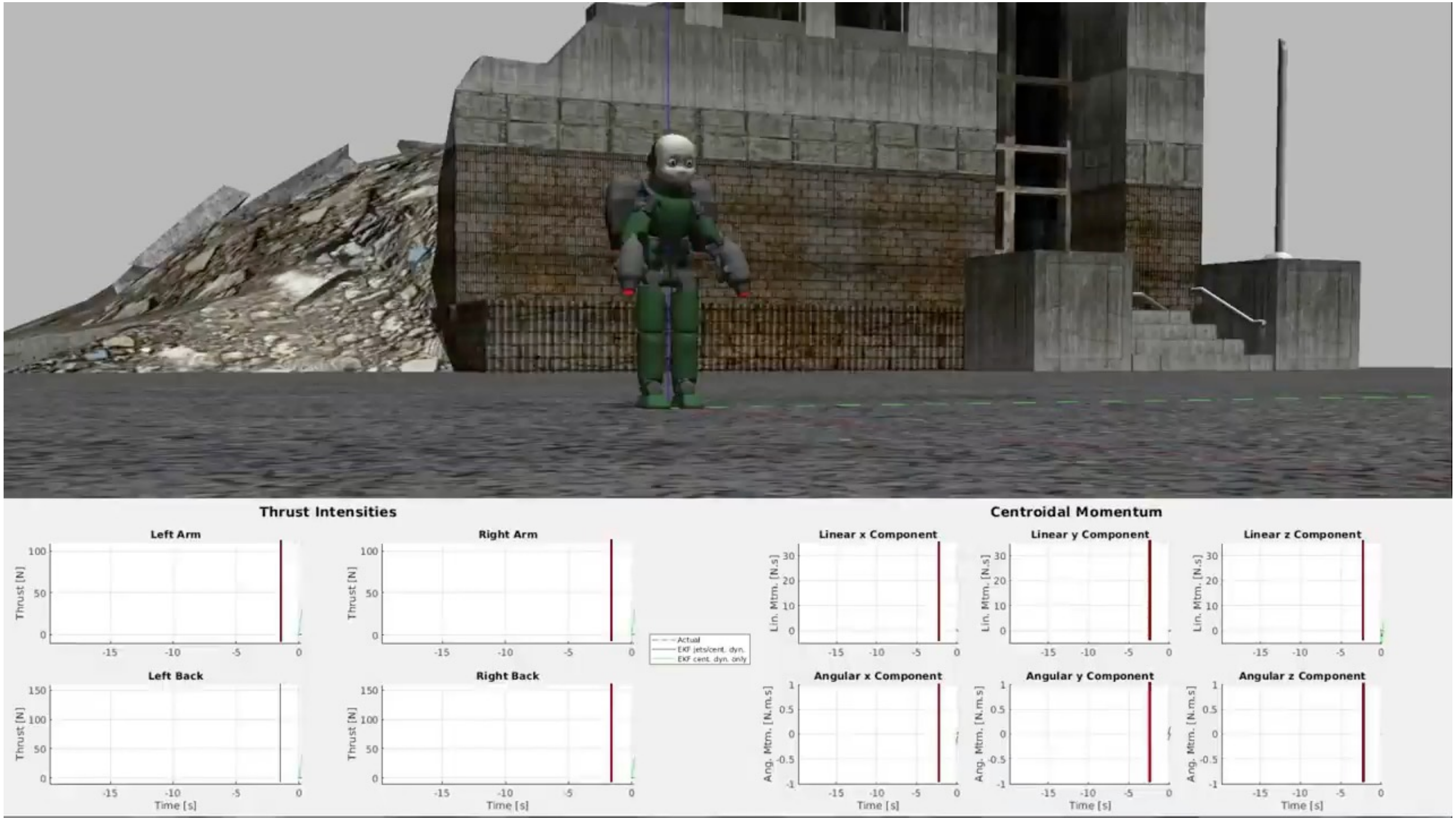
$$p(x_{t+1}|u_{0:t+1}, z_{0:t}) = \mathcal{N}(A\mu_{t|0:t} + Bu_{t+1}, A\Sigma_{t|0:t}A^T + Q_{t+1})$$

Measurement/Correction
(given some z)

Estimate $Bel(x_{t+1})$

$$p(x_{t+1}|u_{0:t+1}, z_{0:t+1})$$
$$= \mathcal{N}(\mu_{t+1|0:t} + K_{t+1}(z_{t+1} - C\mu_{t+1|0:t}), (I - K_{t+1}C)\Sigma_{t+1|0:t})$$

Kalman Filter in Action



Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality *k* and state dimensionality *n*:
  $O(k^{2.376} + n^2)$

Matrix Inversion (Correction)

$$K_{t+1} = \Sigma_{t+1|0:t} C^T (C \Sigma_{t+1|0:t} C^T + R_{t+1})^{-1}$$

Matrix Multiplication (Prediction)

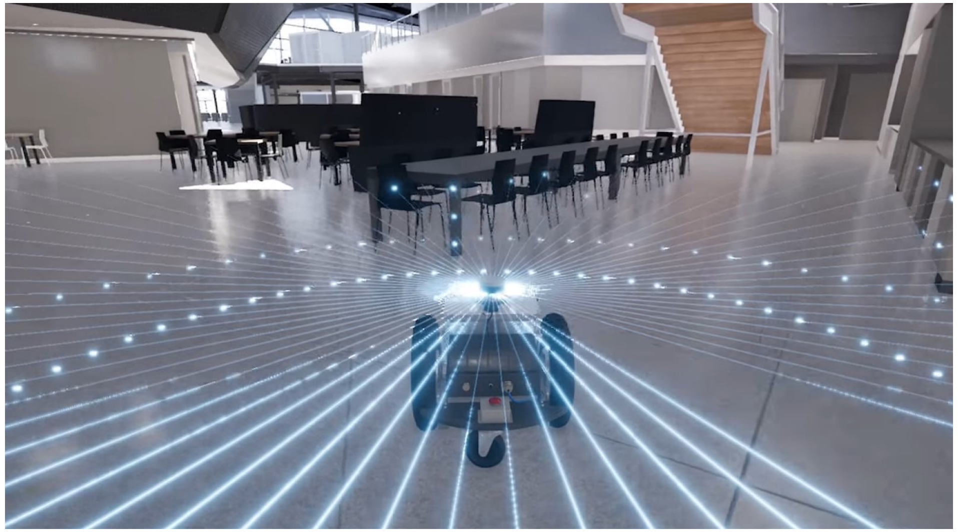
$$p(x_{t+1} | z_{0:t}, u_{0:t+1}) \sim \mathcal{N}(A\mu_{t|0:t} + Bu_t, A\Sigma_{t|0:t}A^T + Q_t)$$

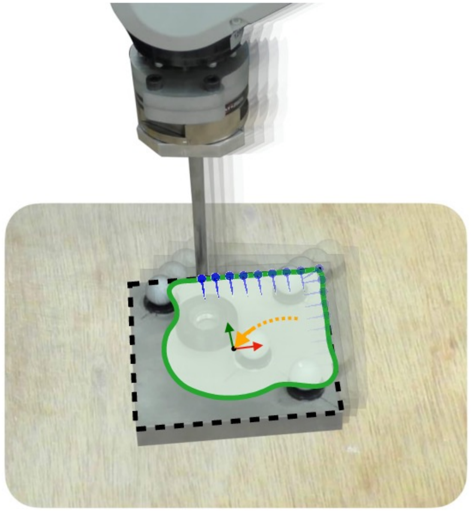
- Optimal for linear Gaussian systems!

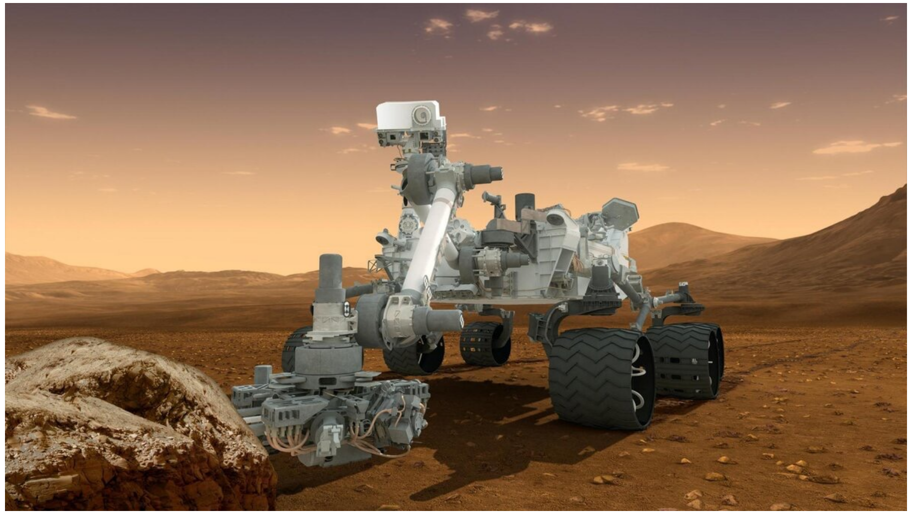
- Most robotics systems are nonlinear! → next time

Why should we care?

Still a very widely used technique for estimation/localization/mapping in real problems







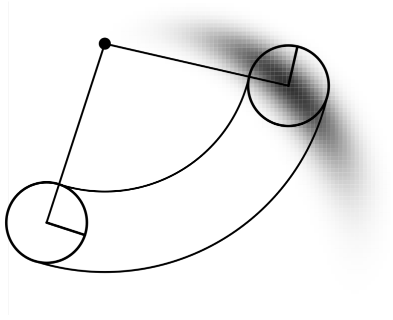
Connecting this back to racecars?

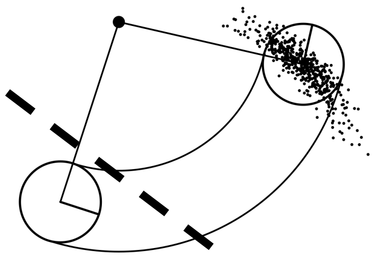
$$\theta_t = \theta_{t-1} + \Delta\theta = \theta_{t-1} + \frac{v}{L}\tan\delta\Delta t$$

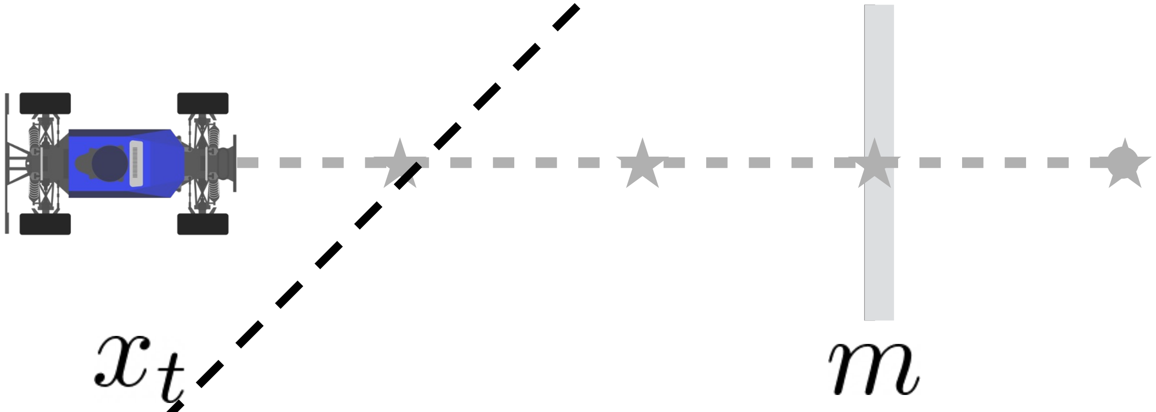
$$x_t = x_{t-1} + \Delta x = x_{t-1} + \frac{L}{\tan\delta}(\sin\theta_t - \sin\theta_{t-1})$$

$$y_t = y_{t-1} + \Delta y = y_{t-1} + \frac{L}{\tan\delta}(\cos\theta_{t-1} - \cos\theta_t)$$

$$p(z_t^k \mid x_t, m) = \begin{pmatrix} z_{\text{hit}} \\ z_{\text{short}} \\ z_{\text{max}} \\ z_{\text{rand}} \end{pmatrix}^T \cdot \begin{pmatrix} p_{\text{hit}}(z_t^k \mid x_t, m) \\ p_{\text{short}}(z_t^k \mid x_t, m) \\ p_{\text{max}}(z_t^k \mid x_t, m) \\ p_{\text{rand}}(z_t^k \mid x_t, m) \end{pmatrix}$$







Not linear! → particle filters next time!

Lecture Outline

Recap

Gaussian Properties

Kalman Filtering

Class Outline

State Estimation
- Robotic System Design
- Filtering
- Localization
- SLAM

Control
- Feedback Control
- PID Control
- MPC
- LQR

Planning
- Search
- Heuristic Search
- Motion Planning
- Lazy Search

Learning
- Imitation Learning
- Policy Gradient
- Actor-Critic
- Model-Based RL