Autonomous Robotics
Winter 2024

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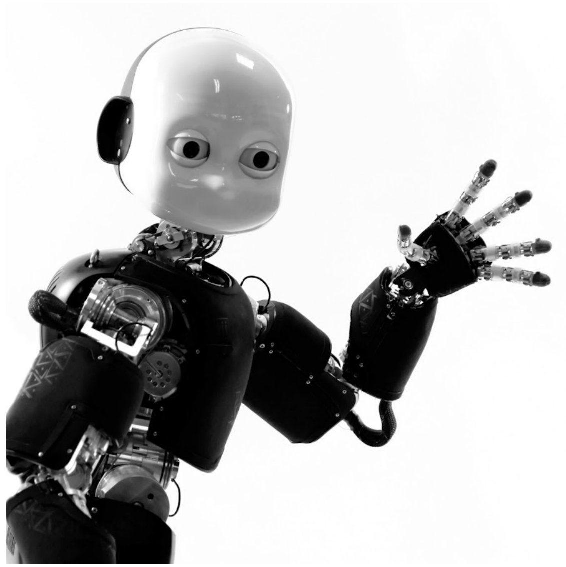

Slides borrowed from many sources – Sidd Srinivasa, Sanjiban Choudhury

Class Outline

State Estimation

- Robotic System Design
- Filtering
- Localization
- SLAM

Control

- Feedback Control
- PID Control
- MPC
- LQR

Planning

- Search
- Heuristic Search
- Motion Planning
- Lazy Search

Learning

- Imitation Learning
- Policy Gradient
- Actor-Critic
- Model-Based RL

Logistics

- HW 3 out now
- Paper readings on Wednesday:
  - Autonomous Automobile Trajectory Tracking for Off-Road Driving:Controller Design, Experimental Validation and Racing, Hoffman et al
  - Sampling-based Model Predictive Control Leveraging Parallelizable Physics Simulations, Pezzato et al

Lecture Outline

Recap + iLQR

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Sampling-based Optimal Control

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Lyapunov Stability

Generalized Problem: Optimal Control

- Minimize sum of costs, subject to dynamics and other constraints

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t)$$

Can be costs like smoothness, preferences, speed

Can be constraints like velocity/acceleration bounds

Linear Quadratic Regulator

- **Linear** system (model)
- **Quadratic** cost function to minimize

$$x_{t+1} = Ax_t + Bu_t$$

$$\sum_t x_t^\top Q x_t + u_t^\top R u_t$$

Turns into a recursion at time-to-go = i

$$K_i = -(R + B^\top P_{i-1} B)^{-1} B^\top P_{i-1} A$$

$$P_i = Q + K_i^\top R K_i + (A + BK_i)^\top P_{i-1} (A + BK_i)$$

$$u = K_i x, \;\; J_i(x) = x^\top P_i x$$

Optimal controller is linear in x

Optimal cost is quadratic in x

RUNTIME: $O(H(n^3 + m^3))$

The LQR algorithm

```
Algorithm OptimalValueControl(A, B, Q, R, time-to-go):

    if time-to-go == 0:
        return 0, Q

    else:
        P_{i-1} = OptimalValueControl(A, B, Q, R, time-to-go - 1)
```

$$K_i = -(R + B^\top P_{i-1} B)^{-1} B^\top P_{i-1} A$$

$$P_i = Q + K_i^\top R K_i + (A + BK_i)^\top P_{i-1} (A + BK_i)$$

return K_i, P_i

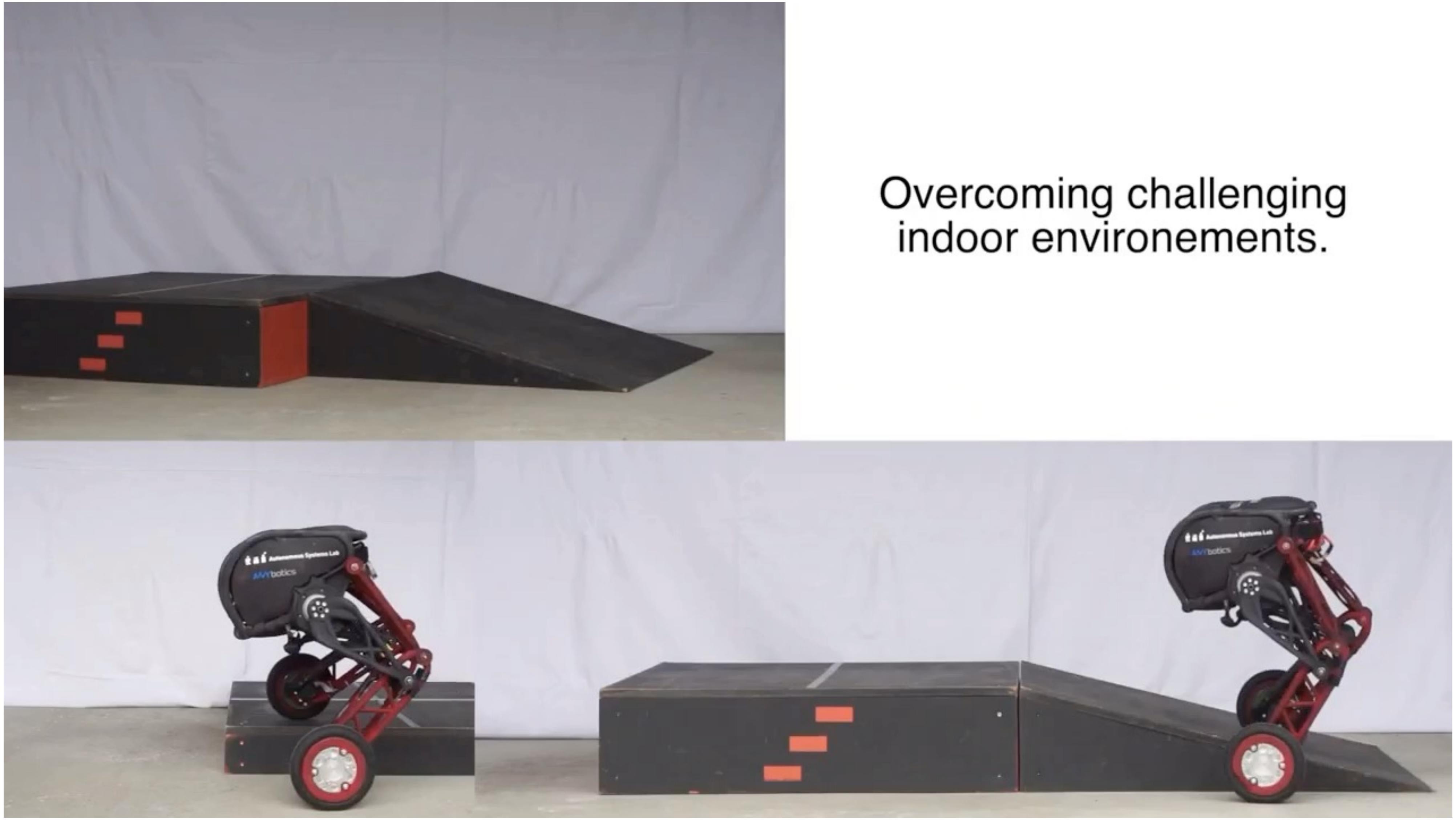
Optimal controller is linear in x

Optimal cost is quadratic in x

LQR in Action: Stanford Helicopter

ABBEEL ET AL., 2006 **HTTPS://YOUTU.BE/0JL04JJJOCC**

LQR in Action



Klemm et al 2020

What if the system is not linear/quadratic?

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

$$\text{s.t. } x_{t+1} = f(x_t, u_t)$$

Non-linear

Non-quadratic

Just use a Taylor expansion! → 1st order for dynamics, 2nd order for cost

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)$$

Dropping higher order terms, when x-a is small enough

Linear function in x

What if the system is not linear/quadratic?

- Let's study a simple case, where cost is quadratic, and there exists optimal tracking actions

$$\exists u_0^*, u_1^*, \ldots, u_{H-1}^* \; : \; \forall t \in \{0, 1, \ldots, H-1\} \; : \; x_{t+1}^* = f(x_t^*, u_t^*)$$

- Problem statement:

$$\begin{aligned} &\min_{u_0, u_1, \ldots, u_{H-1}} \sum_{t=0}^{H-1} (x_t - x_t^*)^\top Q (x_t - x_t^*) + (u_t - u_t^*)^\top R (u_t - u_t^*) \\ &\text{s.t.} \quad x_{t+1} = f(x_t, u_t) \end{aligned}$$

- Transform into linear time varying case (LTV):

$$x_{t+1} \approx f(x_t^*, u_t^*) + \underbrace{\frac{\partial f}{\partial x}(x_t^*, u_t^*)}_{A_t}(x_t - x_t^*) + \underbrace{\frac{\partial f}{\partial u}(x_t^*, u_t^*)}_{B_t}(u_t - u_t^*)$$

$$x_{t+1} - x_{t+1}^* \approx A_t (x_t - x_t^*) + B_t (u_t - u_t^*)$$

What if the system is not linear/quadratic?

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

Non-linear

$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t)$$

Non-quadratic

Use linear/quadratic Taylor expansion
about current nominal states/actions

LQR

$$\sum_t x_t^\top Q x_t + u_t^\top R u_t$$

$$x_{t+1} = A x_t + B u_t$$

$$u = K_i x,$$
$$J_i(x) = x^\top P_i x$$

Lecture Outline

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↓

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↓

Lyapunov Stability

Why might this not be enough?

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t)$$

Non-linear

Non-quadratic

Use linear/quadratic Taylor expansion
about current nominal states/actions

$$\sum_t x_t^\top Q x_t + u_t^\top R u_t$$

$$x_{t+1} = A x_t + B u_t$$

Might be a poor, local approximation!

May not be able to incorporate constraints

Let’s revisit ideas from Bayesian filtering

	Linear Gaussian assumption	Sampling-based approximation
<u>Filtering</u>	Kalman Filtering	Particle Filtering
<u>Control</u>	LQR	Sampling based MPC

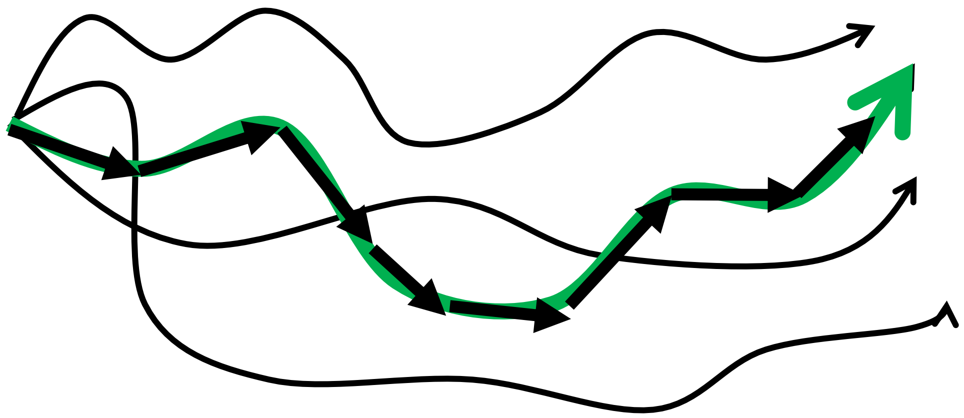
Solving Optimal Control with Sampling

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t)$$

1. Sample a set of K action trajectories of T steps from start state
2. Evaluate each K step action sequence through the model and get per trajectory cost
3. Choose minimum trajectory cost trajectory
4. Execute lowest cost actions

Random Sampling



Can soften by taking softmin rather than argmin

Solving Optimal Control with Sampling – issues?

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t)$$

1. Sample a set of K action trajectories of T steps from start state
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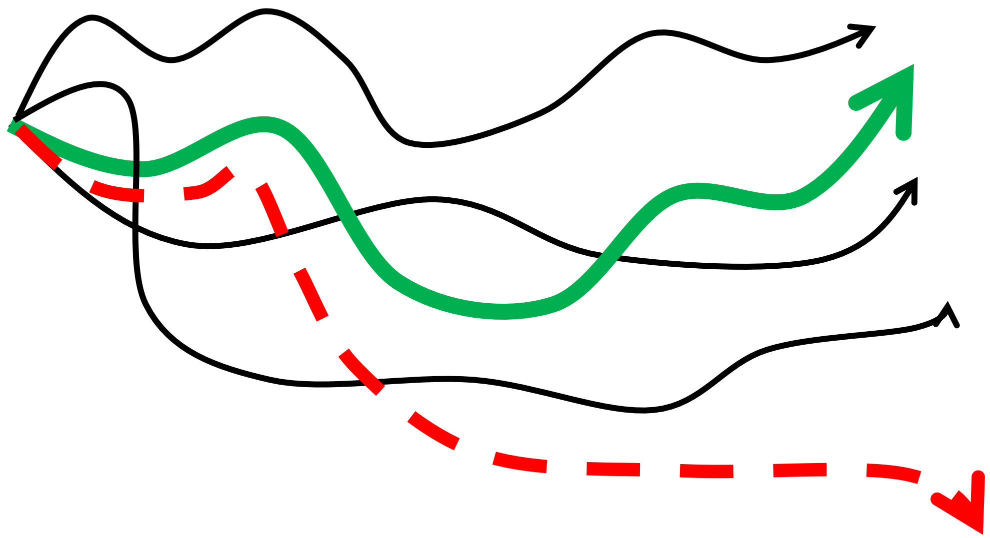
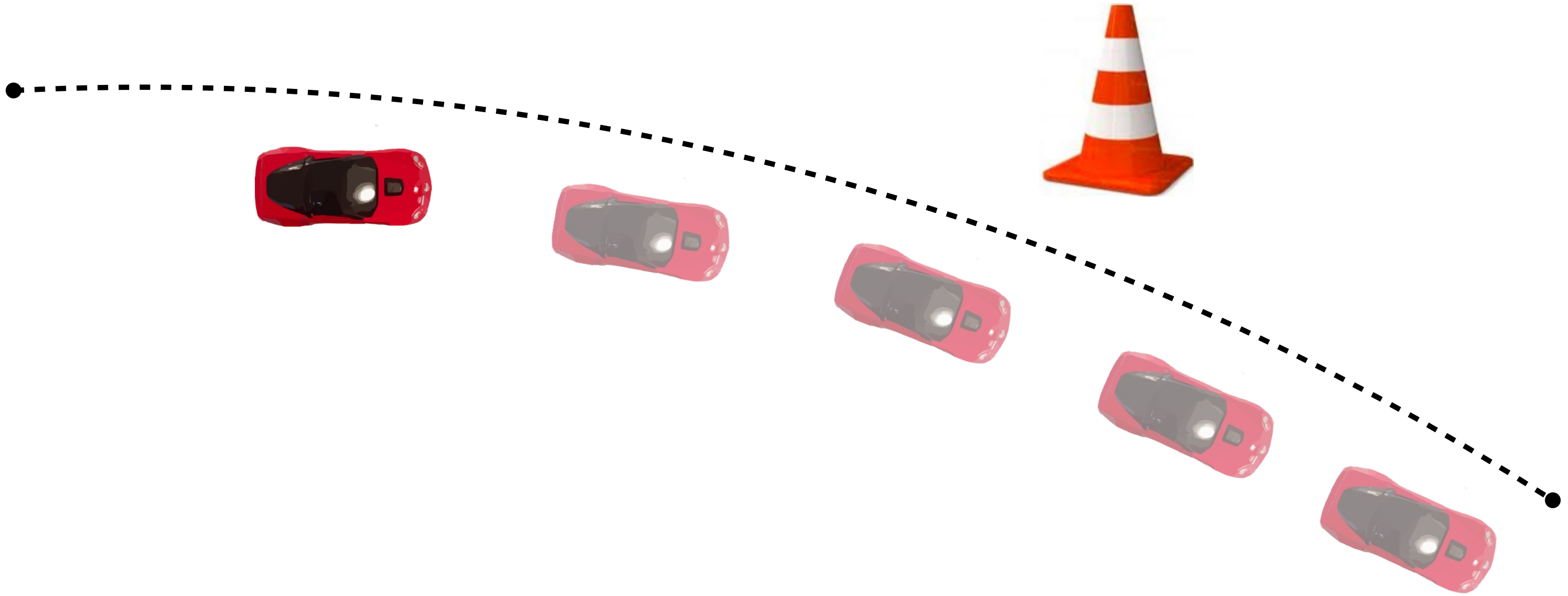


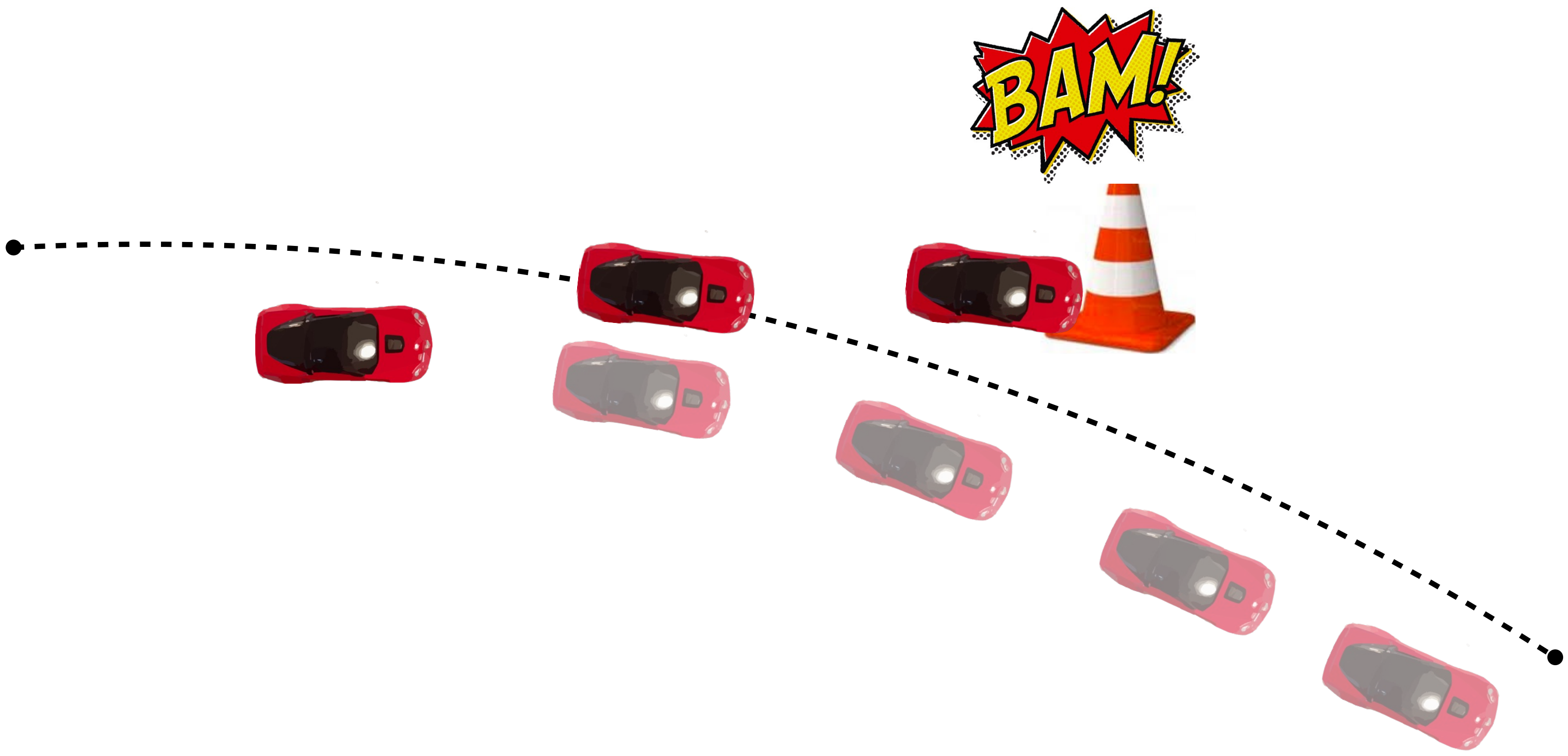
1. Open-loop controller may not be able to deal with unexpected events/divergences
2. Computation of full controller can be expensive: → Do it on the fly!
3. Model might be wrong, errors may accumulate
4. …

Why do we need to replan?



What happens if the controls are planned once and executed?

Why do we need to replan?



What happens if the controls are planned once and executed?

Solving Optimal Control with Sampling – issues?

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

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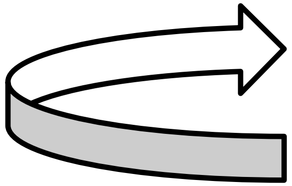
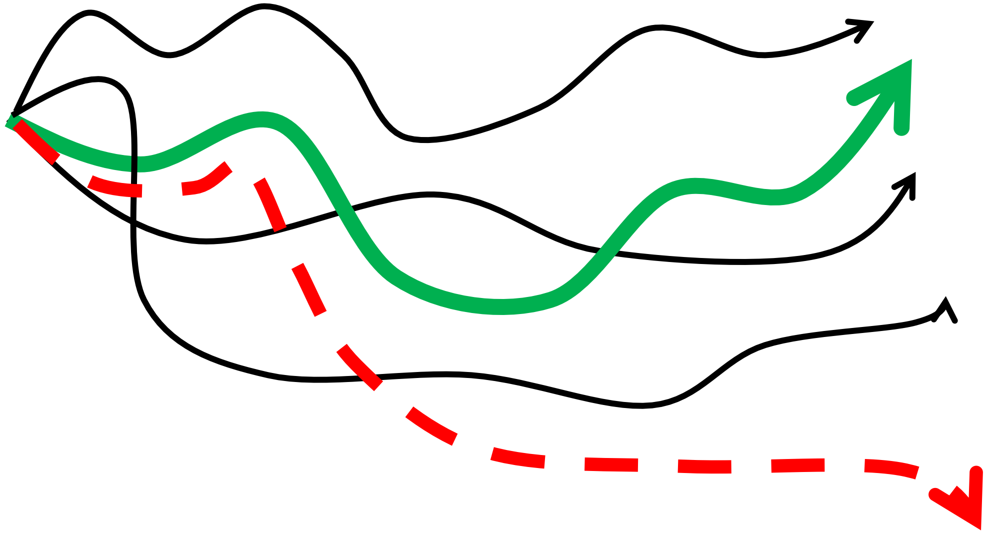
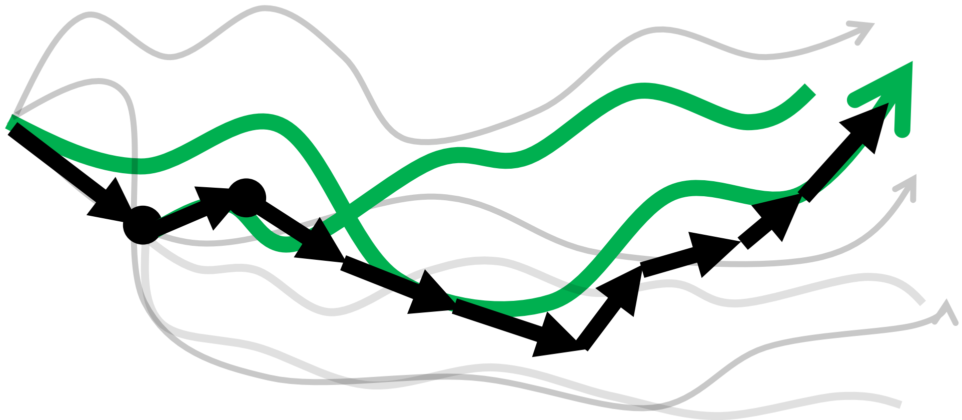


1. Plan with random shooting from s_t
2. Execute the first action a_0 and reach s_{t+1}

A stationary feedback controller may not be able to deal with unexpected events

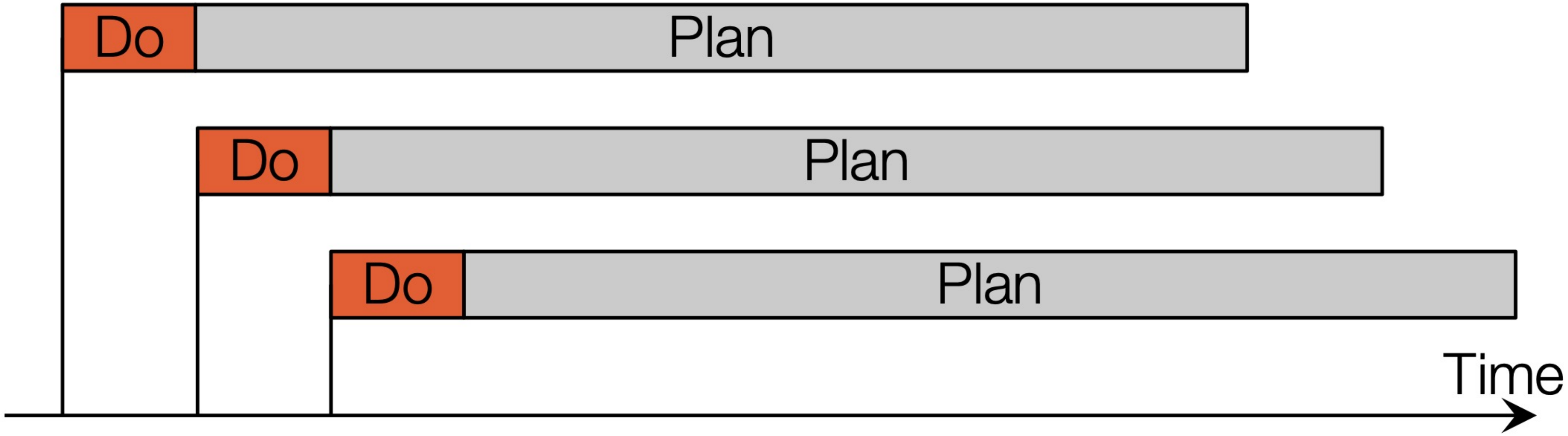


Replanning can help with divergence



Model-Predictive/Receding Horizon Control

General Replanning Framework - MPC

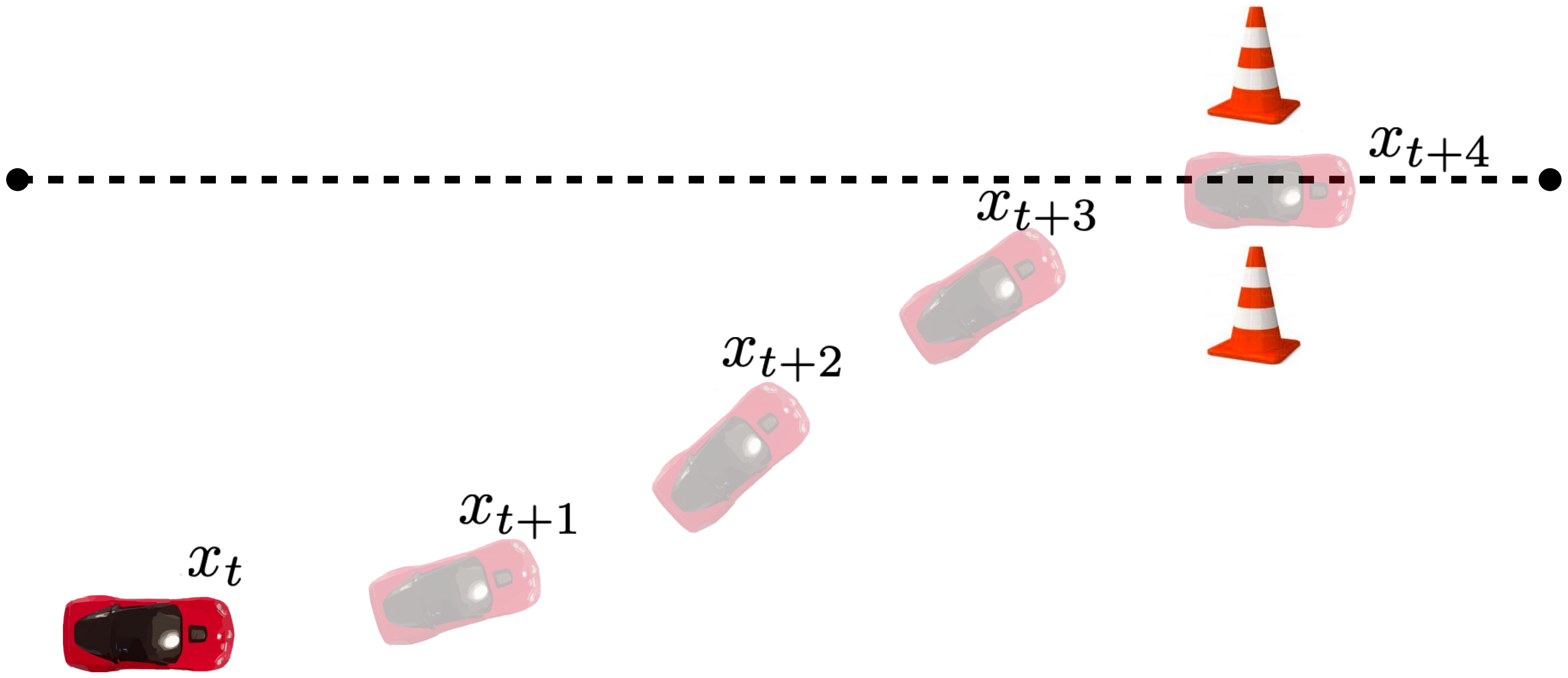


Step 1: Solve optimization problem to a horizon

Step 2: Execute the first control

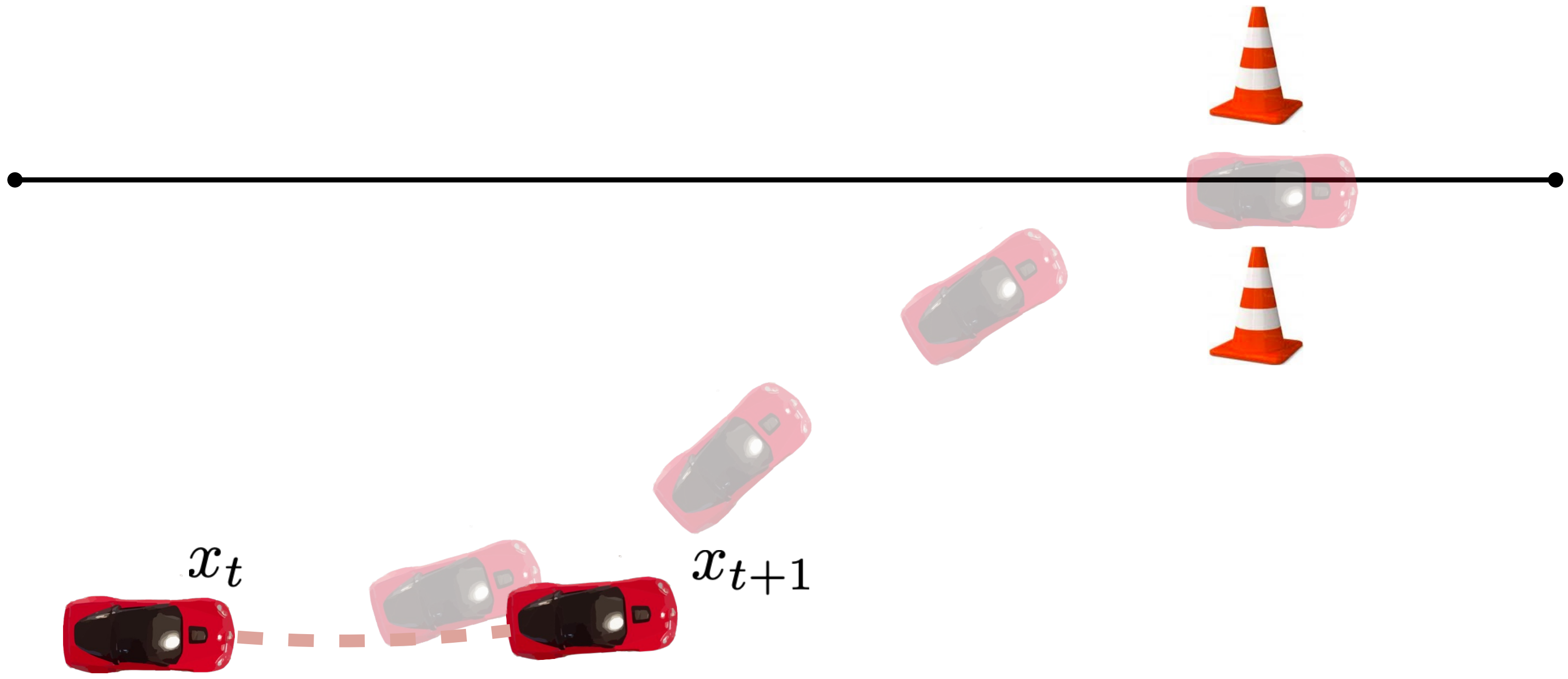
Step 3: Repeat!

How are the controls executed?



Step 1: Solve optimization problem to a horizon

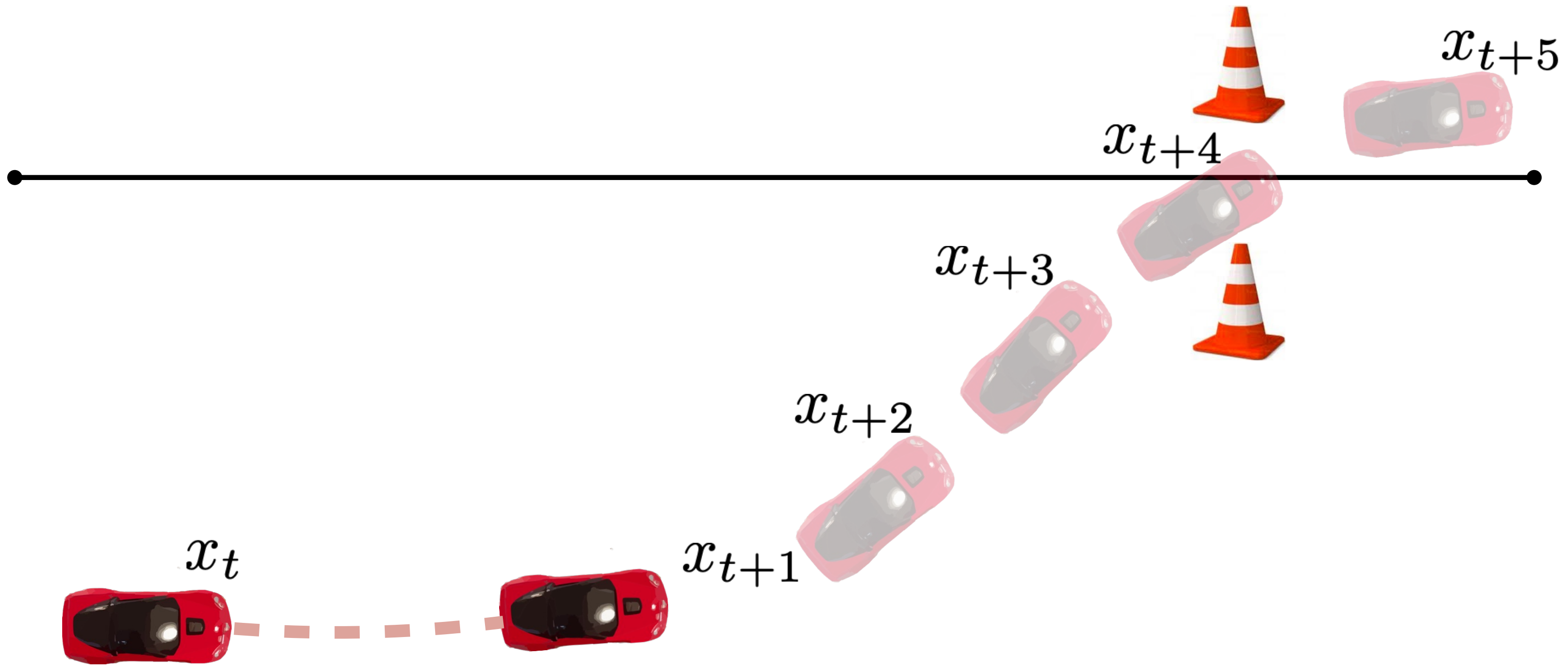
How are the controls executed?



Step 1: Solve optimization problem to a horizon

Step 2: Execute the first control

How are the controls executed?

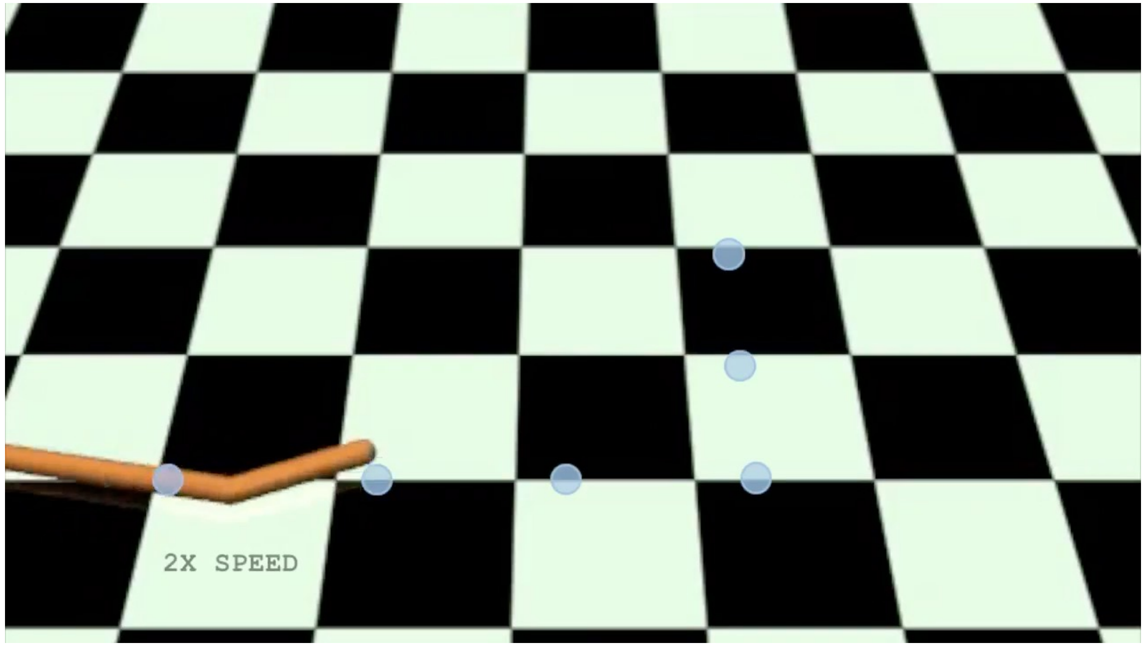


Step 1: Solve optimization problem to a horizon

Step 2: Execute the first control

Step 3: Repeat!

Does it work?



Nagabandi et al

Why might this not work?

$$\min_{u_{1:T}} \sum_{t=1}^{T} c(x_t, u_t)$$

$$\text{s.t.} \quad x_{t+1} = f(x_t, u_t)$$

1. Sample a set of K action trajectories of T steps from start state
2. Evaluate each K step action sequence through the model and get per trajectory cost
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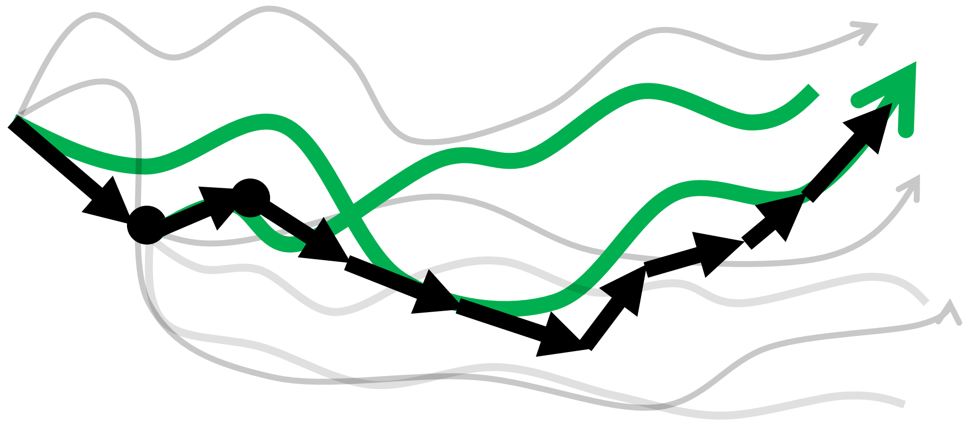
Planning with Shooting + MPC



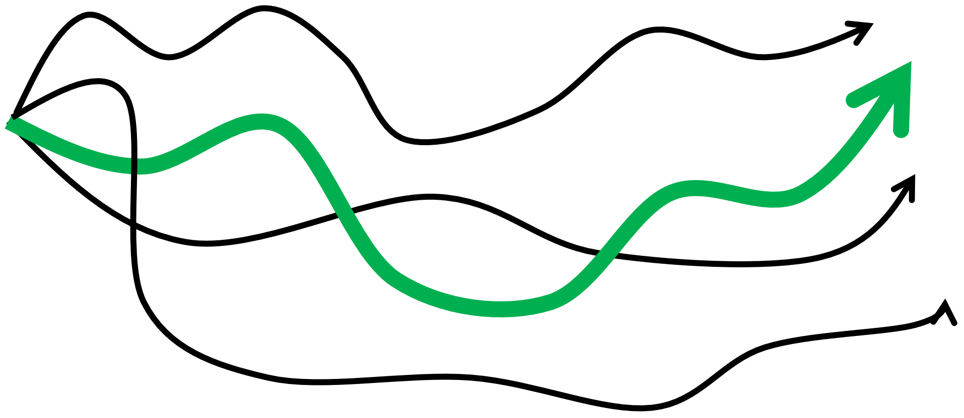
Searching for a needle in a haystack by random shooting, high variance!

Better Sampling Techniques for MPC

Sampled from stationary uniform/gaussian distribution

$$\arg\min_{u_0, u_1, \ldots, u_T} \sum_{t=1}^{T} c(x_t, u_t)$$

$$x_{t+1} = f(x_t, u_t)$$

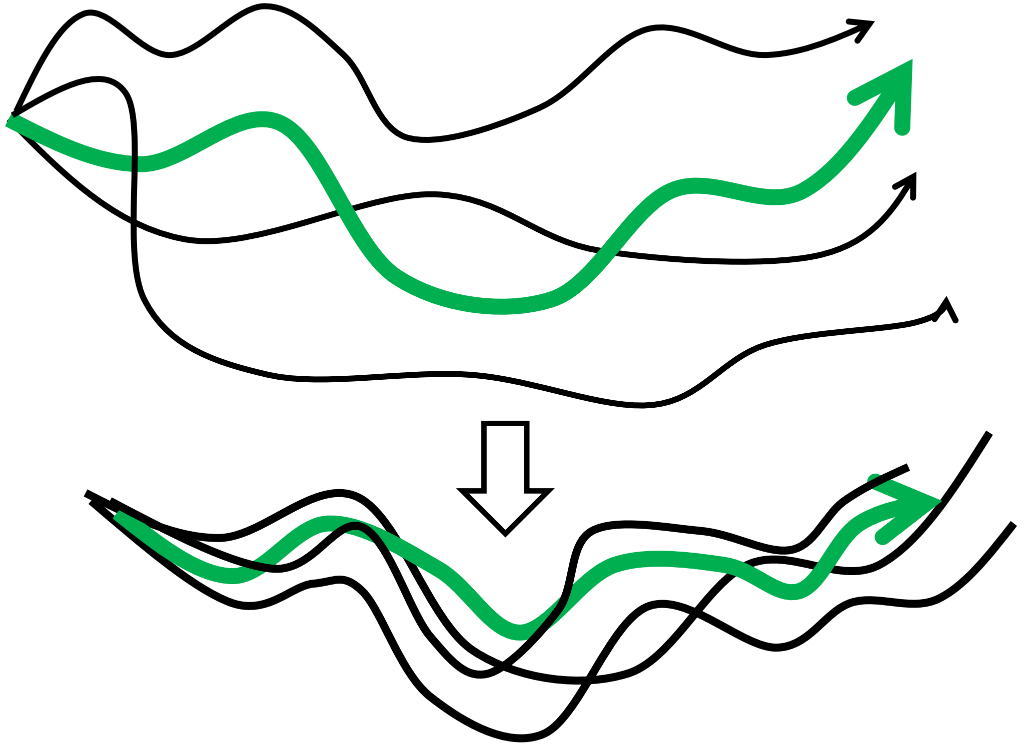


Can we inform the sampling function with the cost function?

Idea: Iteratively upweight sampling distribution around the things that are lower cost

Better Sampling Techniques for Shooting - MPPI

Idea: Iteratively upweight sampling distribution around the things that are lower costs



Referred to as **MPPI**, lower variance!

Sample N action sequences

$$(u_0^j, u_1^j, \ldots, u_T^j)_{j=1}^N \sim p(u)$$

Sample trajectories using these action sequences with the model

$$x_{t+1}^j = f(x_t^j, u_t^j)$$

Update action sampler by upweighting low cost actions

$$p(u) \leftarrow p(u) \frac{\exp \sum_{t=1}^T -c(x_t, u_t)}{Z}$$

Does it work?

Does it work?

Lecture Outline

Recap + iLQR

↓

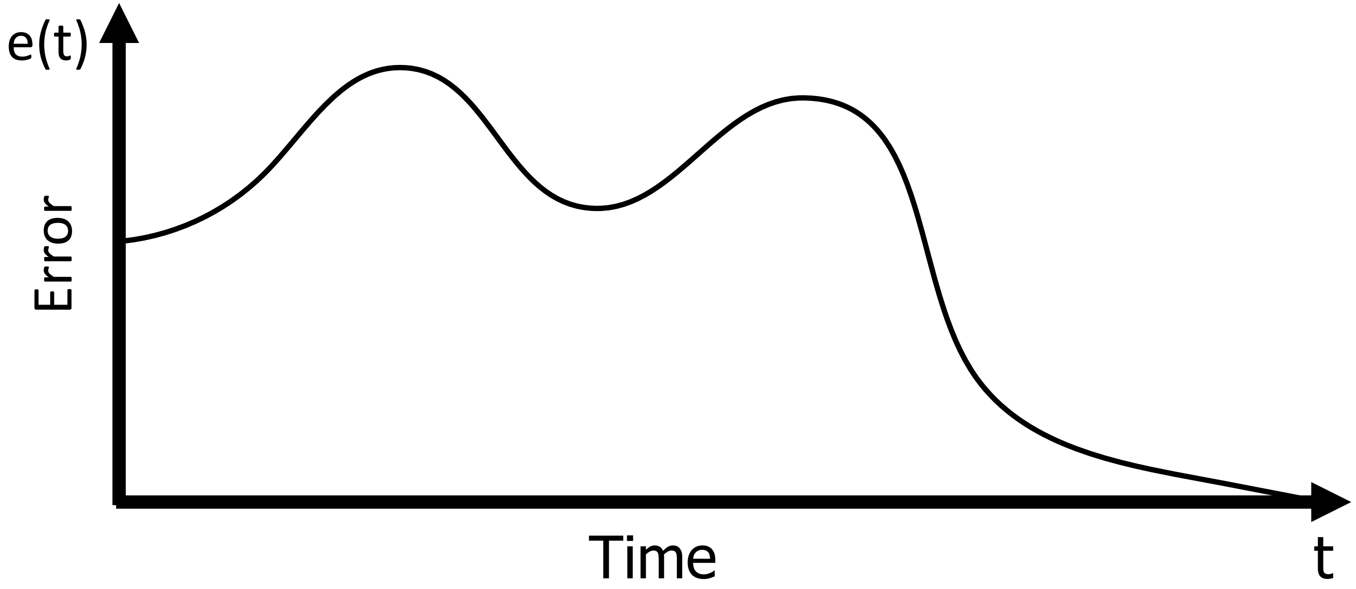
Sampling-based Optimal Control

↓

Lyapunov Stability

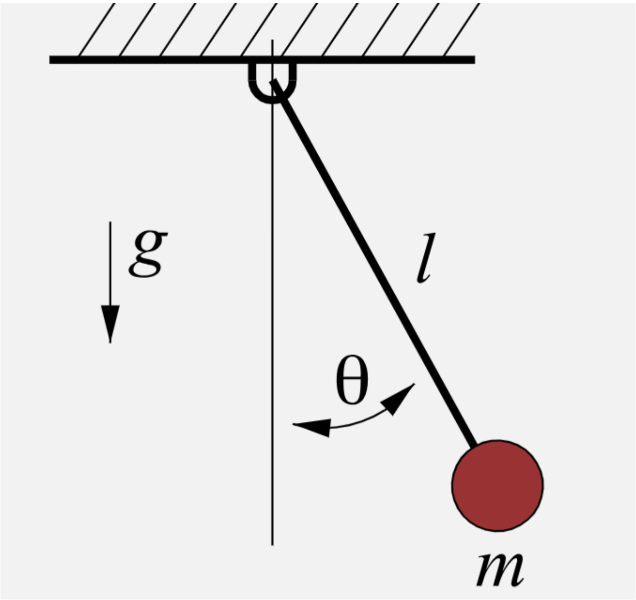
What is stability?

$$\lim_{t \to \infty} e(t) = 0$$



So we want both $e(t) \to 0$ and $\dot{e}(t) \to 0$

Detour: How do we make a pendulum stable?



$$ml^2\ddot{\theta} + mgl\sin\theta = u$$

What control law should we use to stabilize the pendulum, i.e.

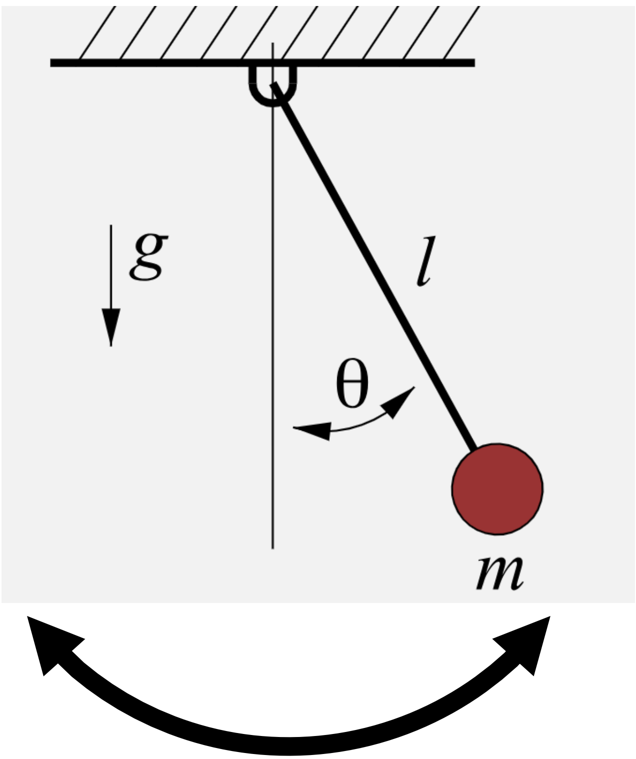
Choose $u = \pi(\theta, \dot{\theta})$ such that $\theta \to 0$

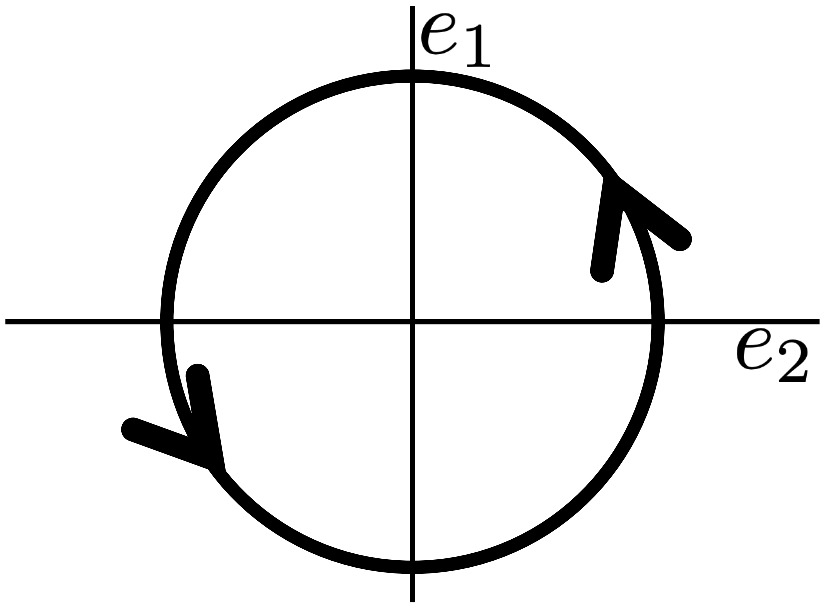
$$\dot{\theta} \to 0$$

How does the passive error dynamics behave?

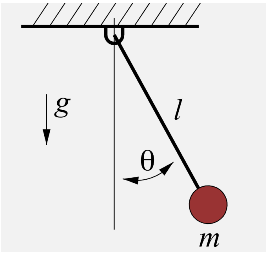
$$e_1 = \theta - 0 = \theta \qquad\qquad e_2 = \dot{\theta} - 0 = \dot{\theta}$$

Set u=0. Dynamics is not stable.





How do we verify if a controller is stable?



$$ml^2\ddot{\theta} + mgl\sin\theta = u$$

Lets pick the following law:

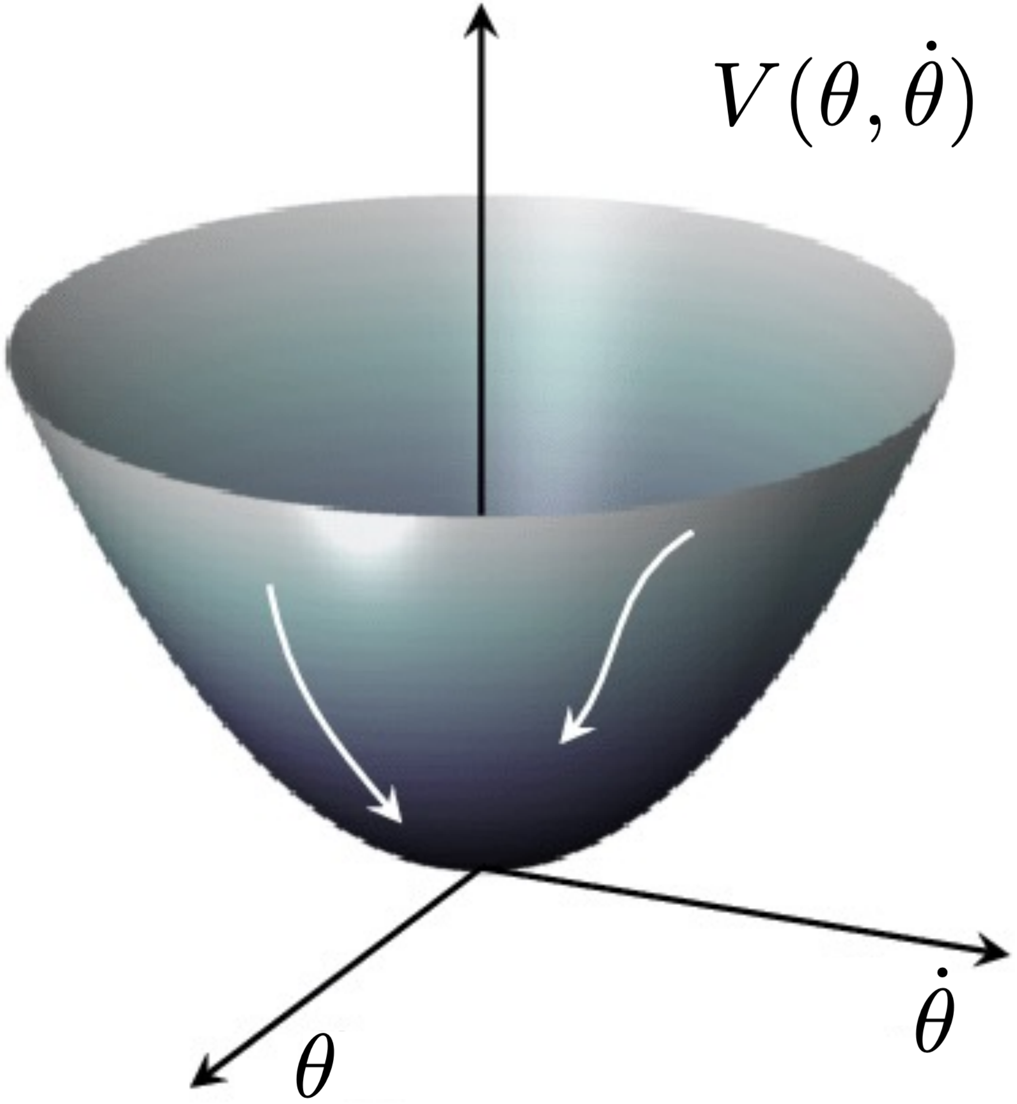
$$u = -K\dot{\theta}$$

Is this stable? How do we know?

We can simulate the dynamics from different start point and check….

but how many points do we check? what if we miss some points?

Key Idea: Think about energy!



Make energy decay to 0 and stay there

$$V(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$

$$> 0$$

$$\begin{aligned}\dot{V}(\theta, \dot{\theta}) &= m l^2 \dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta} \\ &= \dot{\theta}(u - mgl\sin\theta) + mgl(\sin\theta)\dot{\theta} \\ &= \dot{\theta} u\end{aligned}$$

Choose a control law $u = -k\dot{\theta}$

$$\dot{V}(\theta, \dot{\theta}) = -k\dot{\theta}^2 < 0$$

Lyapunov function:

A generalization of energy

Lyapunov function for a closed-loop system

1. Construct an energy function that is always positive

$$V(x) > 0, \forall x$$

Energy is only 0 at the origin, i.e. $V(0) = 0$

2. Choose a control law such that this energy always decreases

$$\dot{V}(x) < 0, \forall x$$

Energy rate is 0 at origin, i.e. $\dot{V}(0) = 0$

No matter where you start, energy will decay and you will reach 0!

Let’s get provable control for our car!

Dynamics of the car

$$\dot{x} = V \cos\theta$$
$$\dot{y} = V \sin\theta$$
$$\dot{\theta} = \frac{V}{B} \tan u$$

Let's get provable control for our car!

Let's define the following Lyapunov function

$$V(e_{ct}, \theta_e) = \frac{1}{2}k_1 e_{ct}^2 + \frac{1}{2}\theta_e^2 \qquad > 0$$

Compute derivative

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} \dot{e_{ct}} + \theta_e \dot{\theta_e}$$

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin\theta_e + \theta_e \frac{V}{B} \tan u$$

Let's get provable control for our car!

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin\theta_e + \theta_e \frac{V}{B} \tan u$$

Trick: Set u intelligently to get this term to always be negative

$$\theta_e \frac{V}{B} \tan u = -k_1 e_{ct} V \sin\theta_e - k_2 \theta_e^2$$

$$\tan u = -\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e$$

$$u = \tan^{-1}\left(-\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e\right)$$

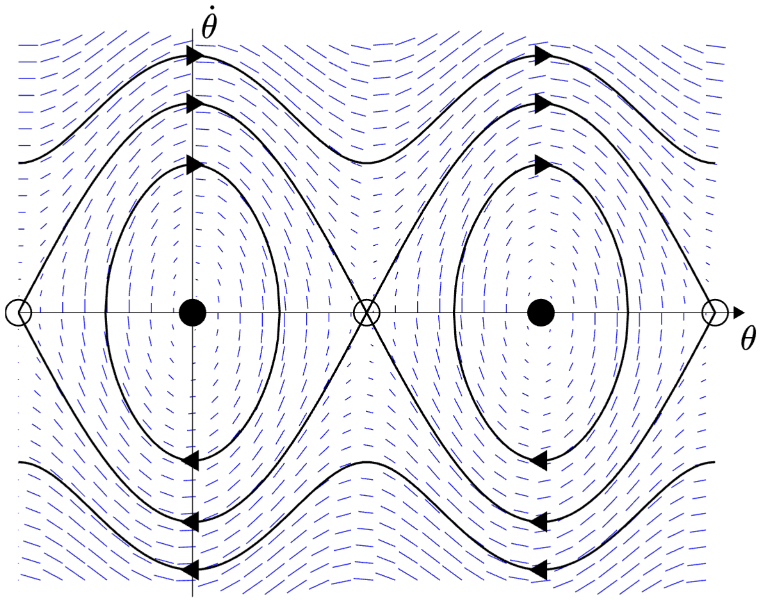
So what's the point of Lyapunov theory?

<u>Option 1:</u>

Use Lyapunov theory to <u>**construct**</u> stable controllers

<u>Option 2:</u>

Use Lyapunov theory to <u>**verify**</u> controllers for stability



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