Iterative LQR & Model Predictive Control

Instructor: Chris Mavrogiannis

TAs: Kay Ke, Gilwoo Lee, Matt Schmittle

*Slides based on or adapted from Sanjiban Choudhury, Drew Bagnell, Steven Boyd

Table of Controllers

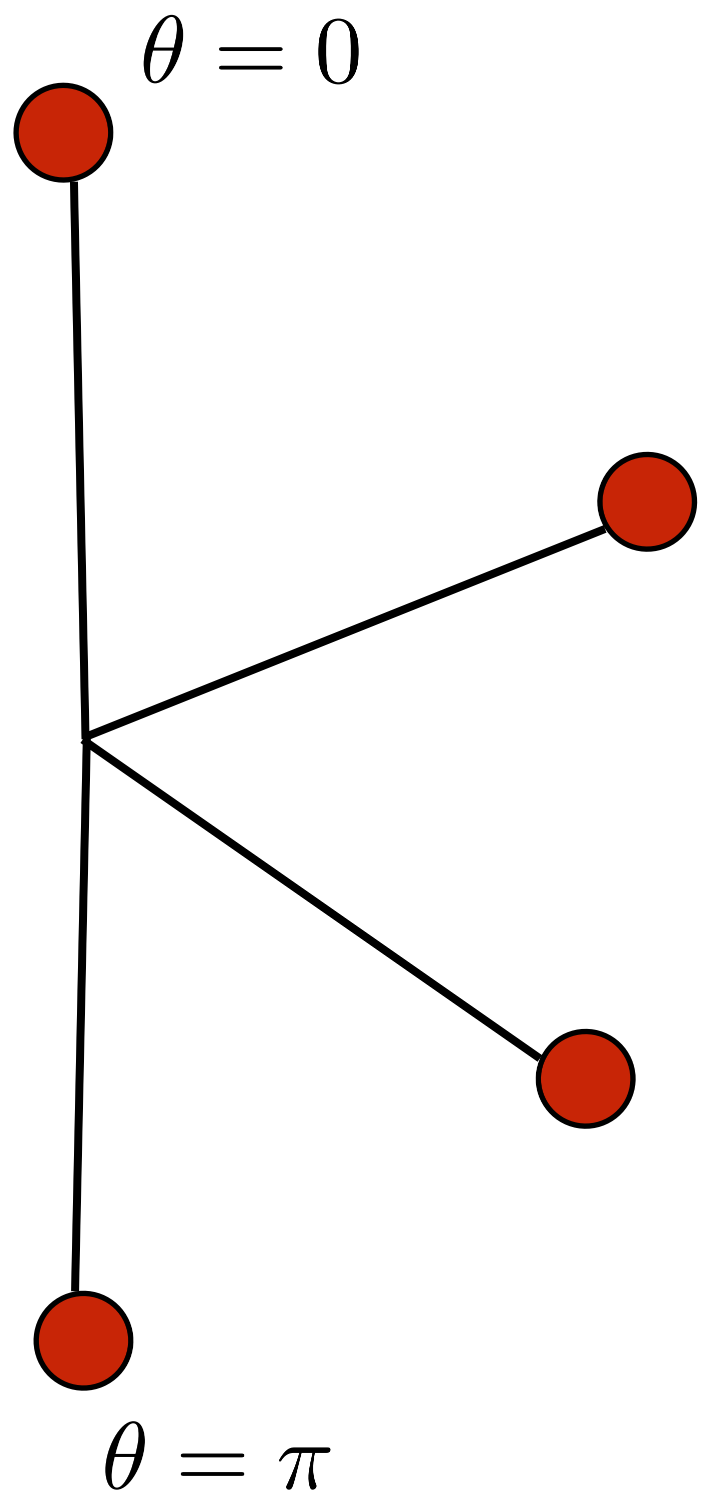
	Control Law	Uses model	Stability Guarantee	Minimize Cost
PID		No	No	No
Pure Pursuit		Circular arcs	Yes - with assumptions	No
Lyapunov		Non-linear	Yes	No
LQR		Linear	Yes	Quadratic

Can we use LQR to swing up a pendulum?

No!

(Large angles imply large linearization error)

But we can track a reference swing up trajectory (small linearization error)



Today's objectives

1. Trajectory following with iLQR
2. General nonlinear trajectory optimization with iLQR
3. Model predictive control (MPC)

LQR for Time-Varying Dynamical Systems

$$x_{t+1} = A_t x_t + B_t u_t$$

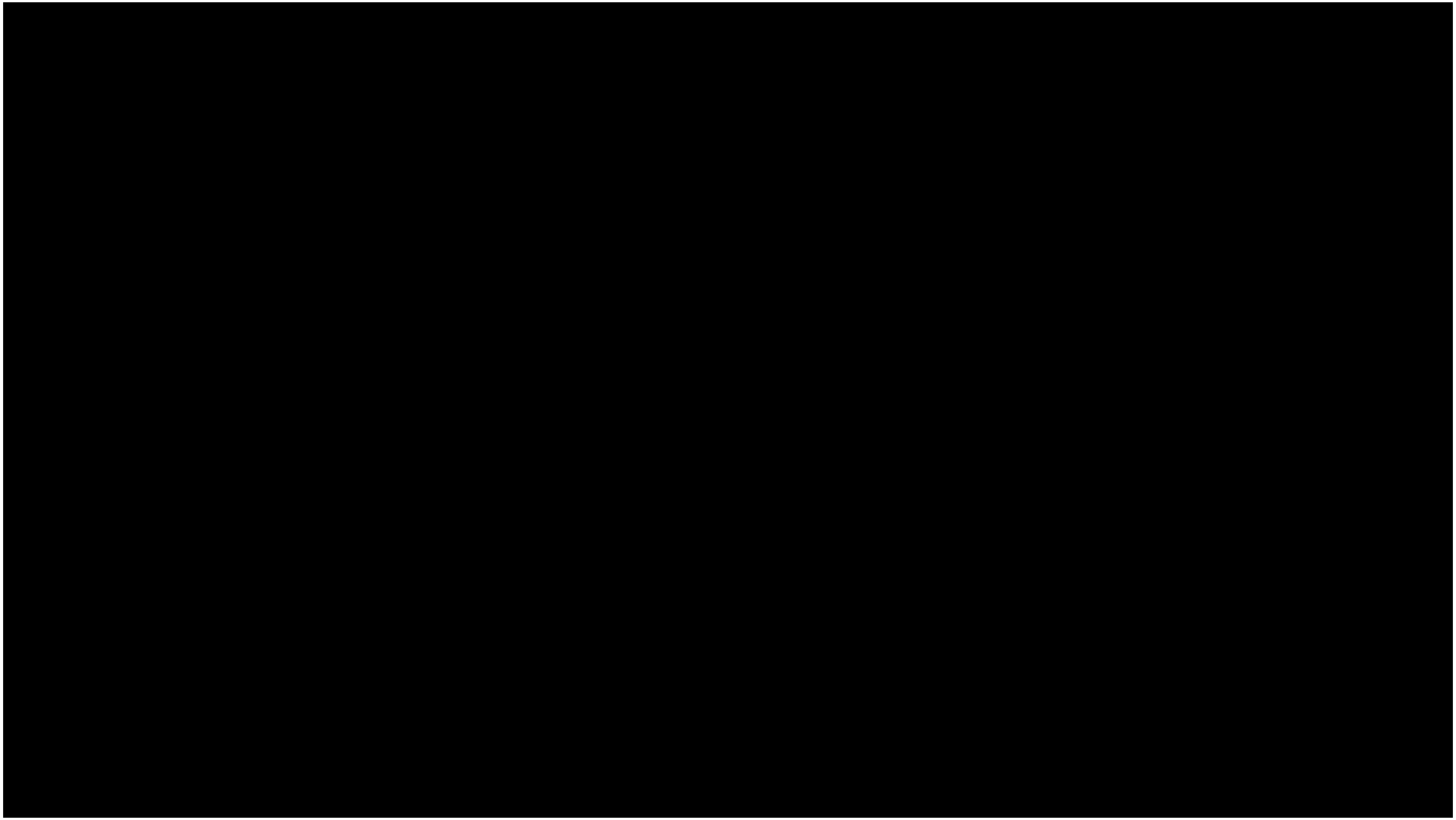
$$c(x_t, u_t) = x_t^T Q_t x_t + u_t^T R_t u_t$$

Straight forward to get LQR equations

$$K_t = -(R_t + B_t^T V_{t+1} B_t)^{-1} B_t^T V_{t+1} A_t$$

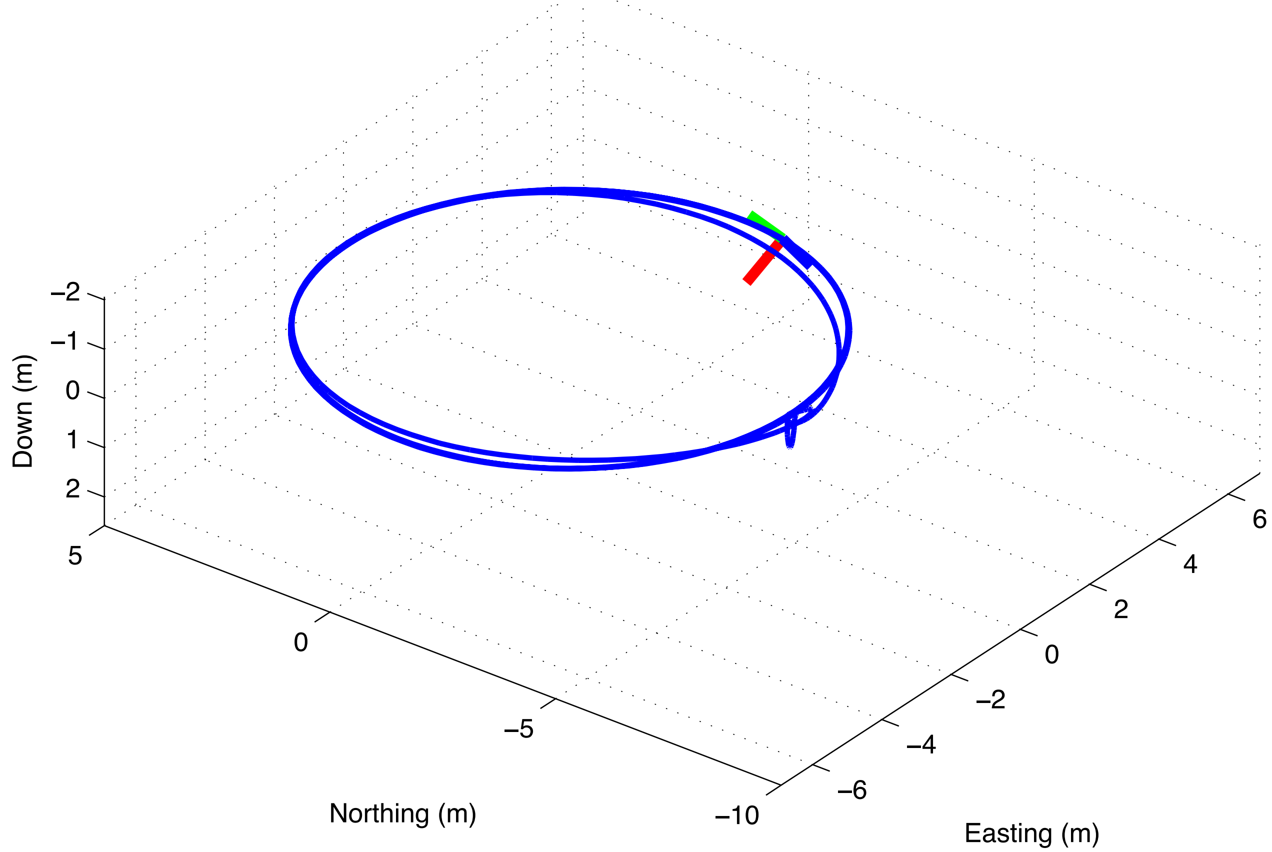
$$V_t = Q_t + K_t^T R_t K_t + (A_t + B_t K_t)^T V_{t+1} (A_t + B_t K_t)$$

Trajectory tracking for stationary rolls?

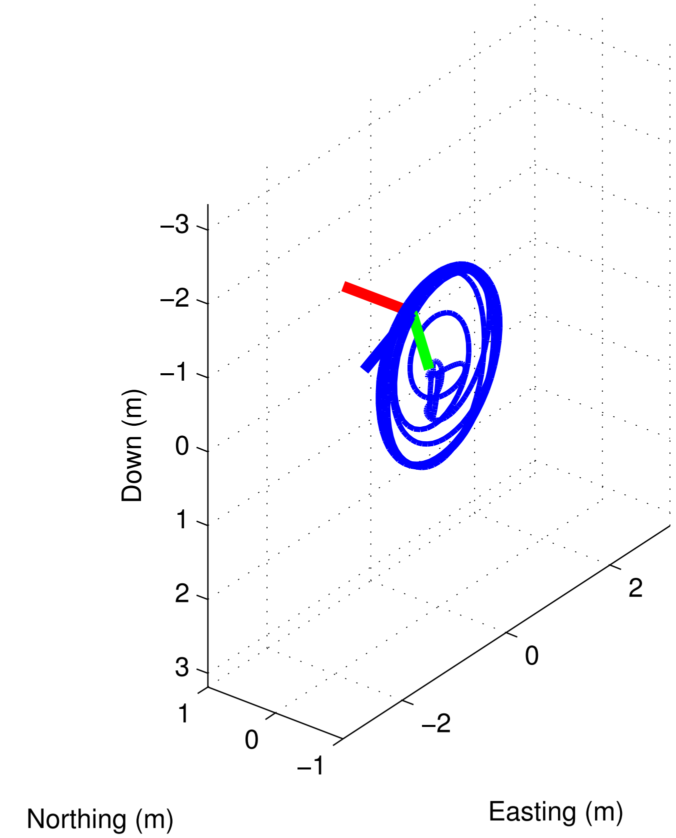


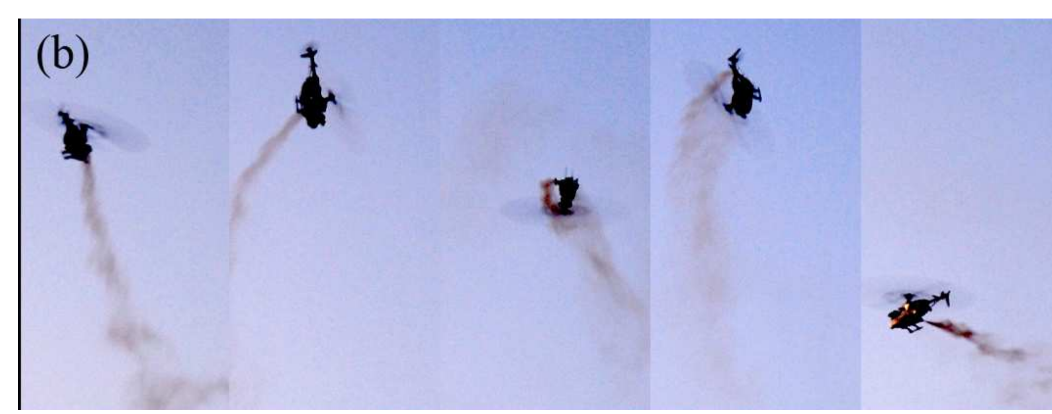
How do we get such behaviors?

Nose-in funnel



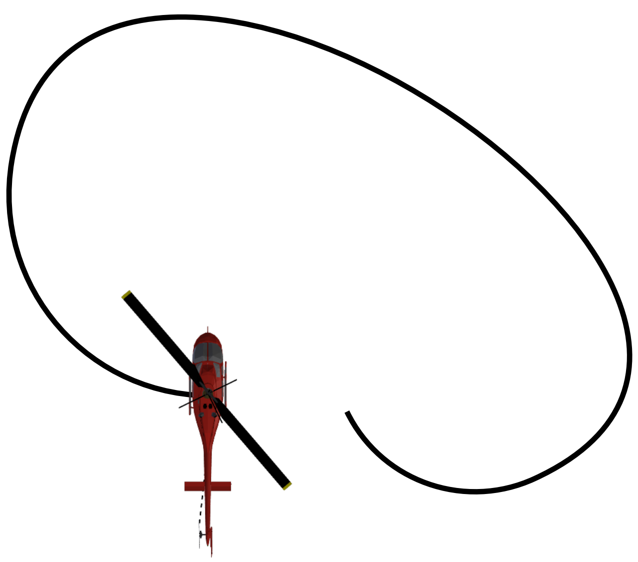
Stationary rolls



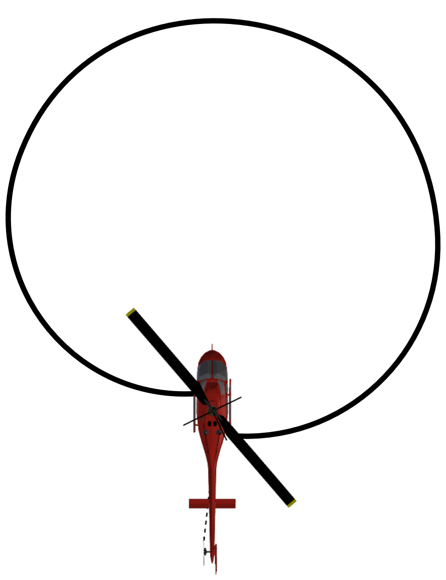


Iterative LQR (iLQR)

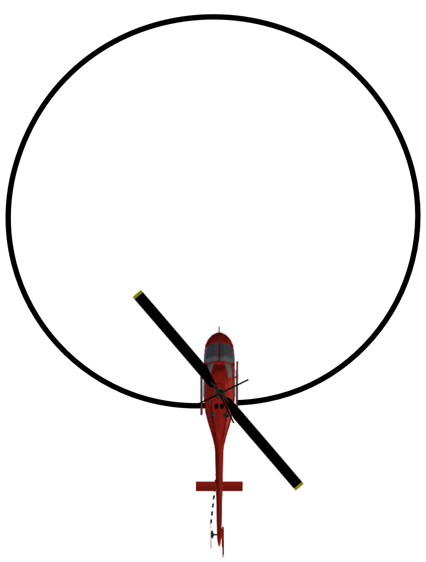
Start by guessing a control sequence, Forward simulate dynamics, Linearize about trajectory, Solve for new control sequence and repeat!



i=0



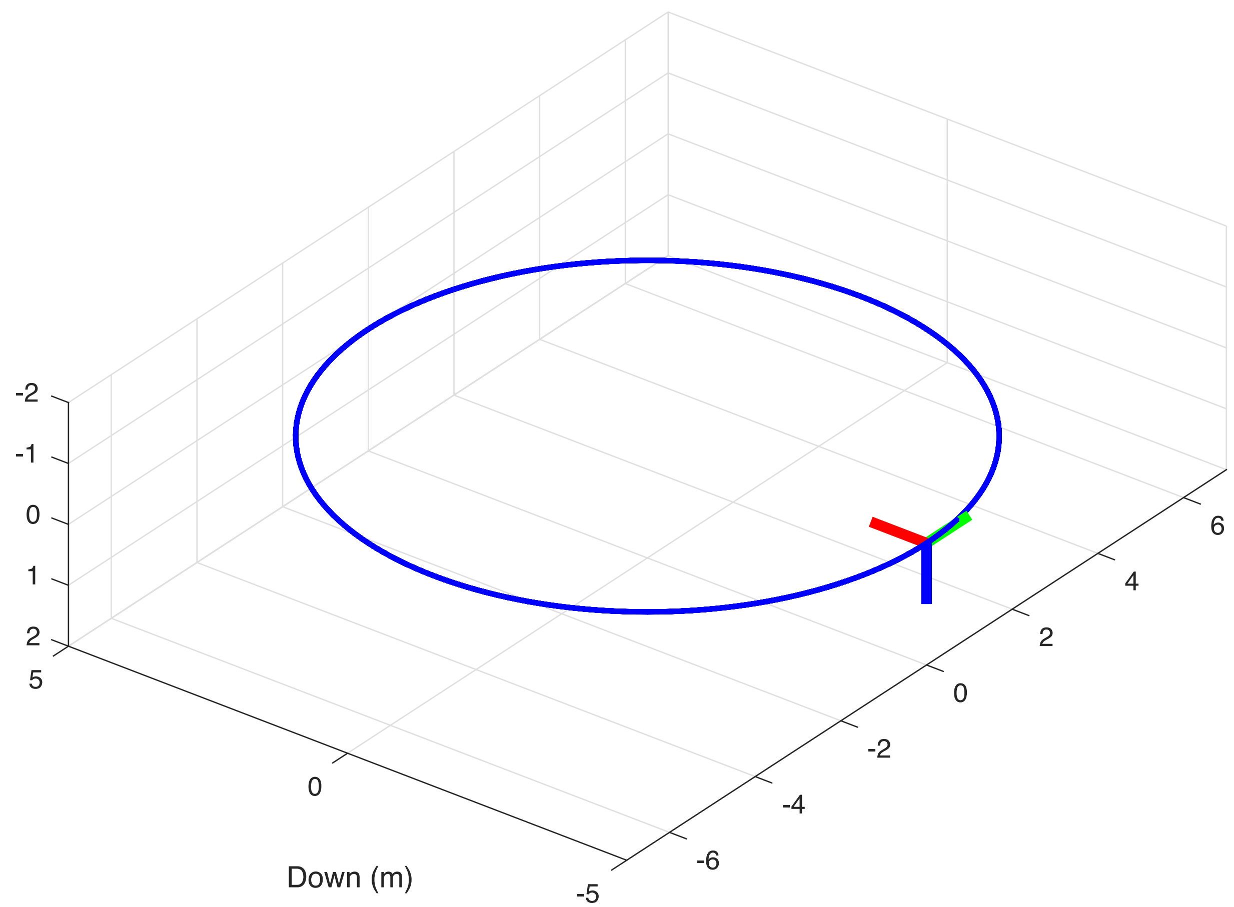
i=10



i=100

Step 1: Get a reference trajectory

$$x_0^{ref}, u_0^{ref}, x_1^{ref}, u_1^{ref}, \ldots, x_{T-1}^{ref}, u_{T-1}^{ref}$$



Note: Simply executing open loop trajectory won't work!

Step 2: Initialize your algorithm

Choose initial trajectory at iteration 0 to linearize about

$$x^0(t), u^0(t) = \{x_0^0, u_0^0, x_1^0, u_1^0, \ldots, x_{T-1}^0, u_{T-1}^0\}$$

It's a good idea to choose the reference trajectory

Initialization is very important!
We will be perturbing this initial trajectory
to account for system dynamics!

Step 3: Linearize your dynamics!

At a given iteration i, we are going to linearize about

$$x_0^i, u_0^i, x_1^i, \dots$$

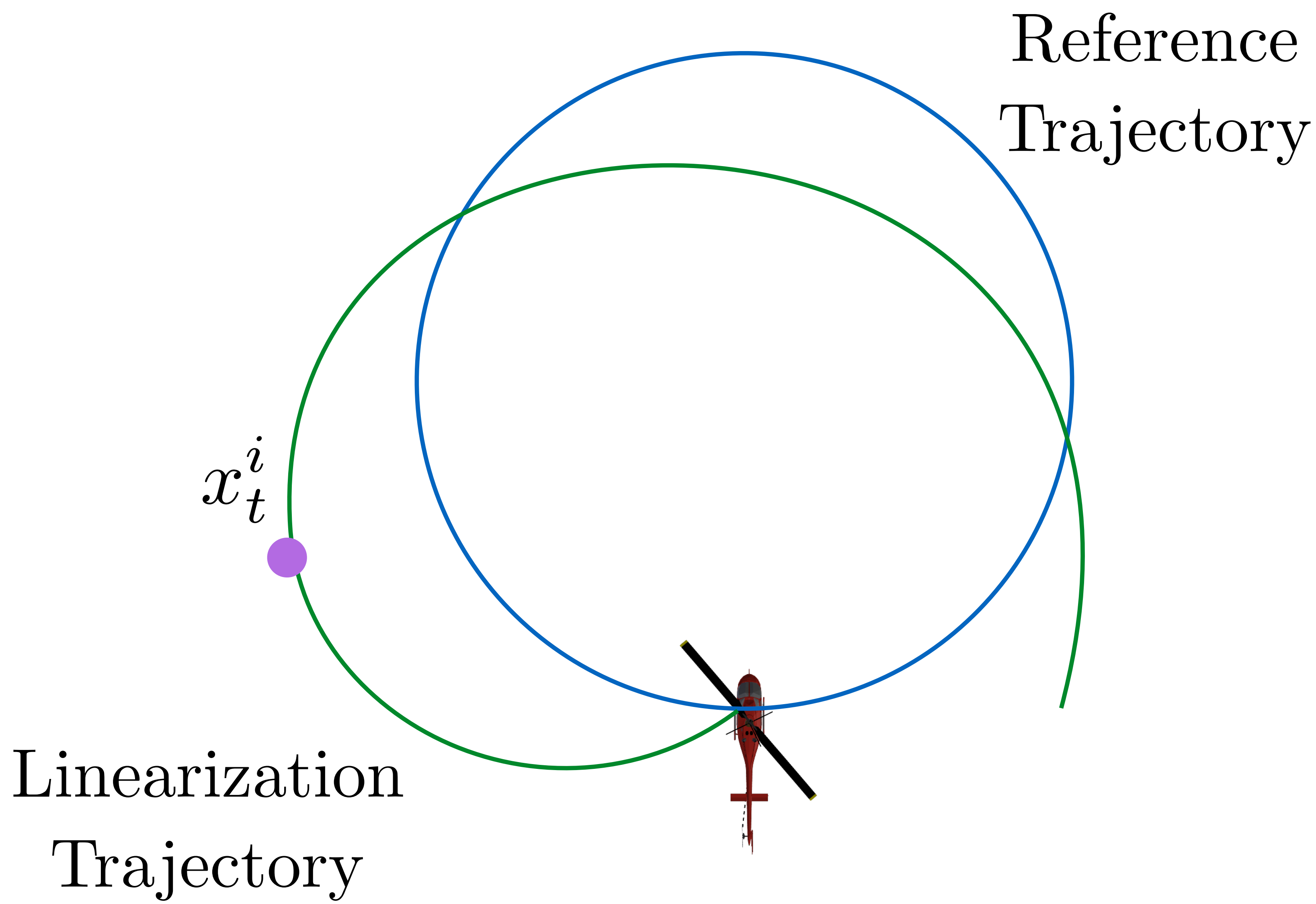
Change of variable - we will track the delta perturbations

$$\delta x_t = x_t - x_t^i$$

$$\delta u_t = u_t - u_t^i$$

Perturbations between current point and i-th trajectory

Step 3: Linearize your dynamics!



Step 3: Linearize your dynamics!

$$\delta x_t = x_t - x_t^i \qquad \delta u_t = u_t - u_t^i$$

Next point (i-th trajectory)

Next point (dynamics)

$$\delta x_{t+1} = A_t \delta x_t + B_t \delta u_t + (f(x_t^i, u_t^i) - x_{t+1}^i)$$

Offset

$$A_t = \left.\frac{\partial f}{\partial x}\right|_{x_t^i} \qquad B_t = \left.\frac{\partial f}{\partial u}\right|_{u_t^i}$$

Step 3: Linearize your dynamics!

Homogeneous coordinate system $\begin{bmatrix} \delta x_t \\ 1 \end{bmatrix}$

Dynamics is now linear!

$$\begin{bmatrix} \delta x_{t+1} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} A_t & f(x_t^i, u_t^i) - x_{t+1}^i \\ 0 & 1 \end{bmatrix}}_{\tilde{A}_t} \begin{bmatrix} \delta x_t \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} B_t \\ 0 \end{bmatrix}}_{\tilde{B}_t} \delta u_t$$

Step 4: Quadricize cost about trajectory

Our cost function is already quadratic

$$c(x_t, u_t) = (x_t - x_t^{ref})^T Q (x_t - x_t^{ref}) + (u_t - u_t^{ref})^T R (u_t - u_t^{ref})$$

$$= \begin{bmatrix} \delta x_t \\ 1 \end{bmatrix}^T \begin{bmatrix} Q & Q(x_t^i - x_t^{ref}) \\ (x_t^i - x_t^{ref})^T Q & (x_t^i - x_t^{ref})^T (x_t^i - x_t^{ref}) \end{bmatrix} \begin{bmatrix} \delta x_t \\ 1 \end{bmatrix}$$

$$\tilde{Q}_t$$

$$+$$

$$\begin{bmatrix} \delta u_t \\ 1 \end{bmatrix}^T \begin{bmatrix} R & R(u_t^i - u_t^{ref}) \\ (u_t^i - u_t^{ref})^T R & (u_t^i - u_t^{ref})^T (u_t^i - u_t^{ref}) \end{bmatrix} \begin{bmatrix} \delta u_t \\ 1 \end{bmatrix}$$

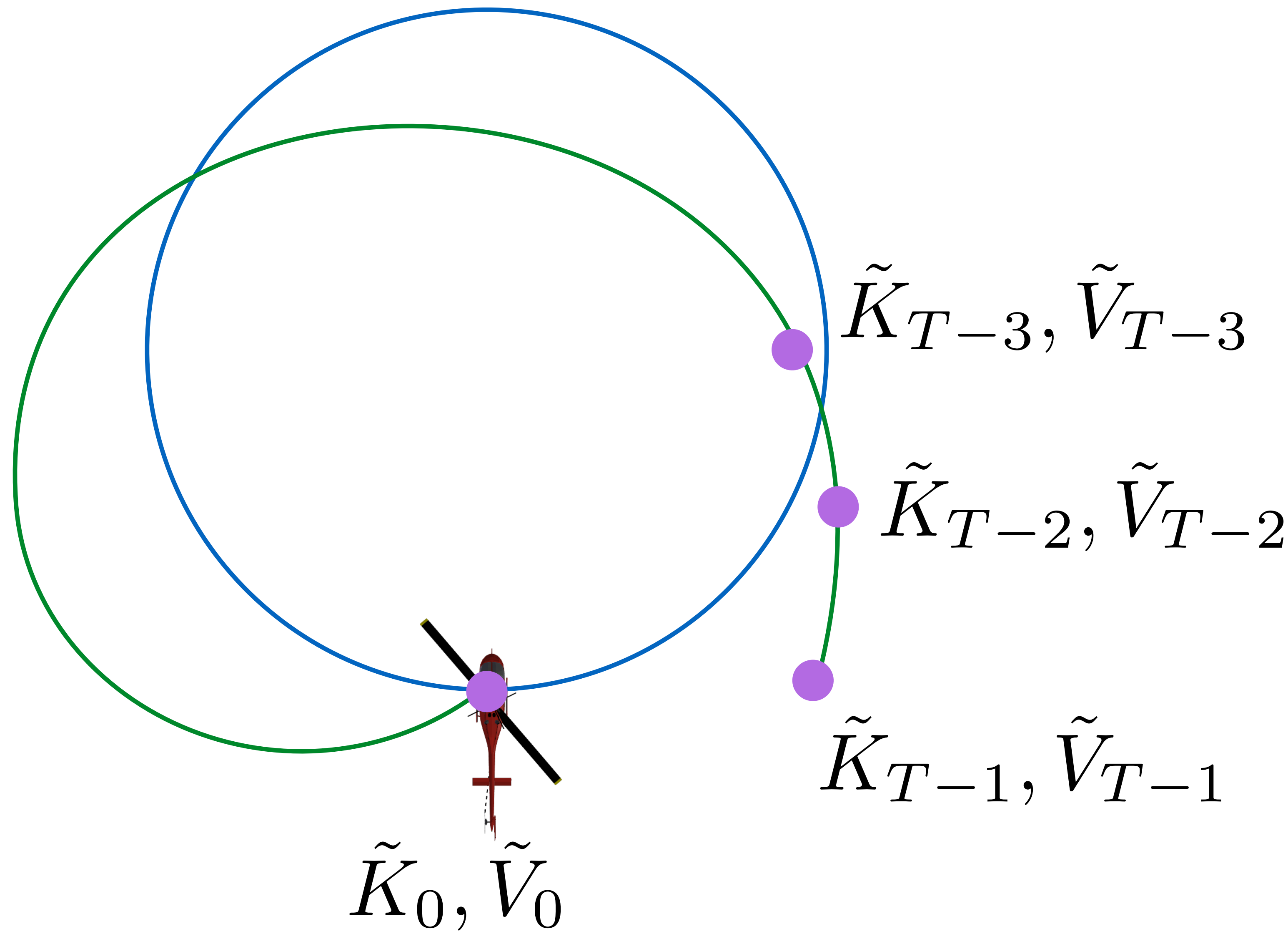
$$\tilde{R}_t$$

We have all the ingredients to call LQR!

$$\tilde{K}_t = -(\tilde{R}_t + \tilde{B}_t^T \tilde{V}_{t+1} \tilde{B}_t)^{-1} \tilde{B}_t^T \tilde{V}_{t+1} \tilde{A}_t$$

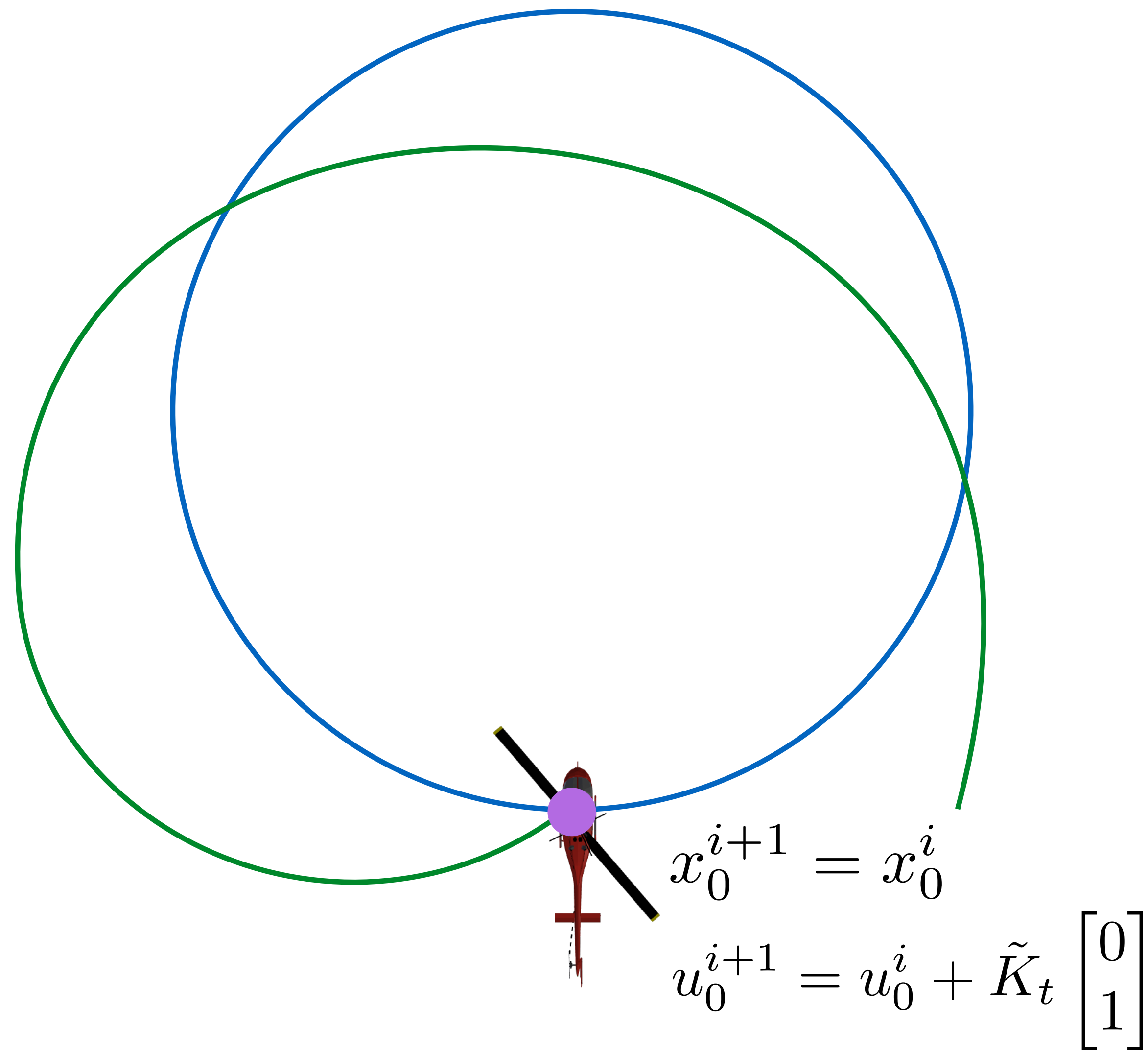
similarly calculate the value function ...

Step 5: Do a backward pass



Calculate controller gains for all time steps

Step 6: Do a forward pass to get new trajectory



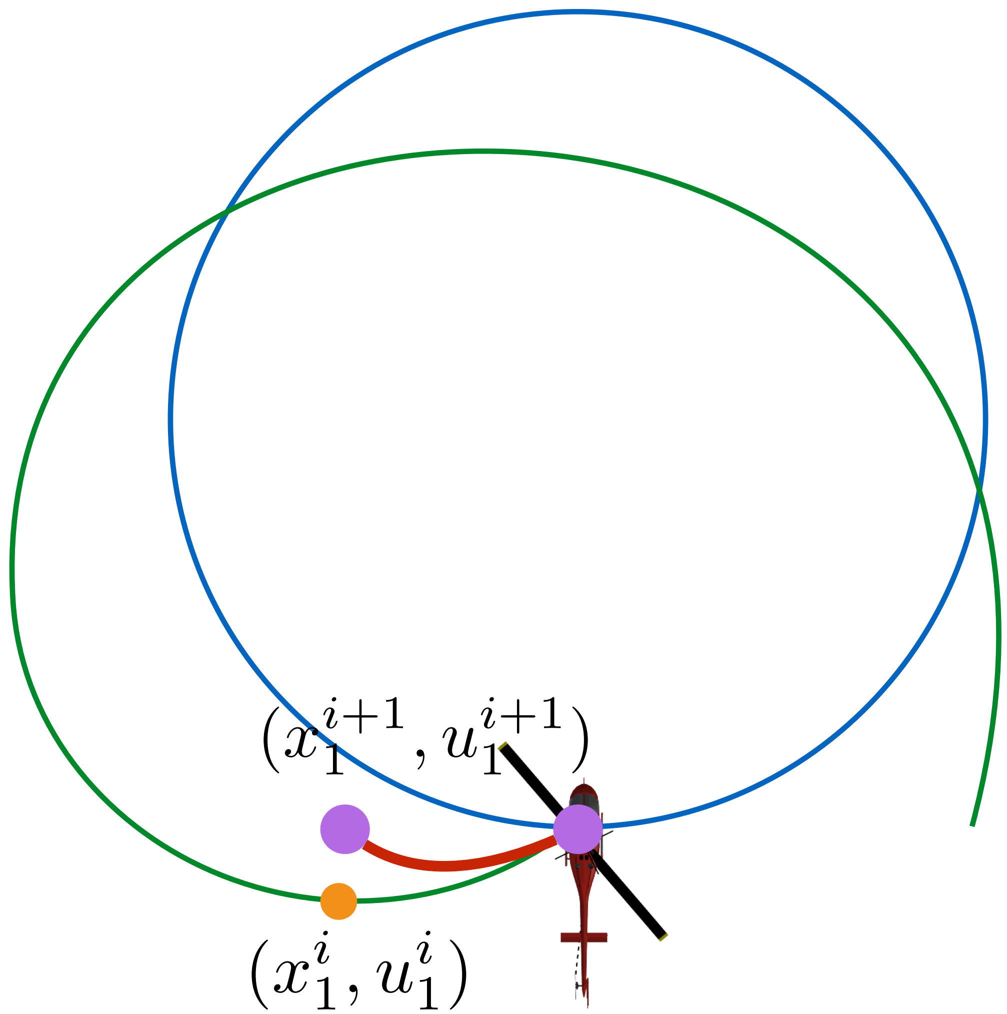
Compute
control action

$$u_t^{i+1} = u_t^i + \tilde{K}_t \begin{bmatrix} x_t^{i+1} - x_t^i \\ 1 \end{bmatrix}$$

Apply dynamics

$$x_t^{i+1} = f(x_t^{i+1}, u_t^{i+1})$$

Step 6: Do a forward pass to get new trajectory



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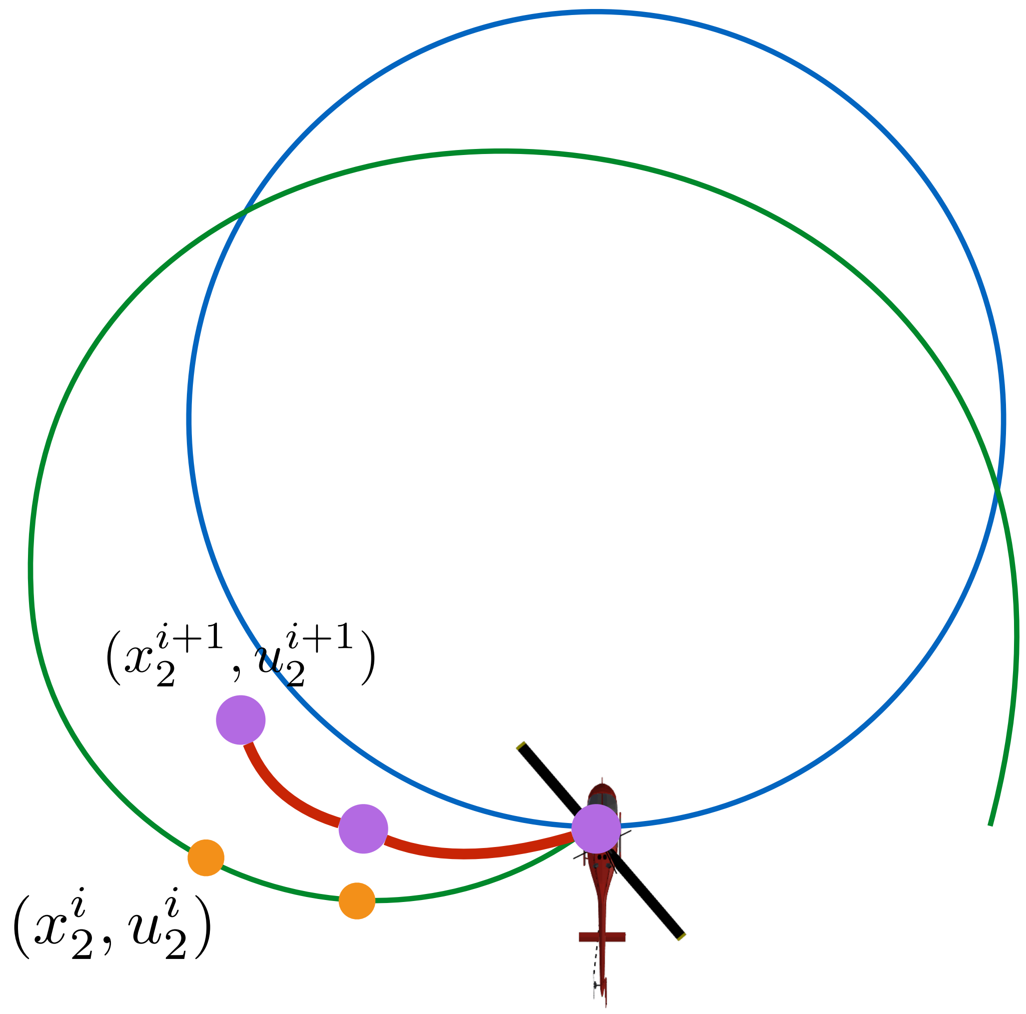

Compute
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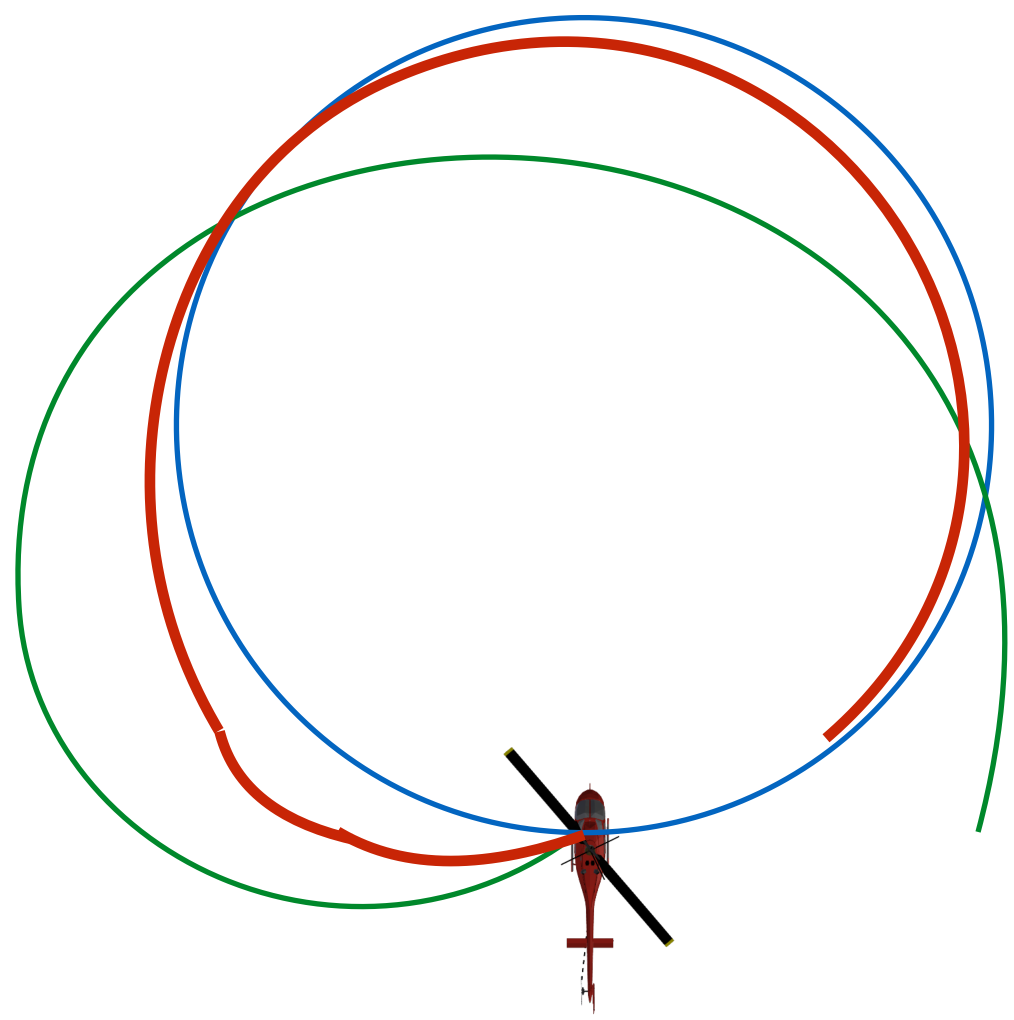
Step 6: Do a forward pass to get new trajectory



Compute
control action

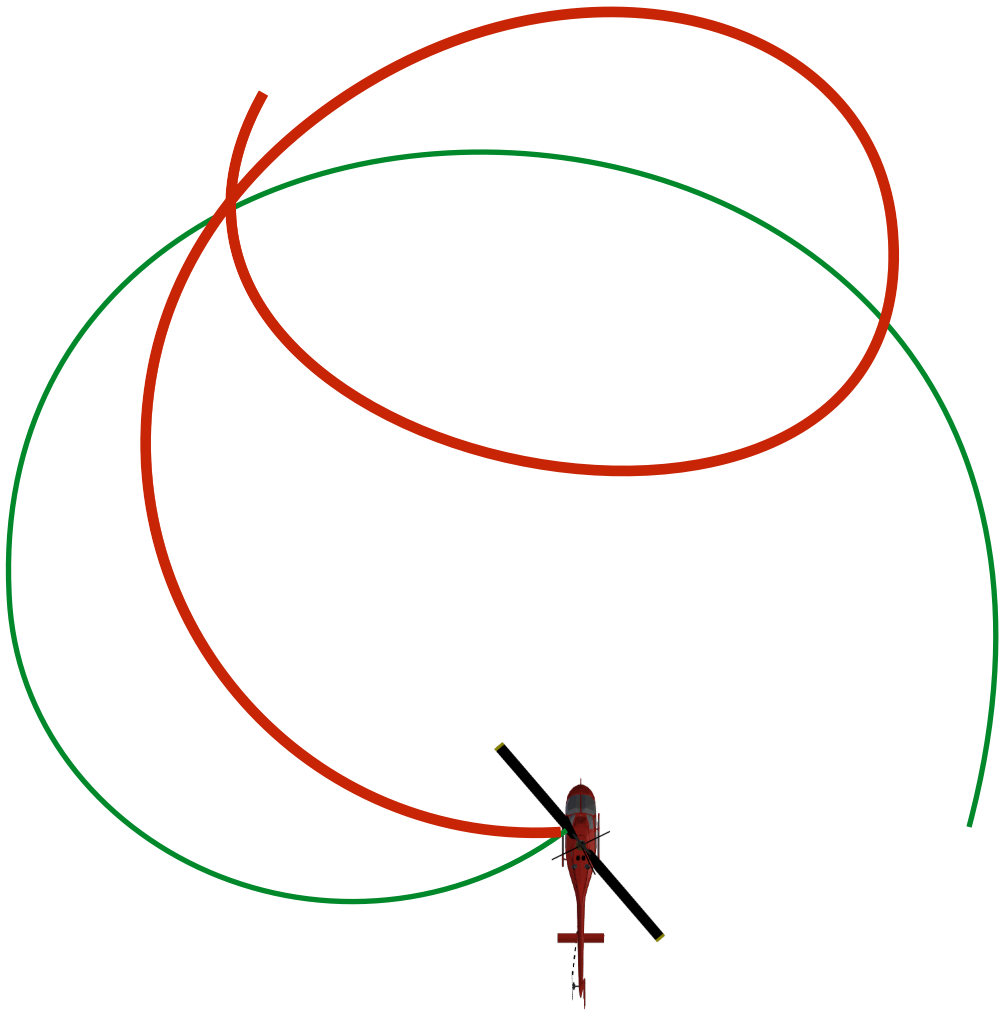
$$u_t^{i+1} = u_t^i + \tilde{K}_t \begin{bmatrix} x_t^{i+1} - x_t^i \\ 1 \end{bmatrix}$$

Apply dynamics

$$x_t^{i+1} = f(x_t^{i+1}, u_t^{i+1})$$

Problem: Forward pass will go bonkers

Why?



Linearization error gets bigger and bigger and bigger

Remedies: Change cost function to penalize deviation from linearization

Questions

1. Can we solve LQR for continuous time dynamics?

   Yes! Refer to Continuous Algebraic Ricatti Equations (CARE)

2. Can LQR handle arbitrary costs (not just tracking)?

   Yes! Just quadricize the cost

3. What if I want to penalize control derivatives?

   No problem! Add control as part of state space

4. Can we handle noisy dynamics?

   Yes! Gaussian noise does not change the answer

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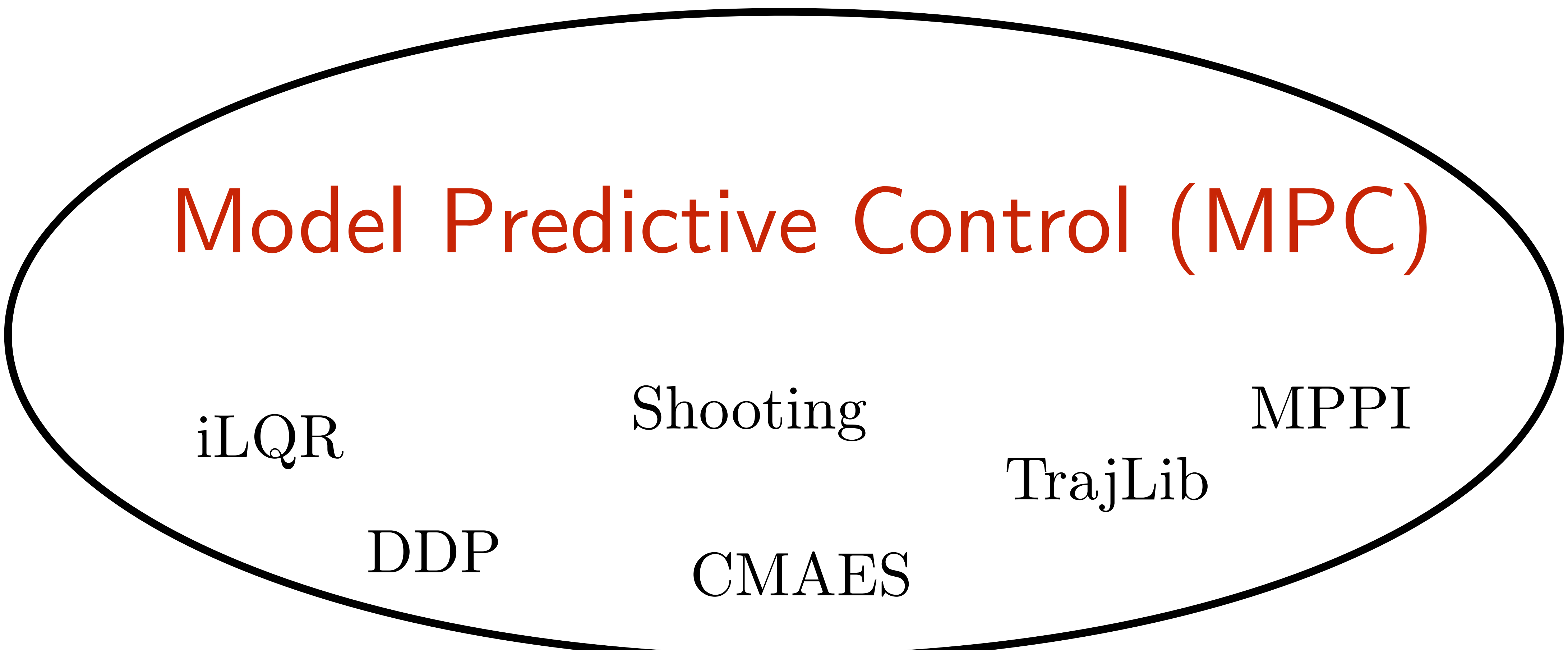
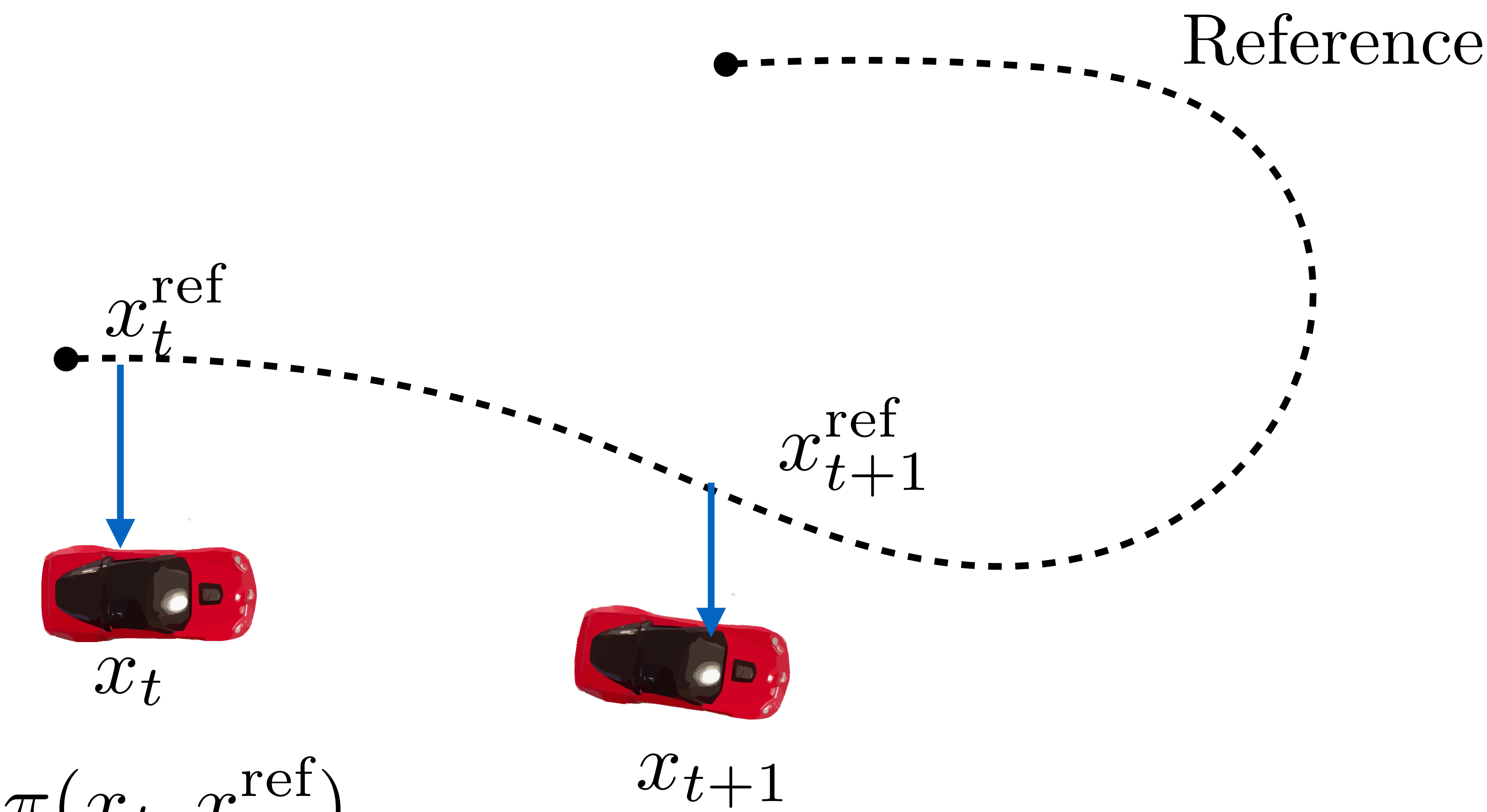
LQR	Linear	Yes	Quadratic
iLQR	Non-linear	Yes	Yes

iLQR is just one technique

It's far from perfect - can't deal with model errors / constraints ...



Recap: Feedback control framework



$$u_t = \pi(x_t, x_t^{\text{ref}})$$

$$u_{t+1} = \pi(x_{t+1}, x_{t+1}^{\text{ref}})$$

Look at current state error and compute control actions

Goal: To drive error to 0 … to optimally drive it to 0

Limitations of this framework

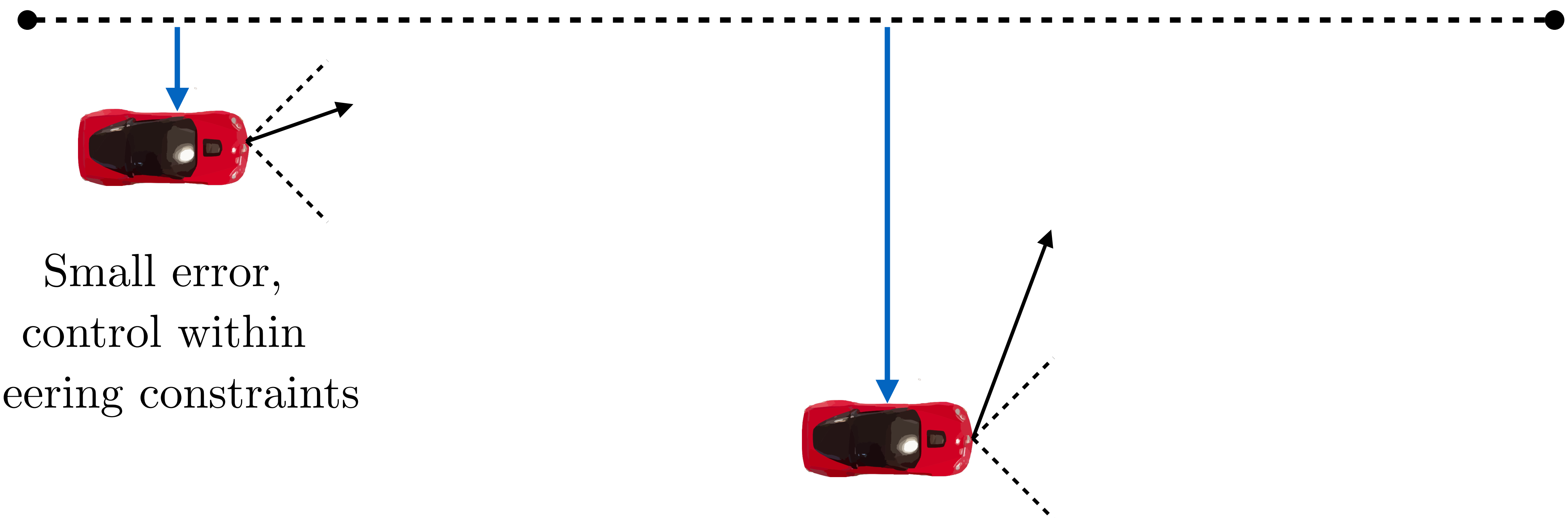
A fixed control law that looks at instantaneous feedback

$$u_t = \underbrace{\pi}_{\text{Fixed}}(x_t, \underbrace{x_t^{\text{ref}}}_{\text{Reference}})$$

Why is it so difficult to create a magic control law?

Problem 1: What if we have constraints?

Simple scenario: Car tracking a straight line



Small error,
control within
steering constraints

Large error,
control violates
steering constraints

We could "clamp control command" …
but what are the implications?

Constraints come as a surprise… bad system response…

General problem: Complex models

Dynamics $$x_{t+1} = f(x_t, u_t)$$

Constraints $$g(x_t, u_t) \leq 0$$

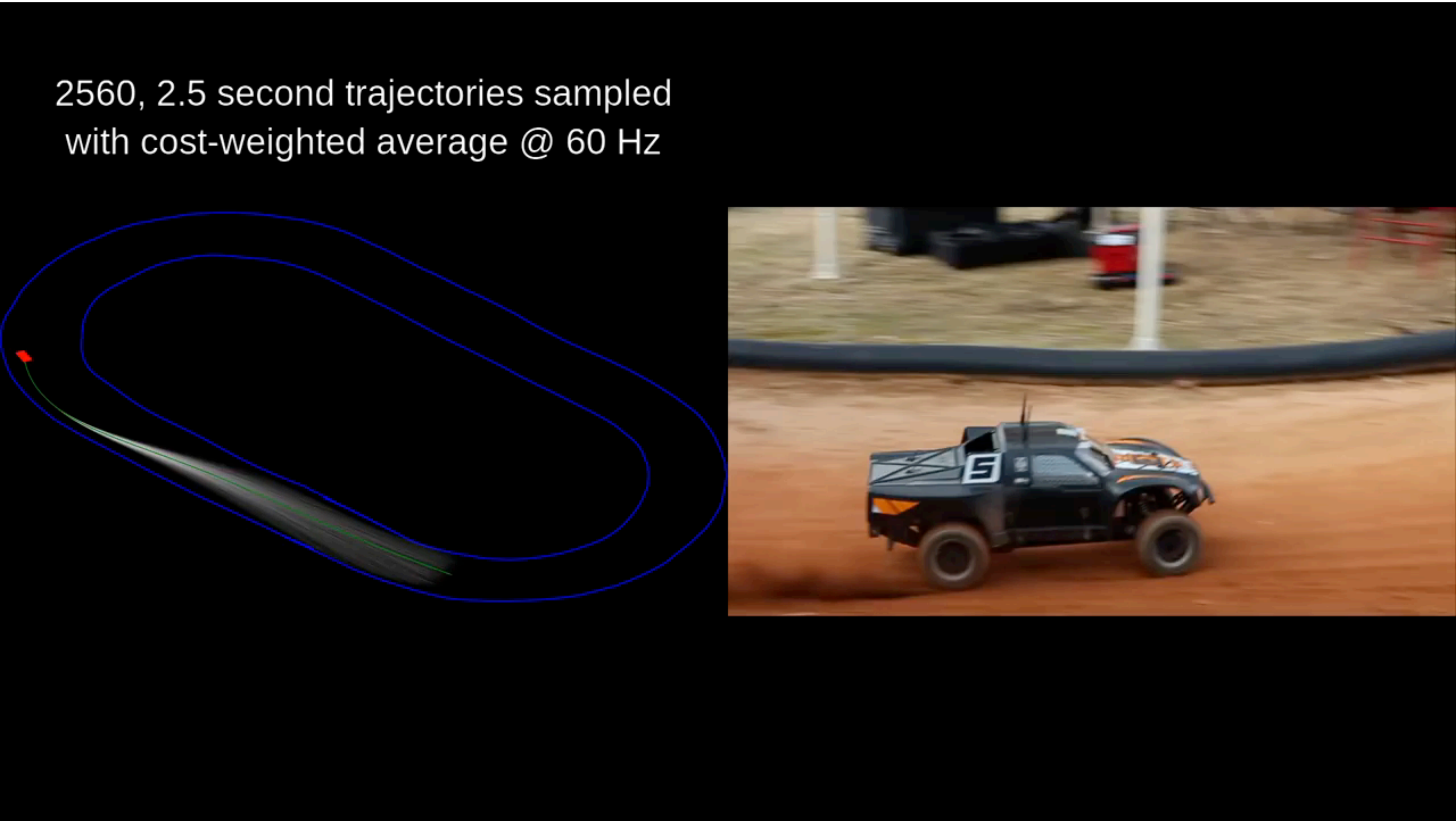
Such complex models imply we need to:

1. Predict the implications of control actions

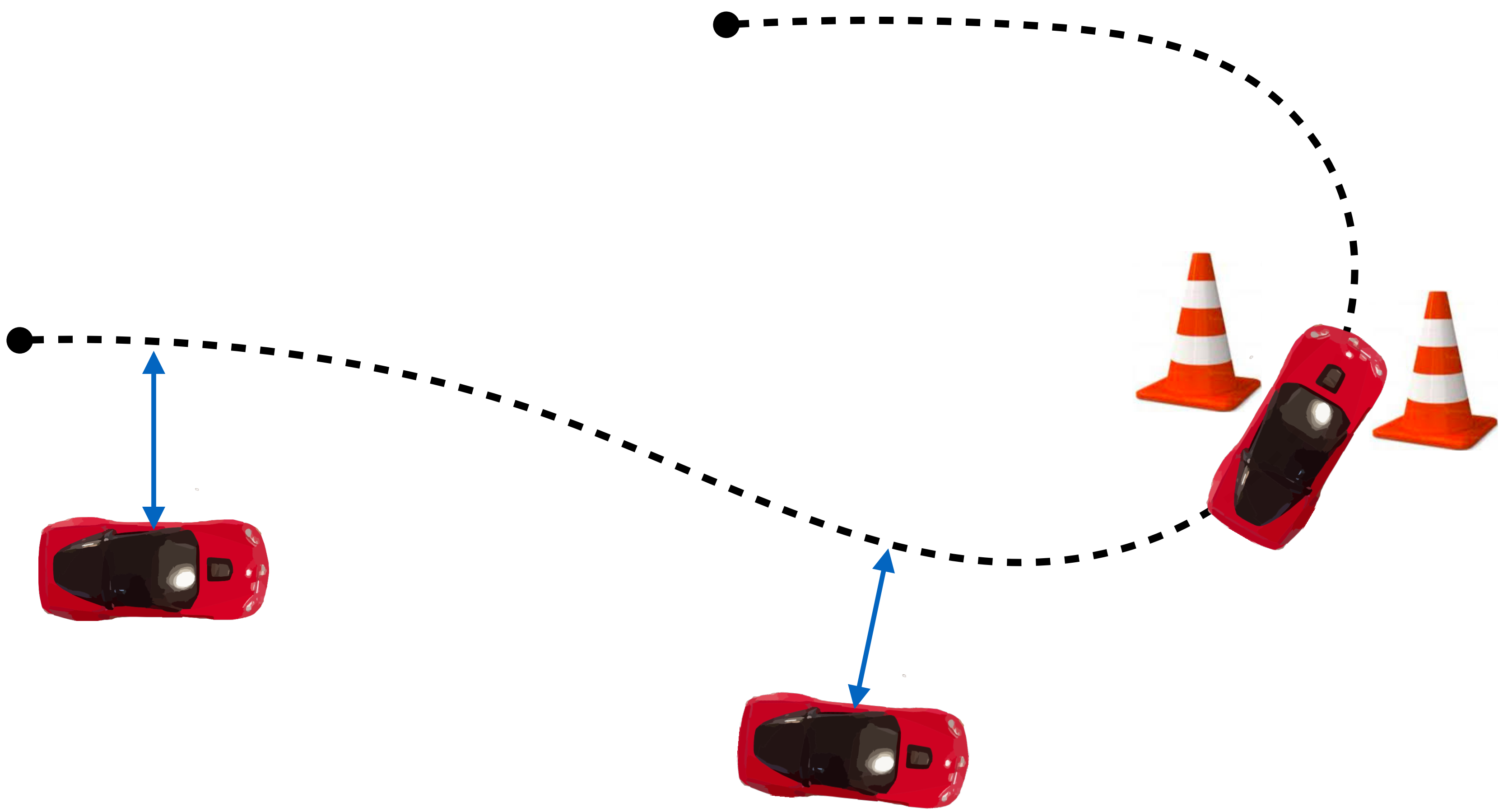
2. Do corrections NOW that would affect the future

3. It may not be possible to find one law - might need to predict

Example: Rough terrain mobility



Problem 2: What if some errors are worse than others?



We need a cost function that penalizes states non-uniformly

Key Idea:

Frame control as an optimization problem

Model predictive control (MPC)

1. Plan a sequence of control actions

2. Predict the set of next states unto a horizon H

3. Evaluate the cost / constraint of the states and controls

4. Optimize the cost

Model predictive control (MPC)

1. At **each** time t, solve the (finite horizon) (planning) problem

$$(\tilde{u}_t, \ldots, \tilde{u}_{t+H-1}) = \min_{u_{t+1},\ldots u_{t+H}} \sum_{k=t}^{t+H-1} J(x_k, u_{k+1})$$

$$\text{s.t. } x_{k+1} = f(x_k, u_{k+1})$$

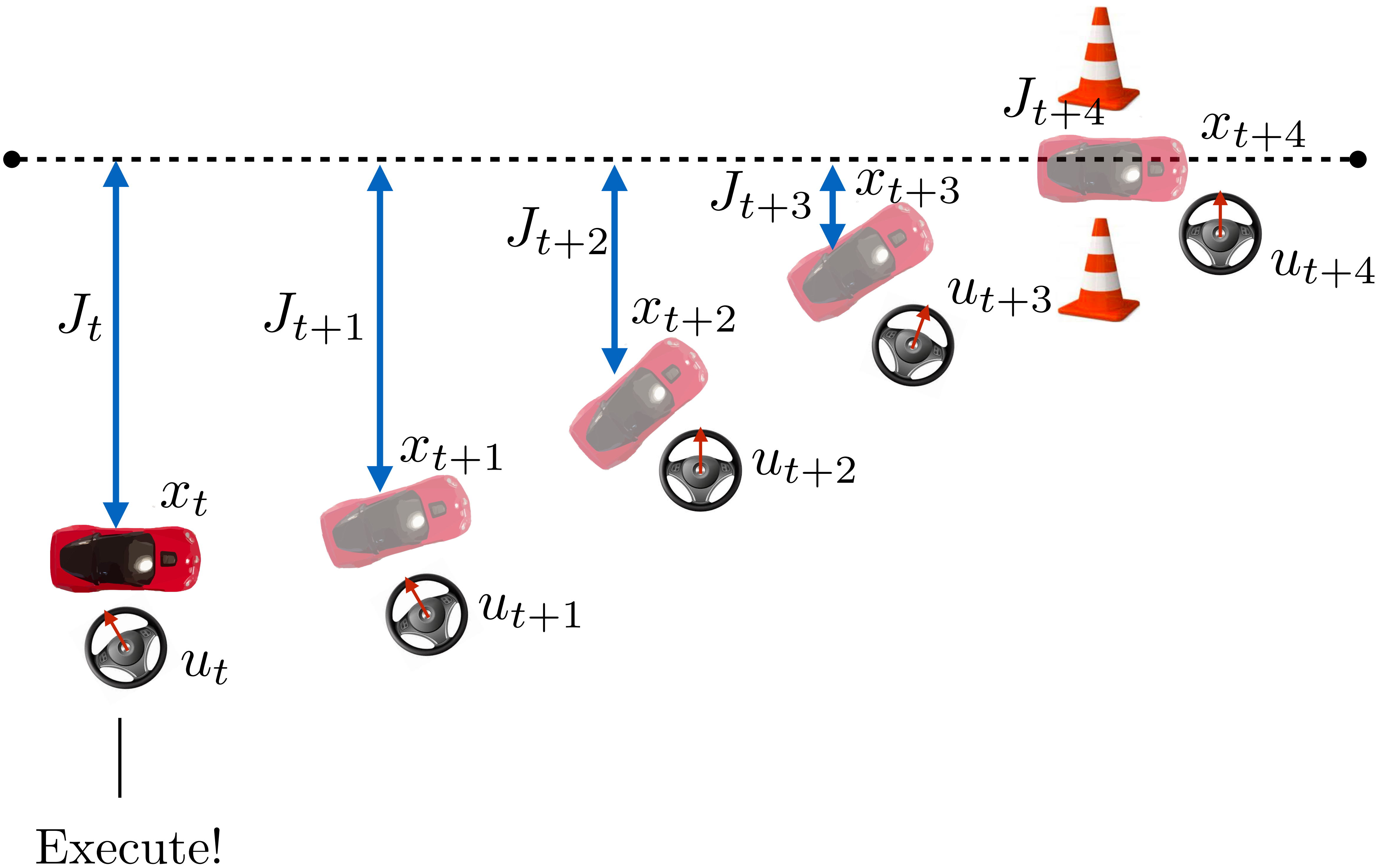
$$g(x_k, u_{k+1}) \leq 0$$

2. Execute $\tilde{u}_t$

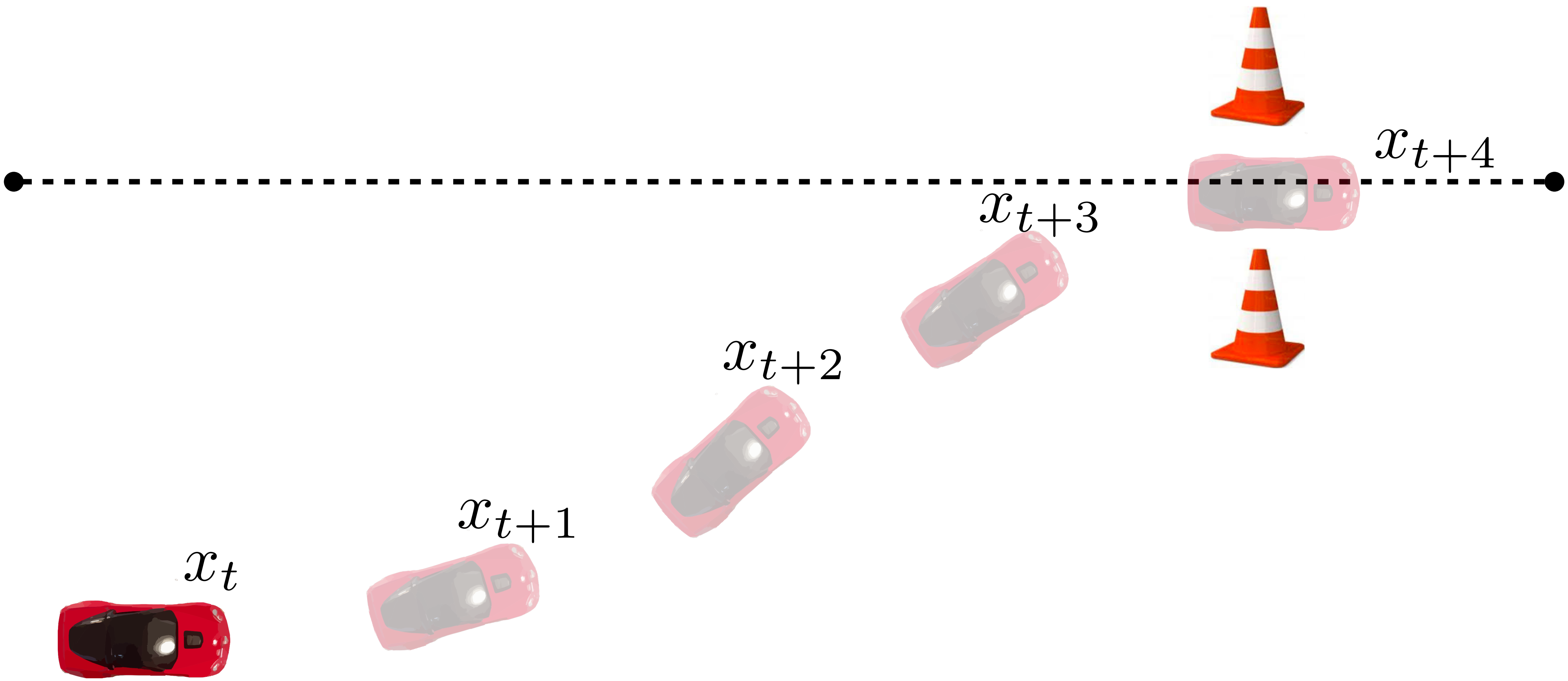
3. Go to step 1

*Slide adapted from Stephen Boyd

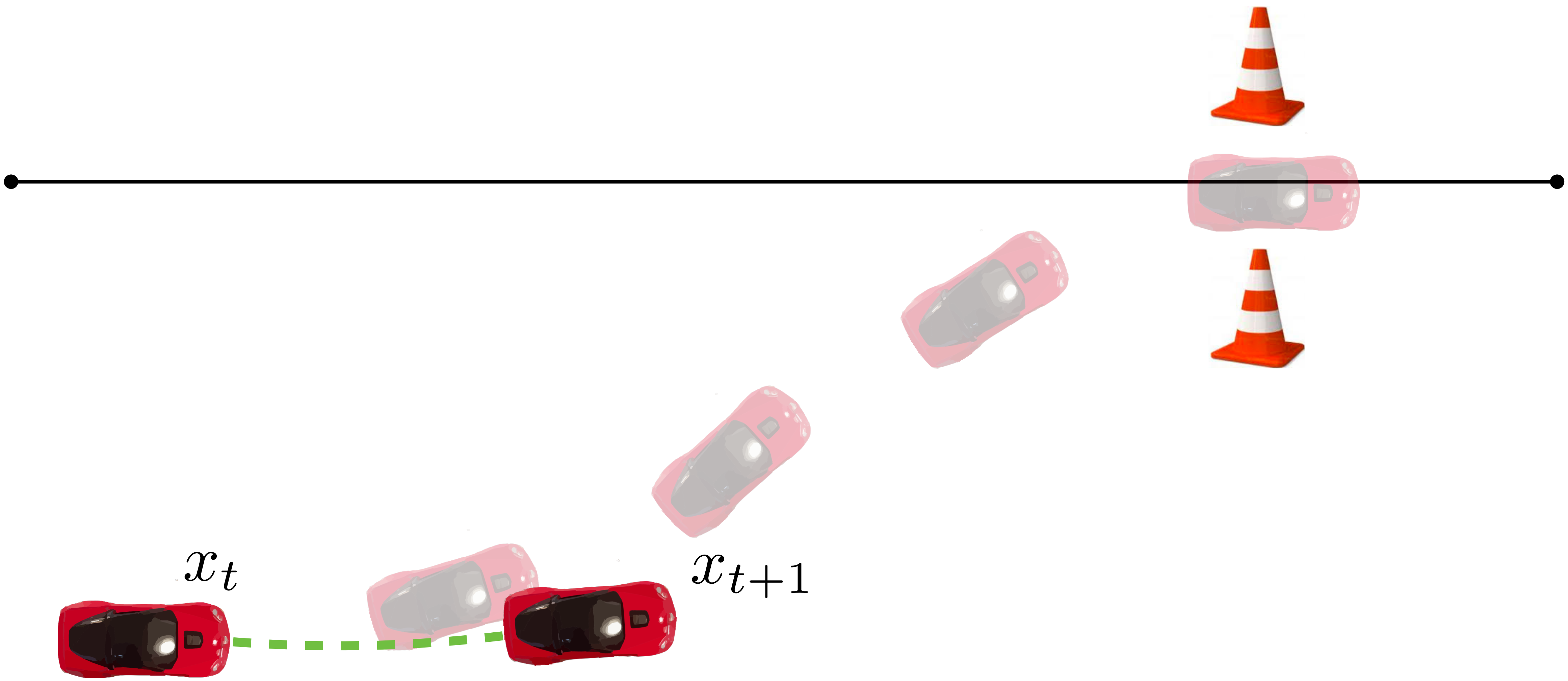
At each time step



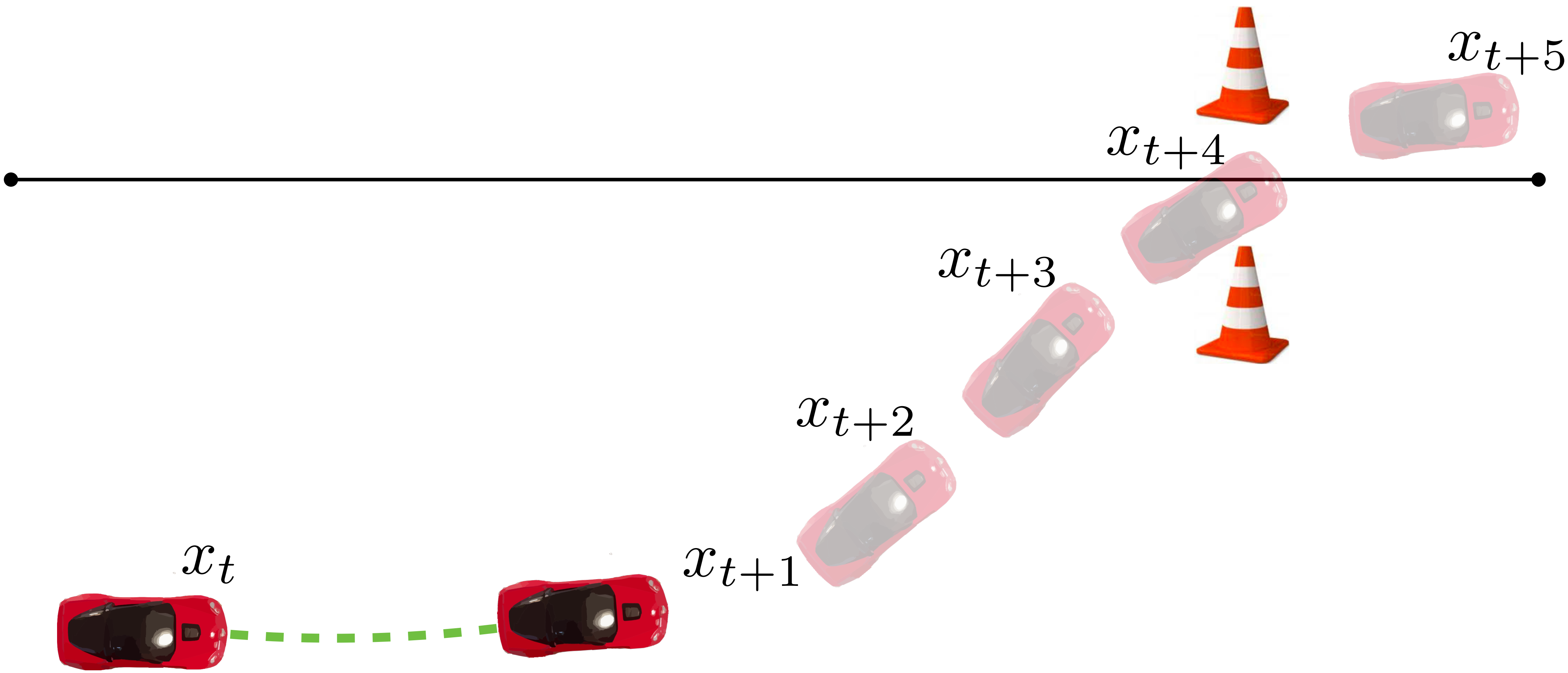
Step 1: Optimize to horizon



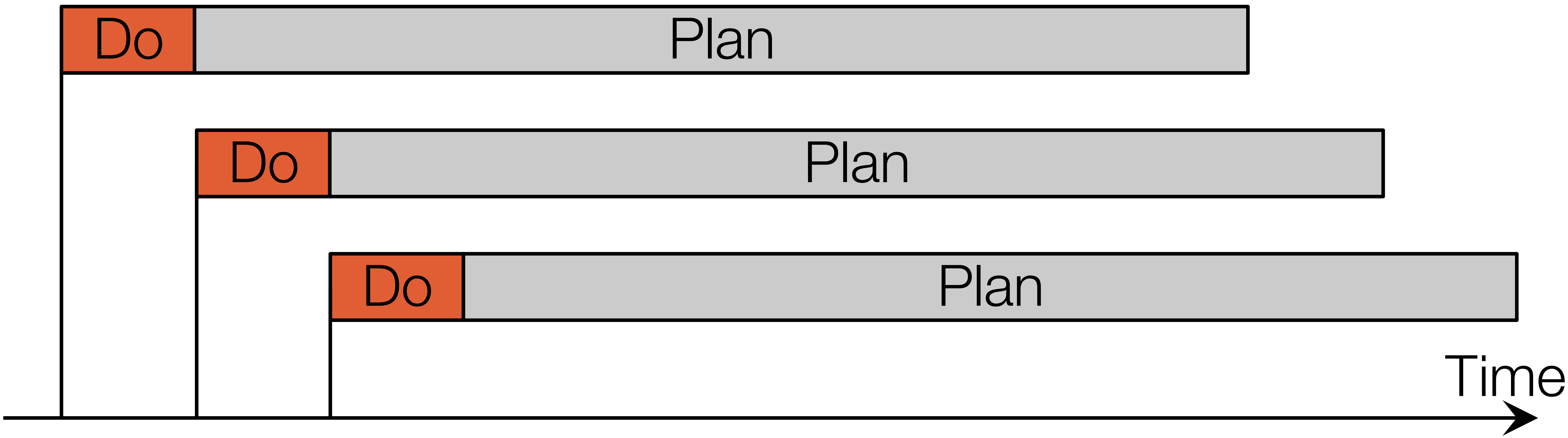
Step 2: Execute first control



Step 3: Repeat!



MPC is a framework

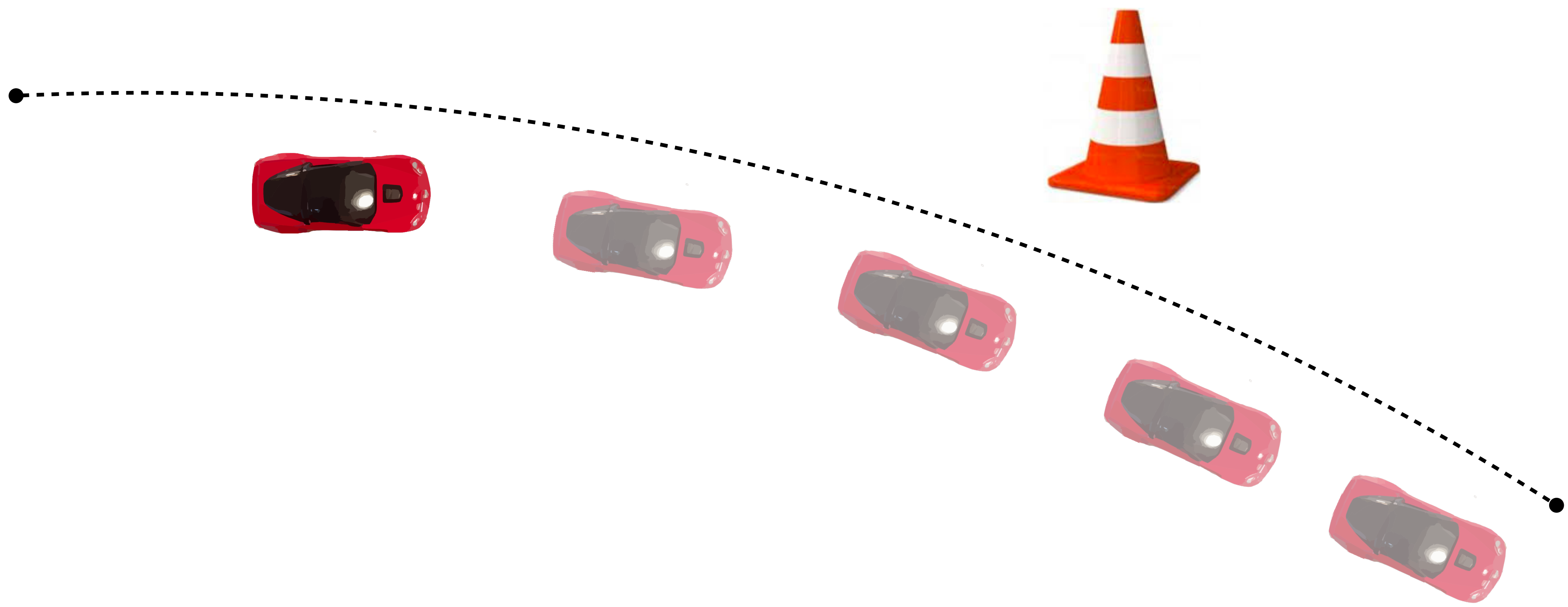


Step 1: Solve optimization problem to horizon

Step 2: Execute the first control

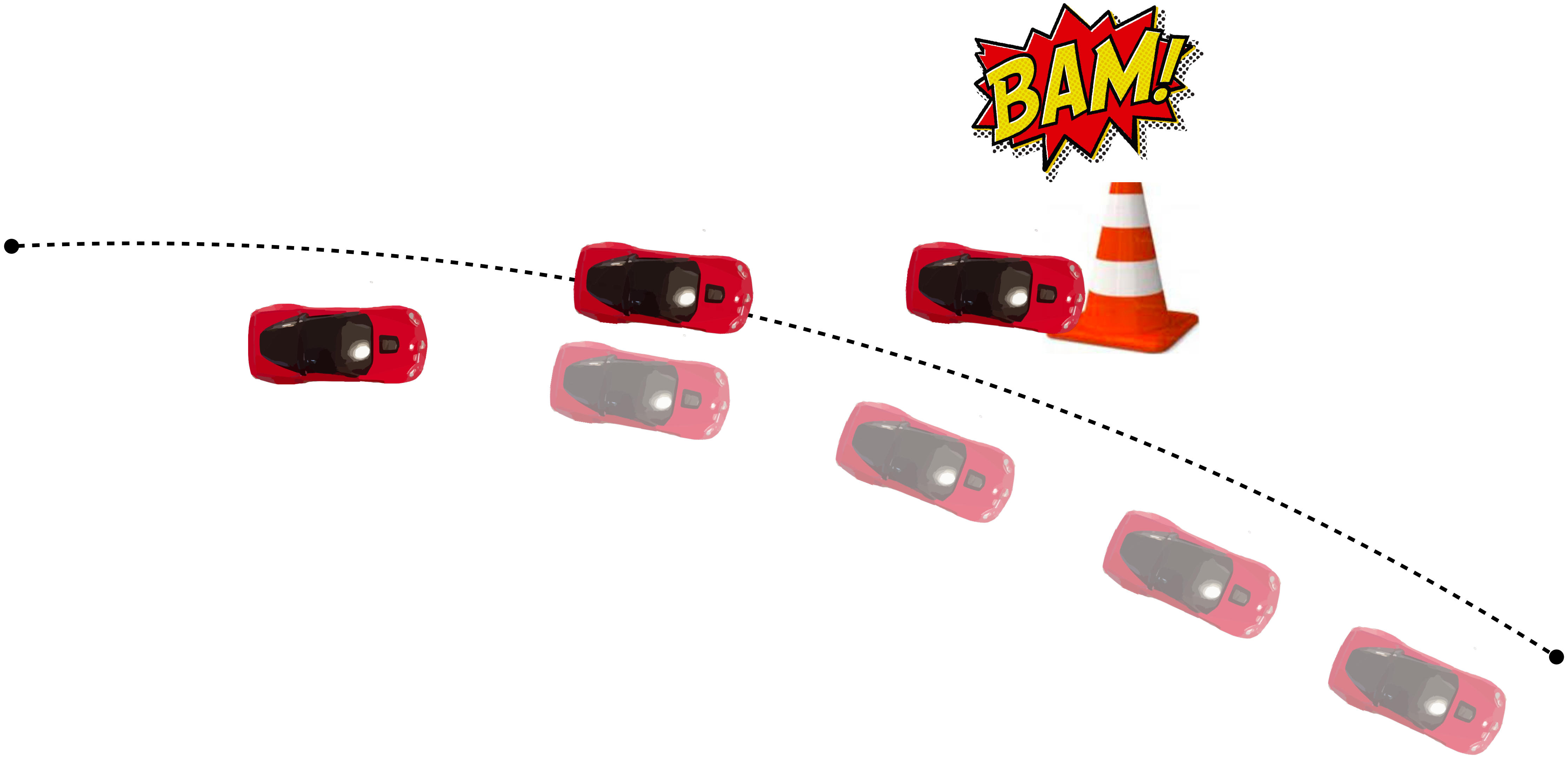
Step 3: Repeat!

Why do we need to replan?



What happens if the controls are planned once and executed?

Why do we need to replan?



What happens if the controls are planned once and executed?

Coming up next: Complex cost functions