Lyapunov Stability

Instructor: Chris Mavrogiannis

TAs: Kay Ke, Gilwoo Lee, Matt Schmittle

*Slides based on or adapted from Sanjiban Choudhury

Logistics

New Office Hours

Chris: Tuesdays at 1:00pm (CSE1 436)

Kay: Tuesdays at 4:00pm (CSE1 022)

Just for this week, Wednesday at 5:00pm

Gilwoo: Thursdays at 4:00pm (CSE1 022)

Schmittle: Fridays at 4:00pm (CSE1 022)

Recap: PID / Pure Pursuit control

	Pros	Cons
PID Control	Simple law that works pretty well!	Tuning parameters! Doesn't understand dynamics
Pure Pursuit	Cars can travel in arc!	No proof of convergence

Can we get some control law that has formal guarantees?

Table of Controllers

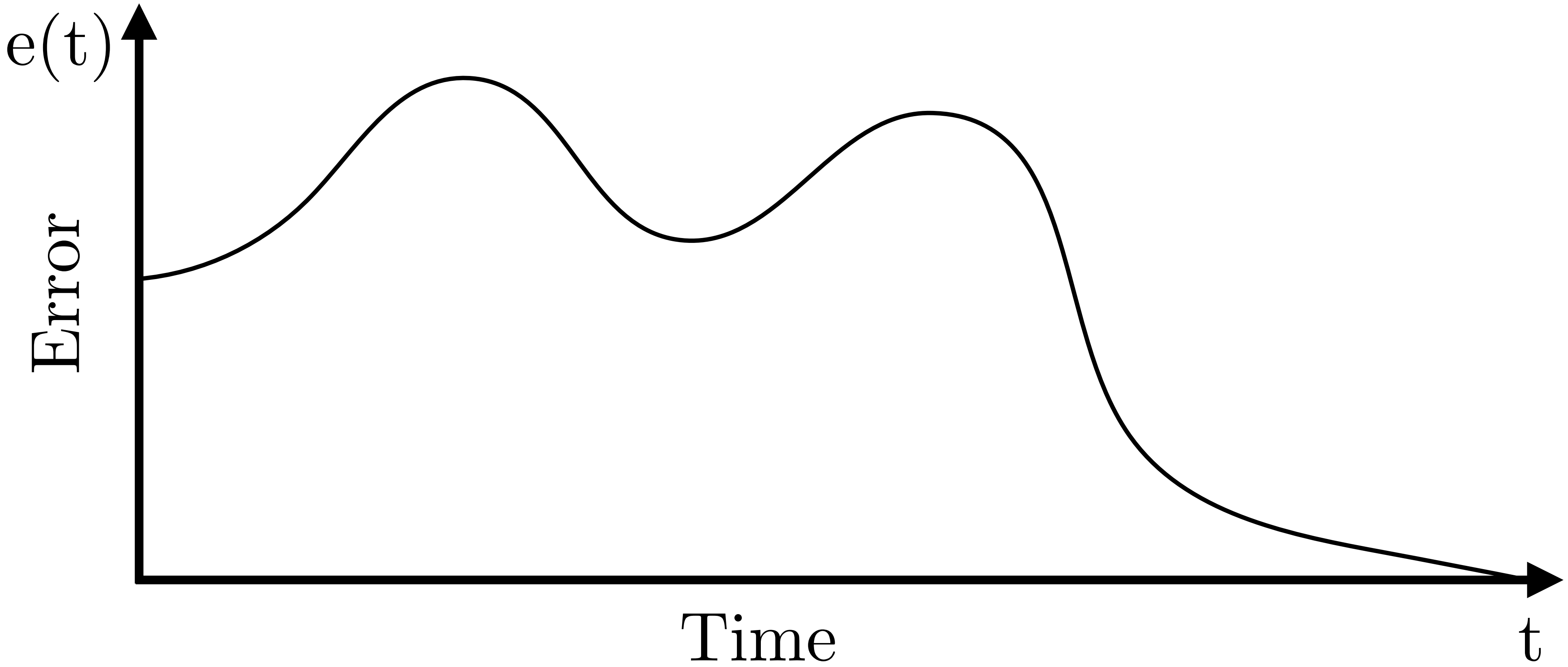
	Control Law	Uses model	Stability Guarantee
PID	$u = K_p e + \ldots$	No	No
Pure Pursuit	$u = \tan^{-1}\left(\frac{2B\sin\alpha}{L}\right)$	Circular arcs	Yes - with assumptions

Stability:

Prove error goes to zero
and stays there

What is stability?

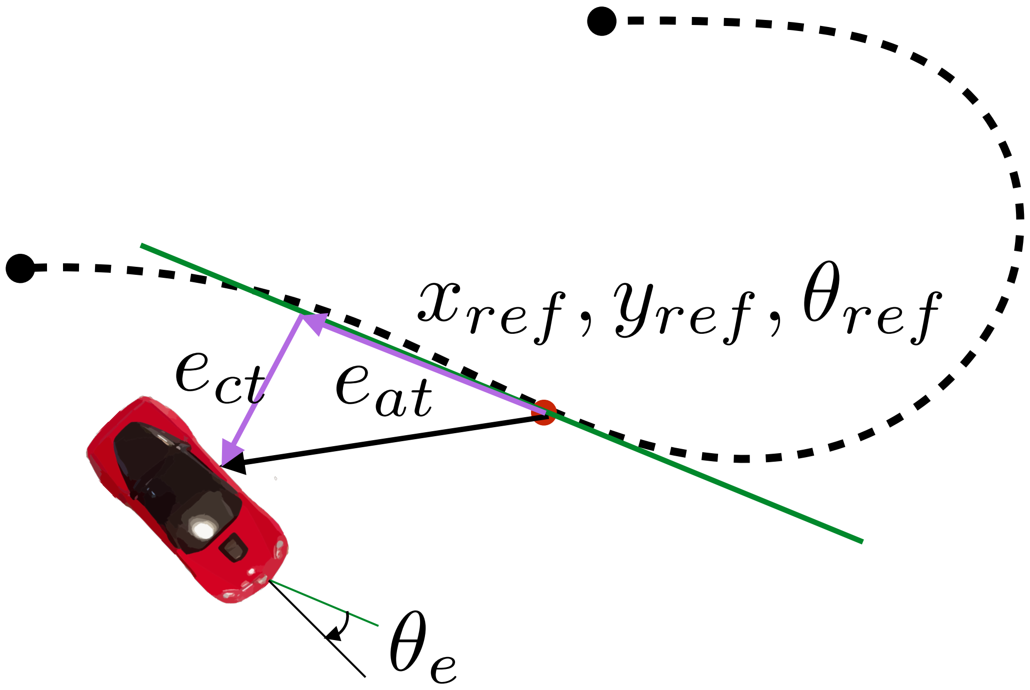
$$\lim_{t \to \infty} e(t) = 0$$



So we want both $e(t) \to 0$ and $\dot{e}(t) \to 0$

Question: Why does the error oscillate?

How does error evolve in time?



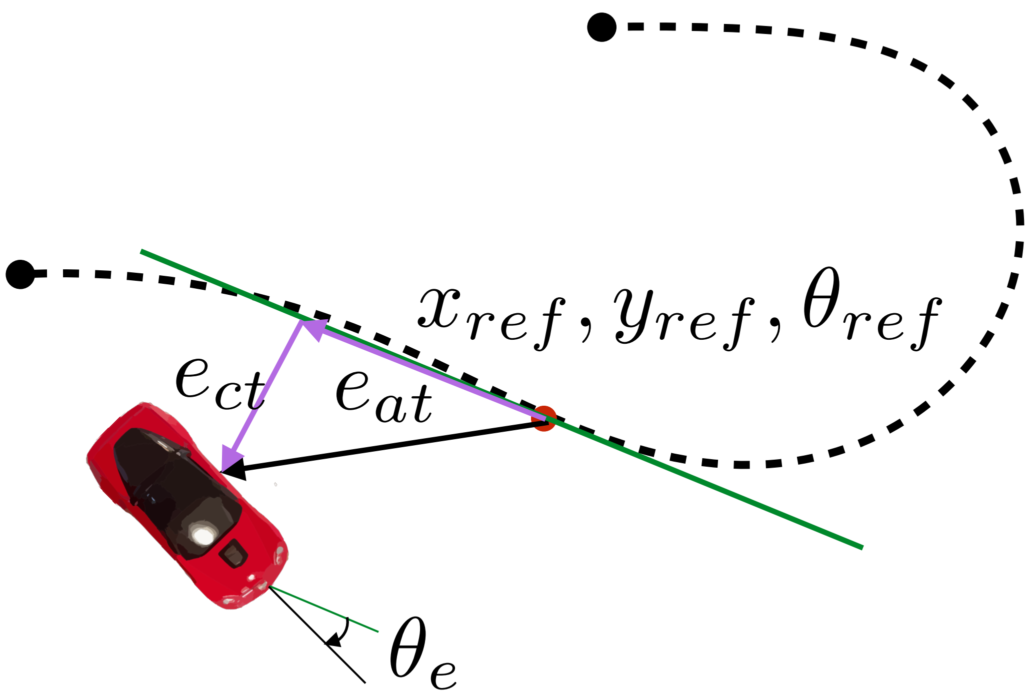
Let's say we were interested in driving both e_{ct} and θ_e to zero

$$e_{ct} = -\sin(\theta_{ref})(x - x_{ref}) + \cos(\theta_{ref})(y - y_{ref}) \qquad \theta_e = \theta - \theta_{ref}$$

$$\dot{e}_{ct} = V \sin \theta_e \qquad \dot{\theta}_e = \omega = u$$

Notice how our control variable affects all the error terms

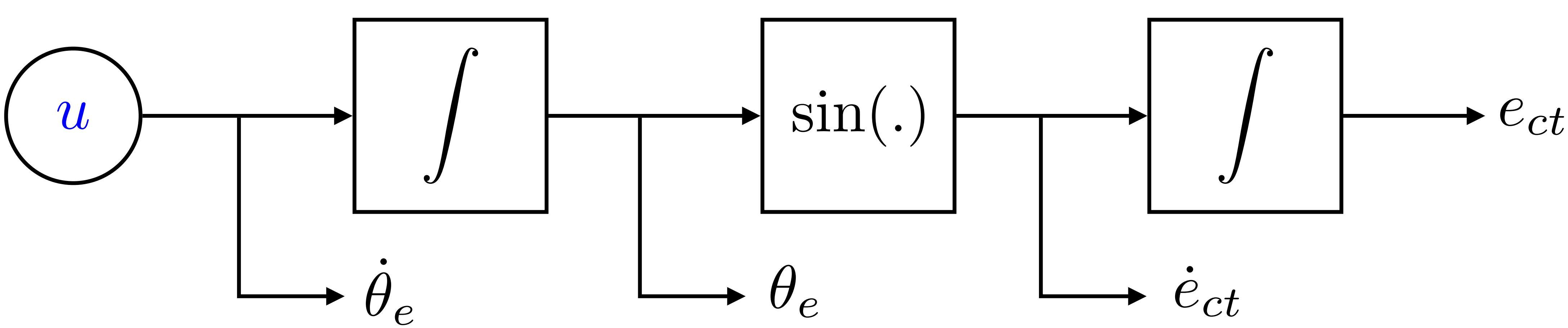
How does error evolve in time?



Let's say we were interested in driving both e_{ct} and θ_e to zero

$$e_{ct} = -\sin(\theta_{ref})(x - x_{ref}) + \cos(\theta_{ref})(y - y_{ref}) \qquad \theta_e = \theta - \theta_{ref}$$

$$\dot{e}_{ct} = V \sin\theta_e \qquad \dot{\theta}_e = \omega = u$$



Why is this tricky?

Is it because of non-linearity?

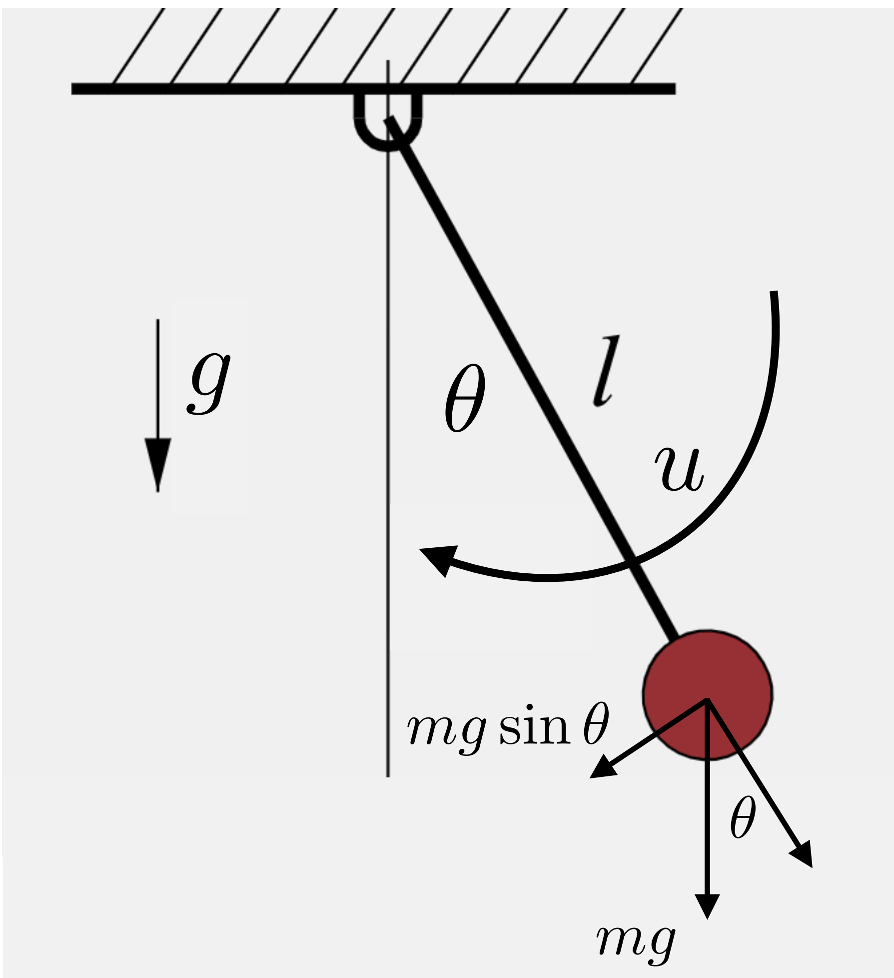
Why is this tricky?

Is it because of non-linearity?

Because of underactuated dynamics...

Fundamental problem with underactuated systems

Example: Pendulum Dynamics



Balance of Moments

$$\Sigma M = J\ddot{\theta}$$

$$J = ml^2$$

$$\Sigma M = -mgl\sin\theta + u$$

$$ml^2\ddot{\theta} = -mgl\sin\theta + u$$

What control law should we use to stabilize the pendulum, i.e.

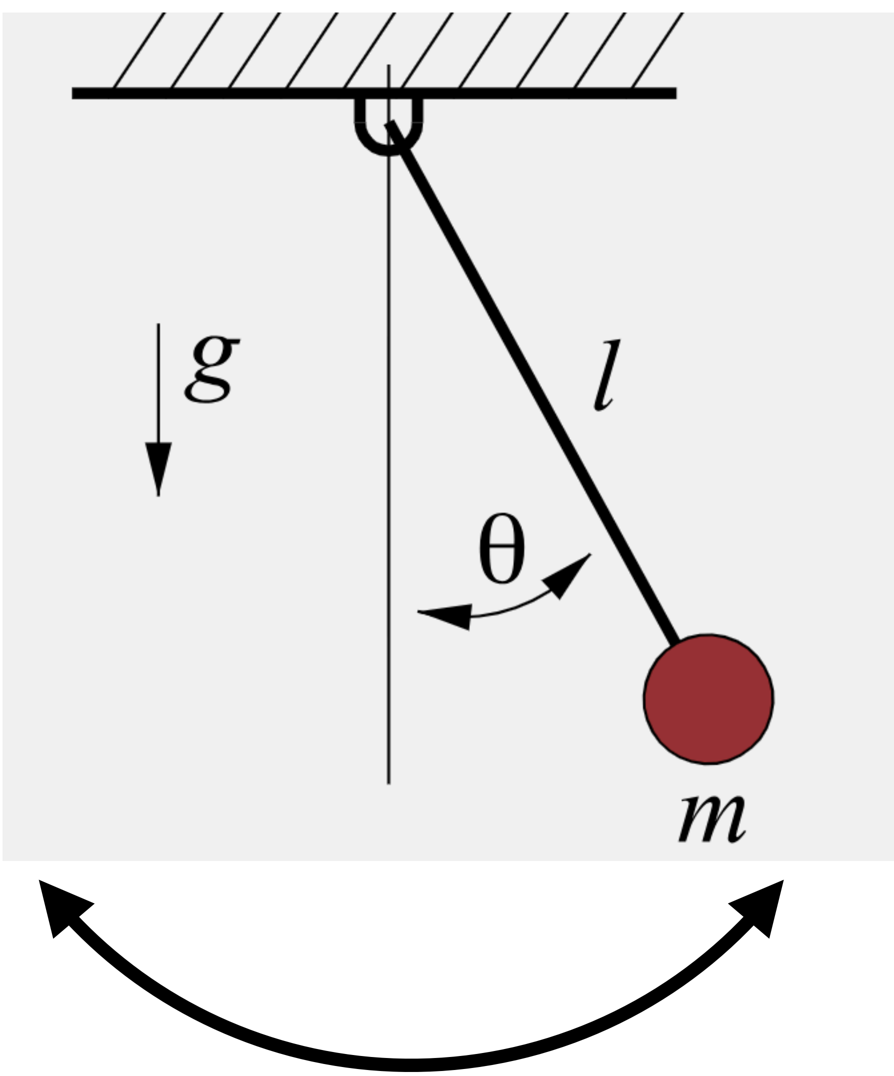
Choose $u = \pi(\theta, \dot{\theta})$ such that $\theta \to 0$

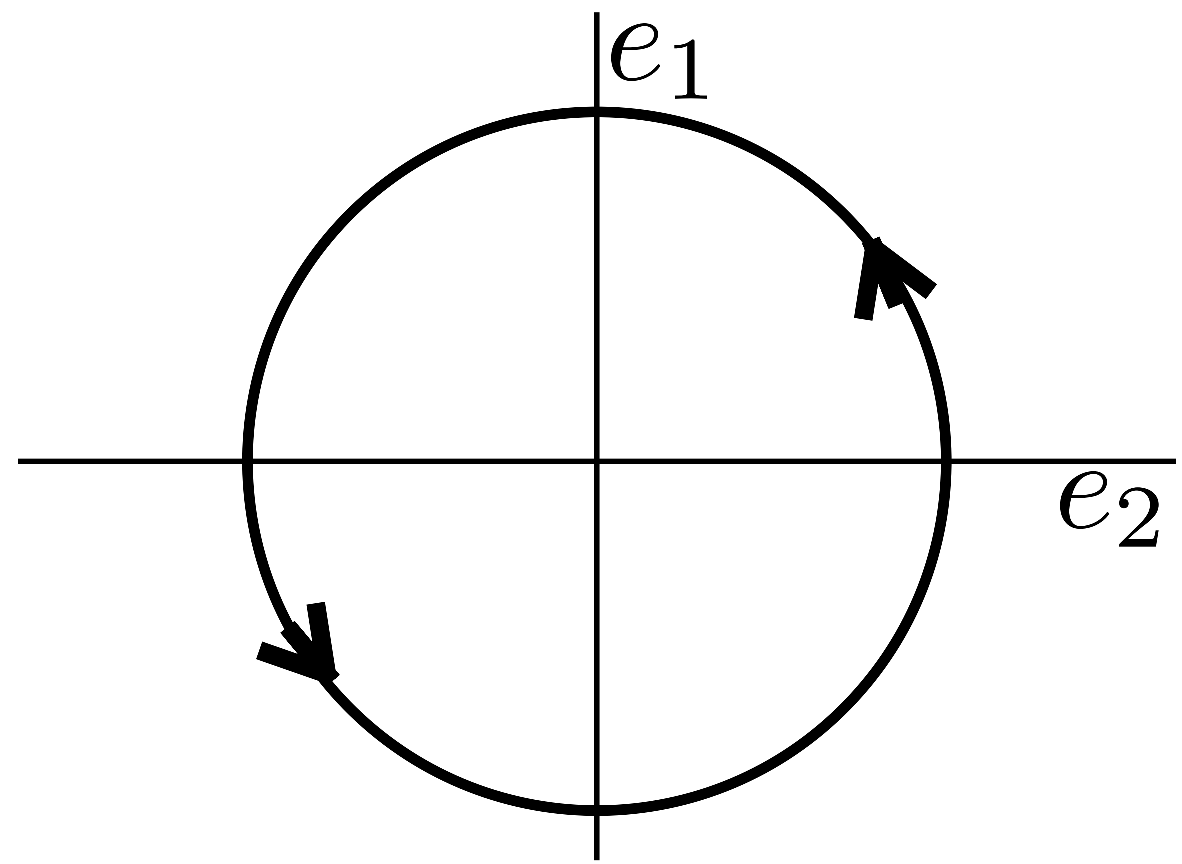
$$\dot{\theta} \to 0$$

How does the passive error dynamics behave?

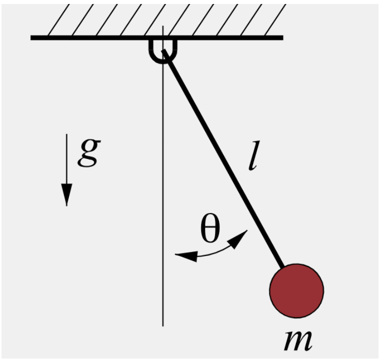
$$e_1 = \theta - 0 = \theta \qquad e_2 = \dot{\theta} - 0 = \dot{\theta}$$

Set u=0. Dynamics is not stable.





How do we verify if a controller is stable?



$$ml^2\ddot{\theta} + mgl\sin\theta = u$$

Lets pick the following law:

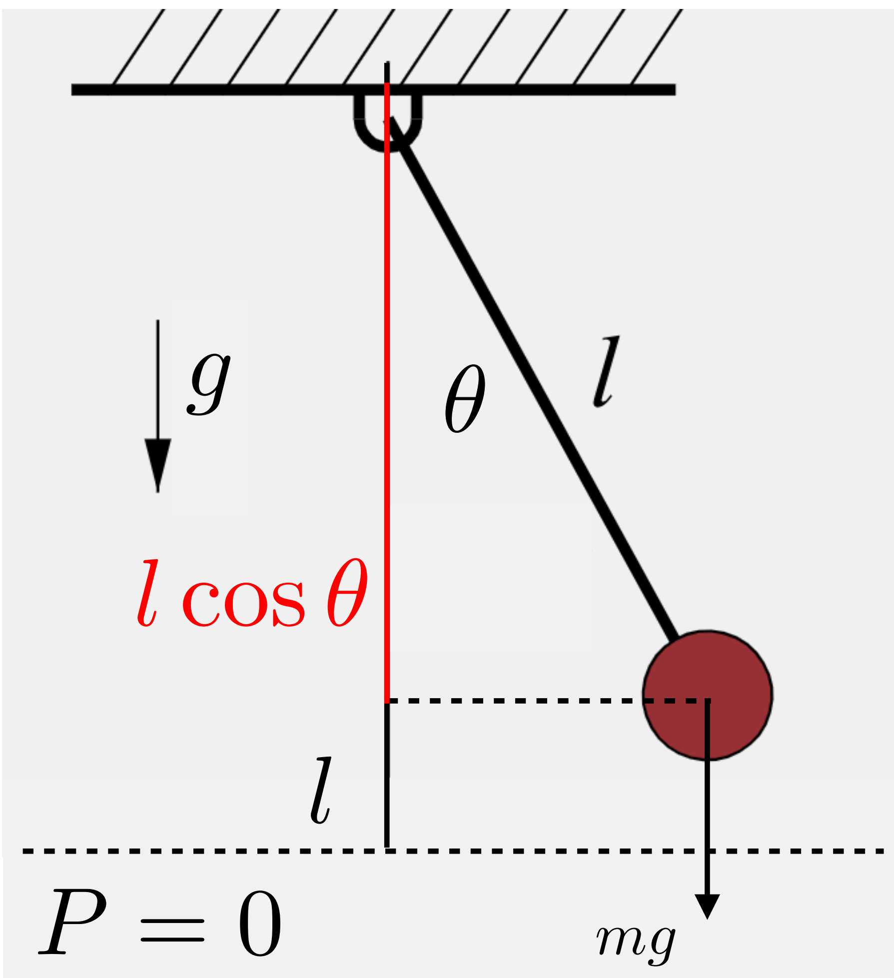
$$u = -K\dot{\theta}$$

Is this stable? How do we know?

We can simulate the dynamics from different start point and check....

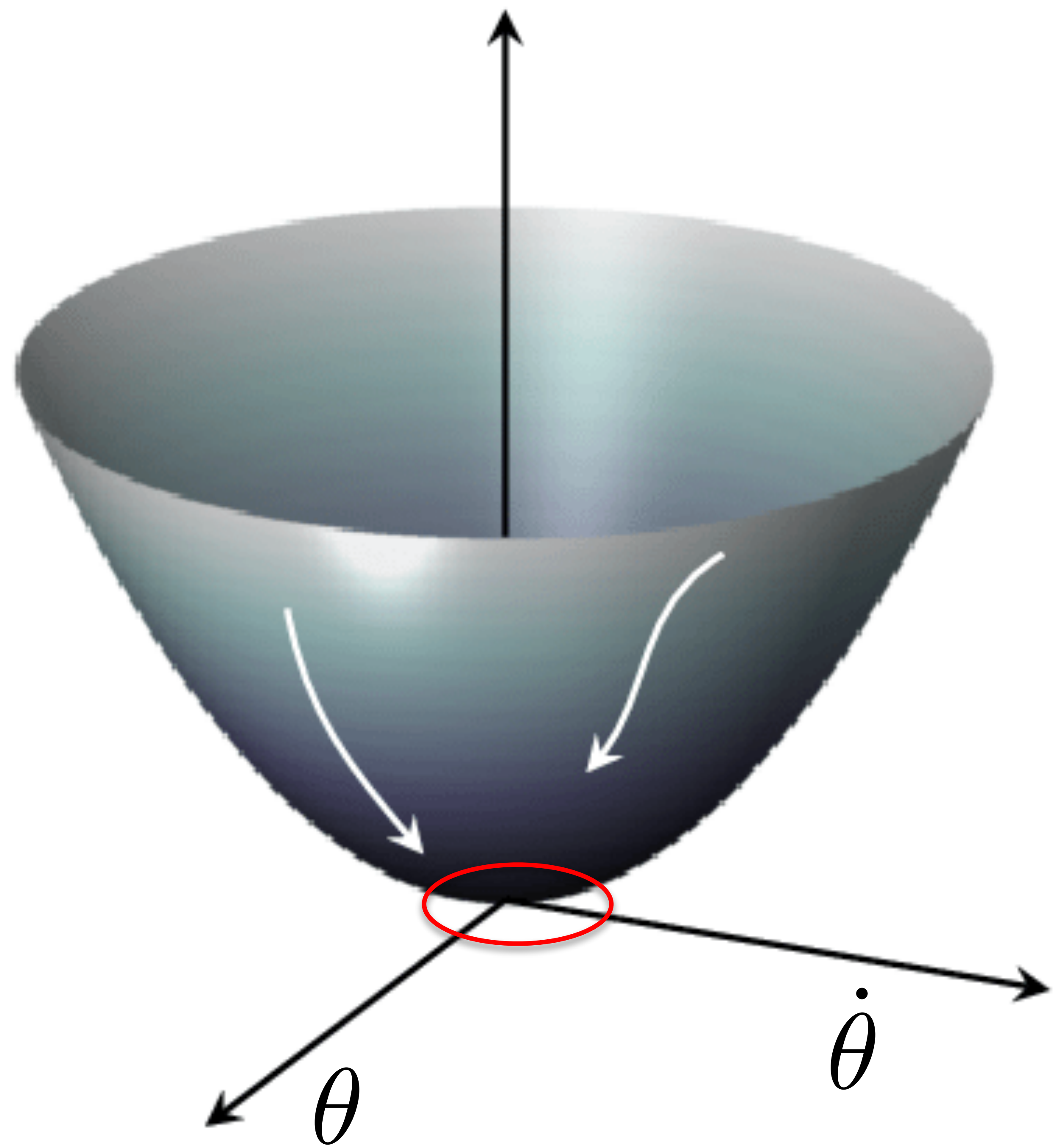
but how many points do we check? what if we miss some points?

Key Idea: Think about energy!



$$K = \frac{1}{2}m(\dot{\theta}l)^2 \quad \text{(Kinetic)}$$

$$P = mgl(1 - \cos\theta) \quad \text{(Potential)}$$



$$V(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta)$$

(Total)

Make energy decay to 0 and stay there

$$V(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$
$$\geq 0$$

$$\dot{V}(\theta, \dot{\theta}) = m l^2 \dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta}$$
$$= \dot{\theta}(u - mgl\sin\theta) + mgl(\sin\theta)\dot{\theta}$$
$$= \dot{\theta} u$$

Choose a control law $u = -k\dot{\theta}$

$$\dot{V}(\theta, \dot{\theta}) = -k\dot{\theta}^2 < 0$$

Lyapunov function:
A generalization of energy

Lyapunov function for a closed-loop system

1. **Construct** an energy function that is always positive

$$V(x) > 0, \forall x$$

Energy is only 0 at the origin, i.e. $V(0) = 0$

2. **Choose** a control law such that this energy always decreases

$$\dot{V}(x) < 0, \forall x$$

Energy rate is 0 at origin, i.e. $\dot{V}(0) = 0$

No matter where you start, energy will decay and you will reach 0!

Let's get provable control for our car!

Dynamics of the car

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \frac{v}{B}\tan u$$

Let's get provable control for our car!

Let's define the following Lyapunov function

$$V(e_{ct}, \theta_e) = \frac{1}{2}k_1 e_{ct}^2 + \frac{1}{2}\theta_e^2 \qquad > 0$$

Compute derivative

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} \dot{e}_{ct} + \theta_e \dot{\theta}_e$$

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin \theta_e + \theta_e \frac{V}{B} \tan u$$

Let's get provable control for our car!

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin\theta_e + \theta_e \frac{V}{B} \tan u$$

Trick: Set u intelligently to get this term to always be negative

$$\dot{V}(e_{ct}, \theta_e) = -k_2 \theta_e^2$$

$$\theta_e \frac{V}{B} \tan u = -k_1 e_{ct} V \sin\theta_e - k_2 \theta_e^2$$

$$\tan u = -\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e$$

$$u = \tan^{-1}\left(-\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e\right)$$

(Advanced Reading)

Bank-to-Turn Control for a Small UAV using Backstepping and Parameter Adaptation

Dongwon Jung and Panagiotis Tsiotras

Coming Up Next: Optimal Control

Assuming we proved stability, how can we handle steering rate, acceleration, jerk, snap constraints?