

# CSE 473: Artificial Intelligence

## Hidden Markov Models



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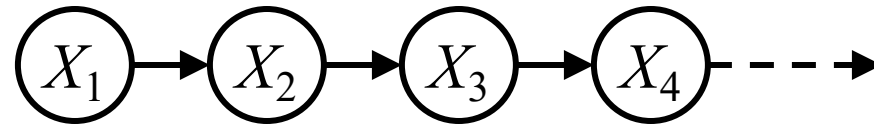
# Reasoning over Time or Space

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- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time (or space) into our models

# Markov Models

- Value of  $X$  at a given time is called the **state**



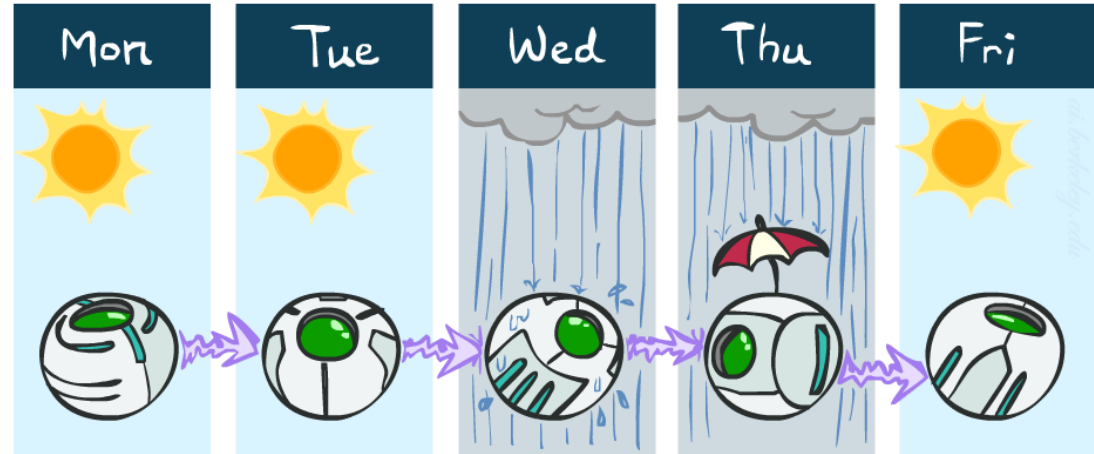
$$P(X_1) \quad P(X_t|X_{t-1})$$

- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

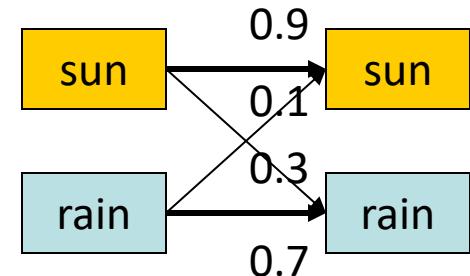
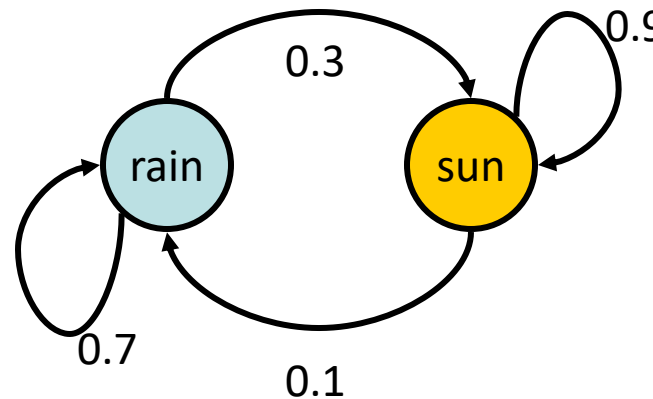
# Example Markov Chain: Weather

- States:  $X = \{\text{rain}, \text{sun}\}$
- Initial distribution: 1.0 sun
- CPT  $P(X_t \mid X_{t-1})$ :

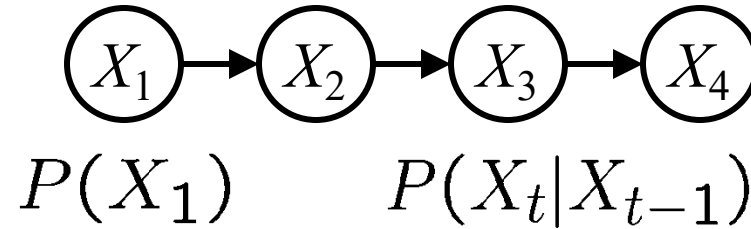
$X_{t-1}$	$X_t$	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7



Two new ways of representing the same CPT



# Joint Distribution of a Markov Model



- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

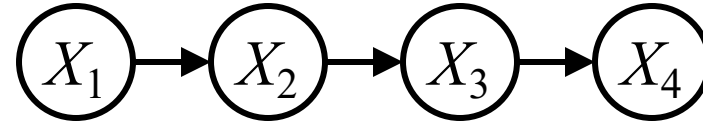
- More generally:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Chain Rule and Markov Models



- From the chain rule, every joint distribution over  $X_1, X_2, X_3, X_4$  can be written as:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)P(X_4|X_1, X_2, X_3)$$

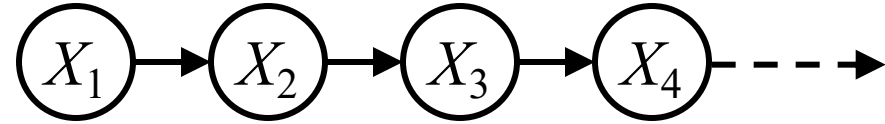
- Assuming that

$$X_3 \perp\!\!\!\perp X_1 \mid X_2 \quad \text{and} \quad X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$$

results in the expression posited on the previous slide:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

# Chain Rule and Markov Models



- From the chain rule, every joint distribution over  $X_1, X_2, \dots, X_T$  can be written as:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_1, X_2, \dots, X_{t-1})$$

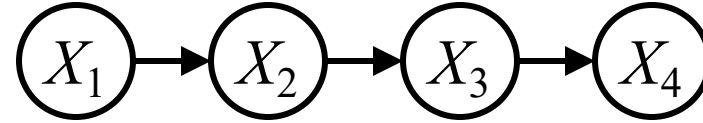
- Assuming that for all  $t$ :

$$X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$$

gives us the expression posited on the earlier slide:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_{t-1})$$

# Implied Conditional Independencies



- We assumed:  $X_3 \perp\!\!\!\perp X_1 \mid X_2$  and  $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

- Do we also have  $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$  ?

- Yes!

- Proof:

$$\begin{aligned} P(X_1 \mid X_2, X_3, X_4) &= \frac{P(X_1, X_2, X_3, X_4)}{P(X_2, X_3, X_4)} \\ &= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}{\sum_{x_1} P(x_1)P(X_2 \mid x_1)P(X_3 \mid X_2)P(X_4 \mid X_3)} \\ &= \frac{P(X_1, X_2)}{P(X_2)} \\ &= P(X_1 \mid X_2) \end{aligned}$$



# Markov Models Recap

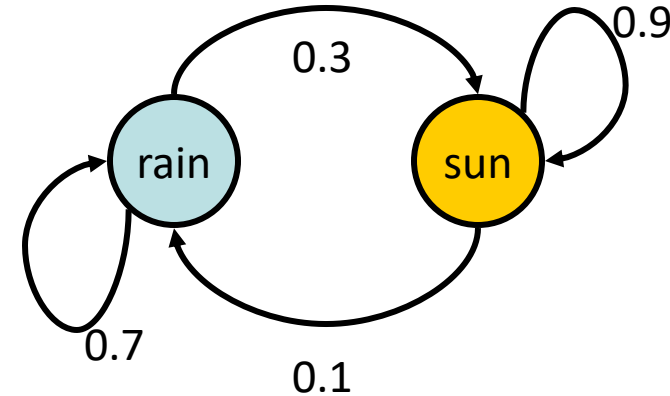
- Explicit assumption for all  $t$  :  $X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Implied conditional independencies: (try to prove this!)
  - Past variables independent of future variables given the present  
i.e., if  $t_1 < t_2 < t_3$  or  $t_1 > t_2 > t_3$  then:  $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption:  $P(X_t \mid X_{t-1})$  is the same for all  $t$

# Example Markov Chain: Weather

- Initial distribution: 1.0 sun



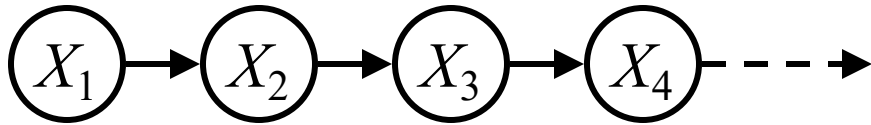
- What is the probability distribution after one step?

$$P(X_2 = \text{sun}) = P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

# Mini-Forward Algorithm

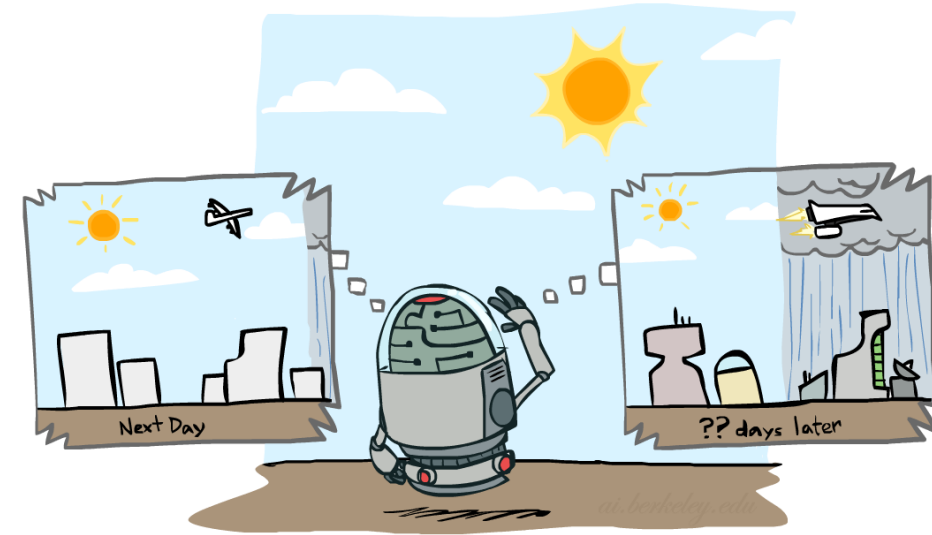
- Question: What's  $P(X)$  on some day  $t$ ?



$P(x_1)$  = known

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1}) \end{aligned}$$

*Forward simulation* ←



# Proof of Mini-Forward Algorithm

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_{t-1})$$

- Question: What's  $P(x_3)$ ?

$$P(x_3) = \sum_{x_1} \sum_{x_2} P(x_1, x_2, x_3)$$

[Inference by enumeration]

$$= \sum_{x_1} \sum_{x_2} P(x_1) P(x_2 | x_1) P(x_3 | x_2)$$

[Def. of Markov model]

$$= \sum_{x_2} P(x_3 | x_2) \sum_{x_1} P(x_1) P(x_2 | x_1)$$

[Factoring: basic algebra]

$$= \sum_{x_2} P(x_3 | x_2) P(x_2)$$

[Def. of Markov model]

# Proof of Mini-Forward Algorithm

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_{t-1})$$

- Question: What's  $P(X_T)$ ?

$$P(x_T) = \sum_{x_1, \dots, x_{T-1}} P(x_1, \dots, x_T)$$

[Inference by enumeration]

$$= \sum_{x_1, \dots, x_{T-1}} P(x_1) \prod_{t=2}^T P(x_t | x_{t-1})$$

[Def. of Markov model]

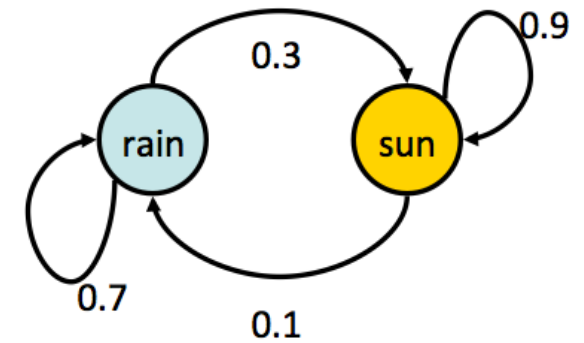
$$= \sum_{x_{T-1}} P(x_T | x_{T-1}) \sum_{x_1, \dots, x_{T-2}} P(x_1) \prod_{t=2}^{T-1} P(x_t | x_{t-1})$$

[Factoring: basic algebra]

$$= \sum_{x_{T-1}} P(x_T | x_{T-1}) P(x_{T-1})$$

[Def. of Markov model]

# Example: Runs of Mini-Forward Algorithm



- From initial observation of sun

$$\begin{array}{ccccc}
 \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle & \xrightarrow{\quad} \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & P(X_\infty)
 \end{array}$$

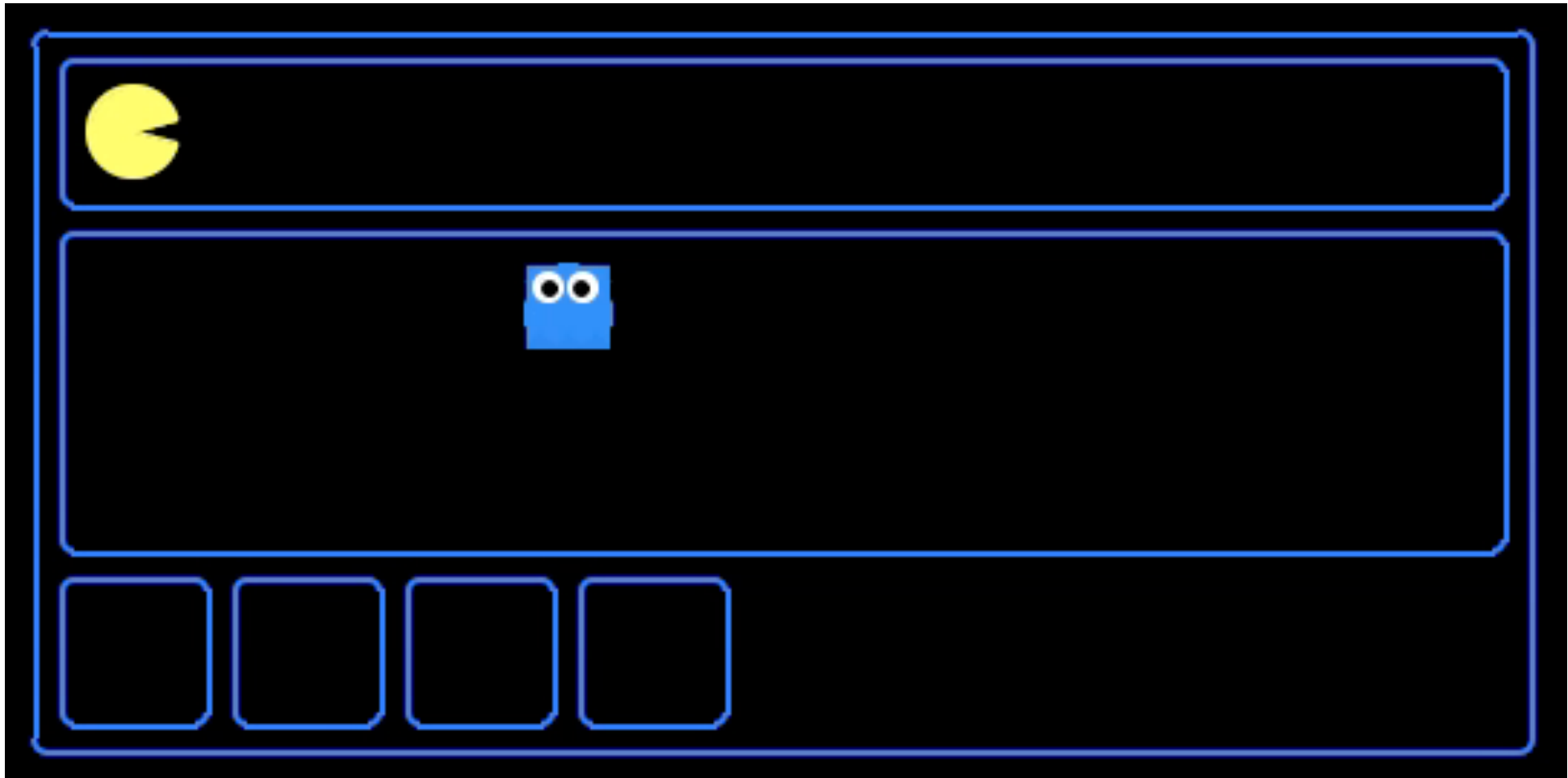
- From initial observation of rain

$$\begin{array}{ccccc}
 \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle & \xrightarrow{\quad} \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & P(X_2) & P(X_3) & P(X_4) & P(X_\infty)
 \end{array}$$

- From yet another initial distribution  $P(X_1)$ :

$$\begin{array}{ccc}
 \left\langle \begin{array}{c} p \\ 1 - p \end{array} \right\rangle & \dots & \xrightarrow{\quad} \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\
 P(X_1) & & P(X_\infty)
 \end{array}$$

# Mini-Forward Algorithm



# Stationary Distributions

- For most chains:

- Influence of the initial distribution gets less and less over time.
- The distribution we end up in is independent of the initial distribution

- Stationary distribution:

- The distribution we end up with is called the **stationary distribution**  $P_\infty$  of the chain
- It satisfies

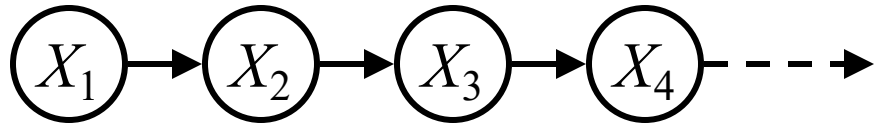
$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$





# Example: Stationary Distributions

- Question: What's  $P(X)$  at time  $t = \text{infinity}$ ?



$$P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain})$$

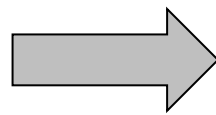
$$P_{\infty}(\text{sun}) = 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 3P_{\infty}(\text{rain})$$

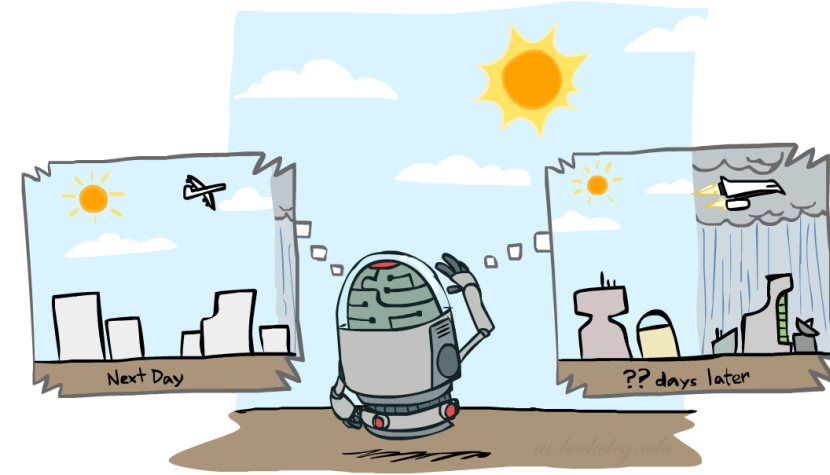
$$P_{\infty}(\text{rain}) = 1/3P_{\infty}(\text{sun})$$

Also:  $P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$



$$P_{\infty}(\text{sun}) = 3/4$$

$$P_{\infty}(\text{rain}) = 1/4$$

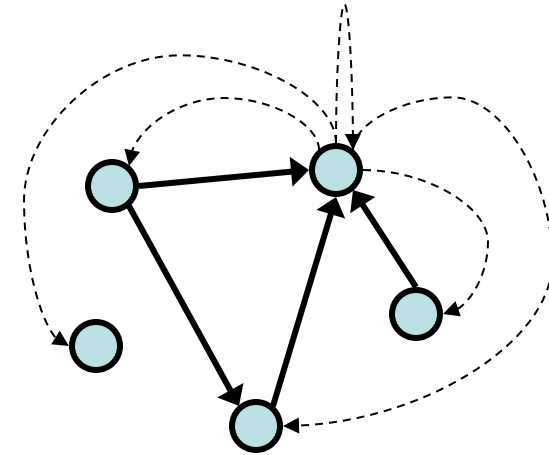


$X_{t-1}$	$X_t$	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

# Application of Stationary Distribution: Web Link Analysis

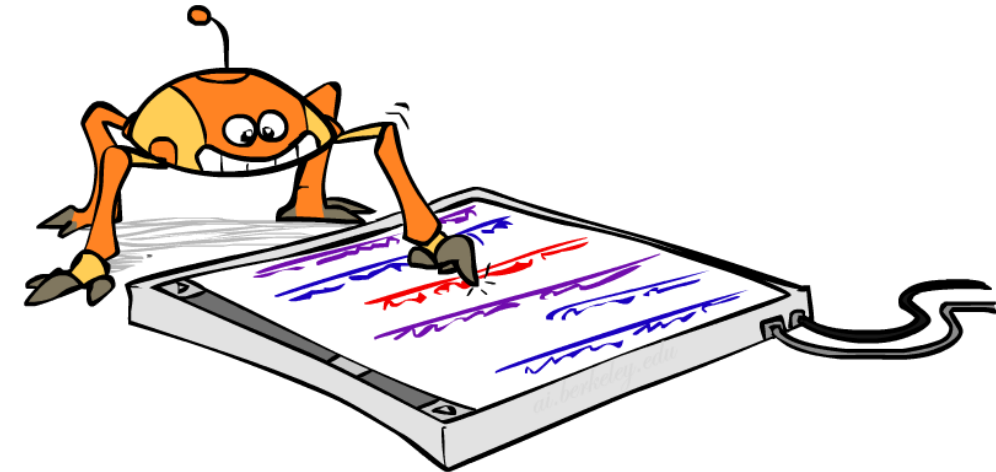
- PageRank over a web graph

- Each web page is a state
- Initial distribution: uniform over pages
- Transitions:
  - With prob.  $c$ , uniform jump to a random page (dotted lines, not all shown)
  - With prob.  $1-c$ , follow a random outlink (solid lines)



- Stationary distribution

- Will spend more time on highly reachable pages
- E.g. many ways to get to the Acrobat Reader download page
- Somewhat robust to link spam
- Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



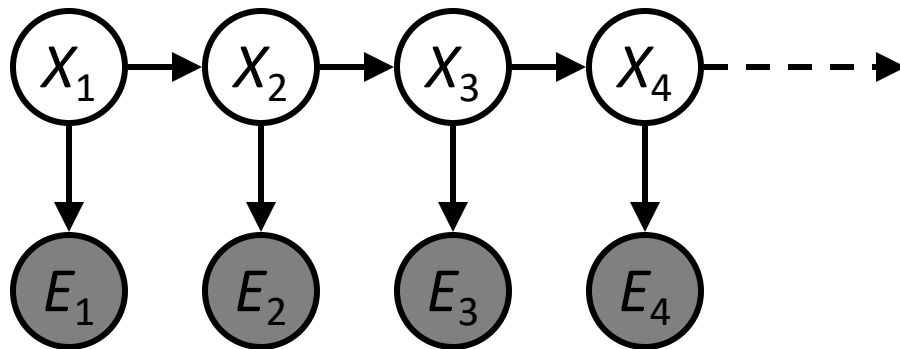
# Hidden Markov Models

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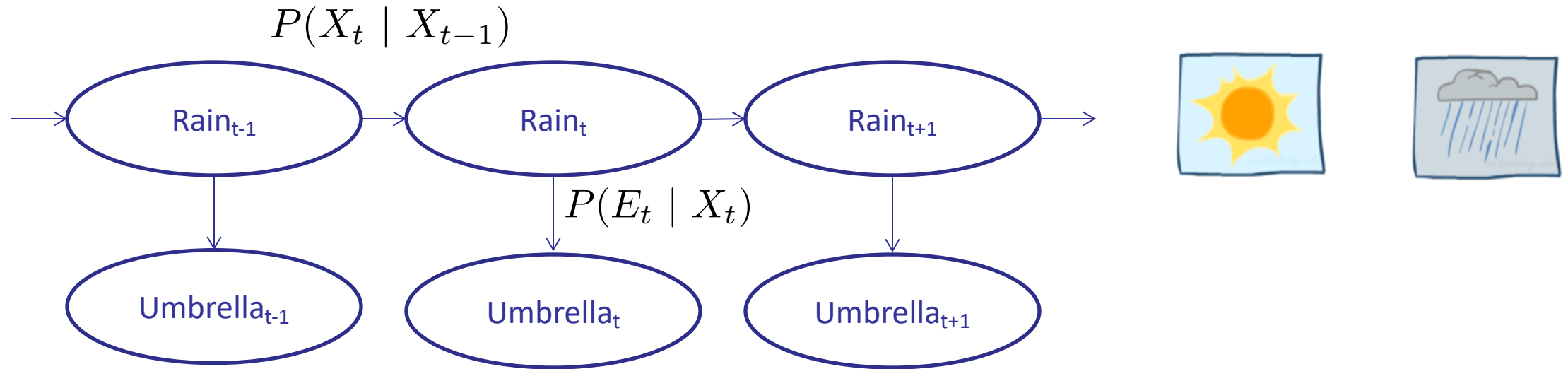


# Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states  $X$
  - You observe outputs (effects) at each time step



# Example: Weather HMM



- An HMM is defined by:

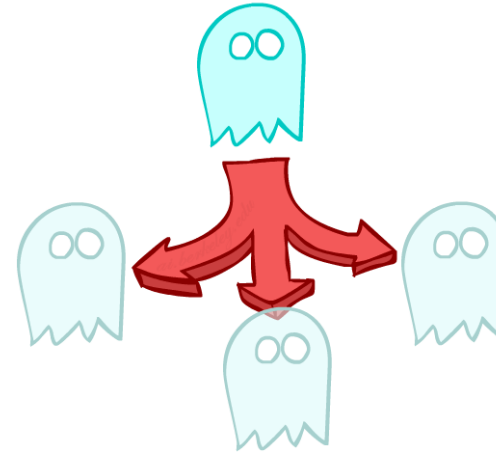
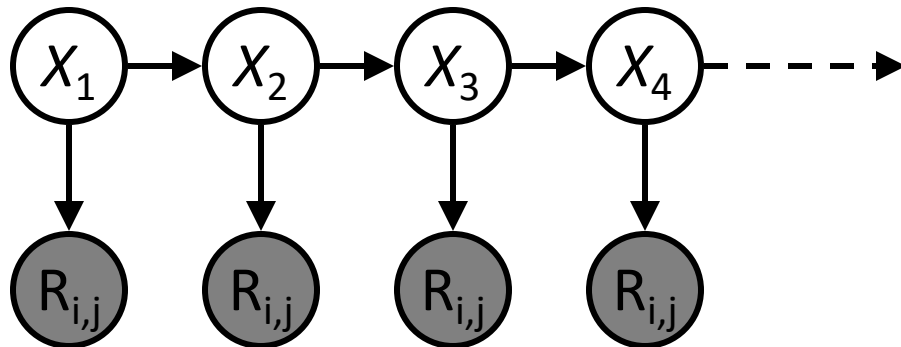
- Initial distribution:  $P(X_1)$
- Transitions:  $P(X_t | X_{t-1})$
- Emissions:  $P(E_t | X_t)$

$R_t$	$R_{t+1}$	$P(R_{t+1}   R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$R_t$	$U_t$	$P(U_t   R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

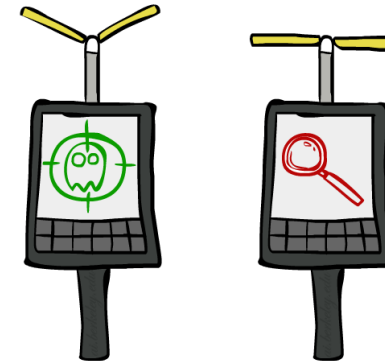
# Example: Ghostbusters HMM

- $P(X_1)$  = uniform
- $P(X|X')$  = usually move clockwise, but sometimes move in a random direction or stay in place
- $P(R_{ij}|X)$  = same sensor model as before: red means close, green means far away.



1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

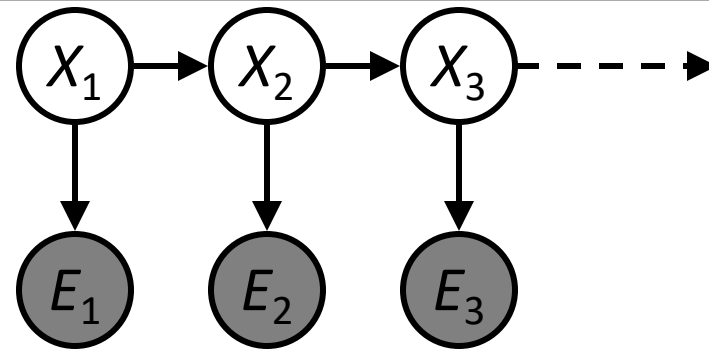
$P(X_1)$



1/6	1/6	1/2
0	1/6	0
0	0	0

$P(X|X' = \langle 1, 2 \rangle)$

# Joint Distribution of an HMM



- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

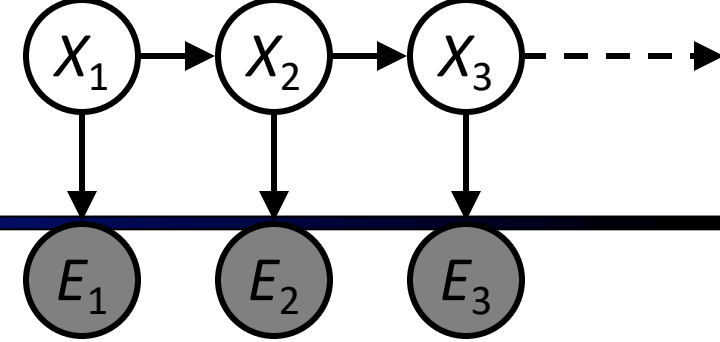
- More generally:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Chain Rule and HMMs



- From the chain rule, *every* joint distribution over  $X_1, E_1, X_2, E_2, X_3, E_3$  can be written as:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2) \\ P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)$$

- Assuming that

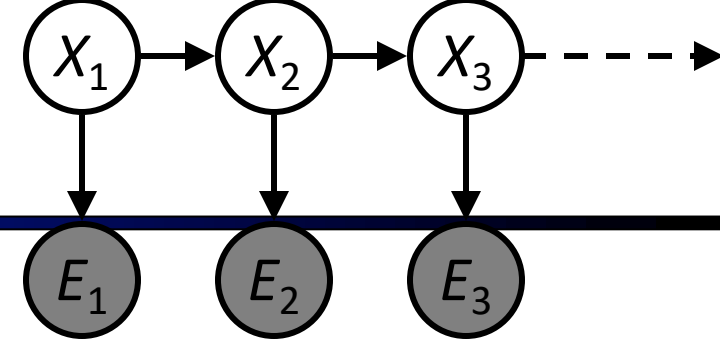
$$X_2 \perp\!\!\!\perp E_1 \mid X_1, \quad E_2 \perp\!\!\!\perp X_1, E_1 \mid X_2, \quad X_3 \perp\!\!\!\perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp\!\!\!\perp X_1, E_1, X_2, E_2 \mid X_3$$

gives us the expression posited on the previous slide:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$



# Chain Rule and HMMs



- From the chain rule, *every* joint distribution over  $X_1, E_1, \dots, X_T, E_T$  can be written as:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_1, E_1, \dots, X_{t-1}, E_{t-1})P(E_t|X_1, E_1, \dots, X_{t-1}, E_{t-1}, X_t)$$

- Assuming* that for all  $t$ :

- State independent of all past states and all past evidence given the previous state, i.e.:

$$X_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1}$$

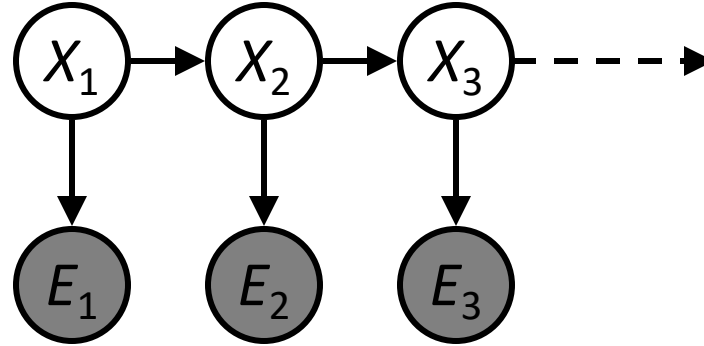
- Evidence is independent of all past states and all past evidence given the current state, i.e.:

$$E_t \perp\!\!\!\perp X_1, E_1, \dots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t$$

gives us the expression posited on the earlier slide:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

# Implied Conditional Independencies



- Many implied conditional independencies, e.g.,

$$E_1 \perp\!\!\!\perp X_2, E_2, X_3, E_3 \mid X_1$$

- To prove them

- Approach 1: follow similar (algebraic) approach to what we did in the Markov models lecture
- Approach 2: directly from the graph structure (3 lectures from now)
  - Intuition: If path between U and V goes through W, then  $U \perp\!\!\!\perp V \mid W$  [Some fineprint later]

# Real HMM Examples

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- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation options
- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

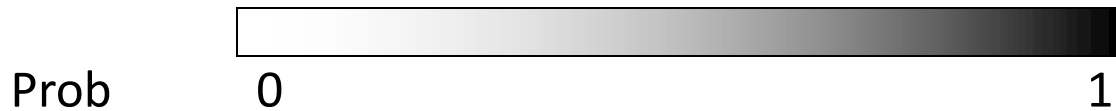
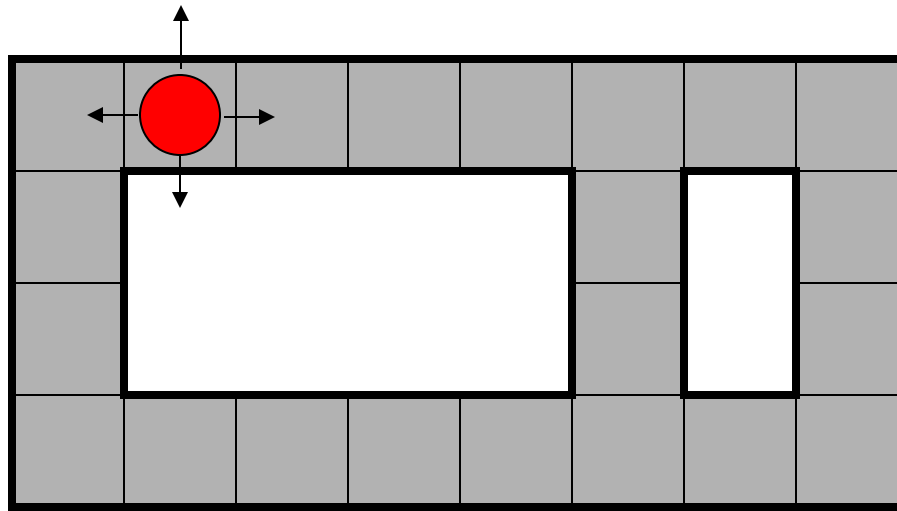
# Filtering / Monitoring

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- Filtering, or monitoring, is the task of tracking the distribution  $B_t(X) = P_t(X_t \mid e_1, \dots, e_t)$  (the belief state) over time
- We start with  $B_1(X)$  in an initial setting, usually uniform
- As time passes, or we get observations, we update  $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

# Example: Robot Localization

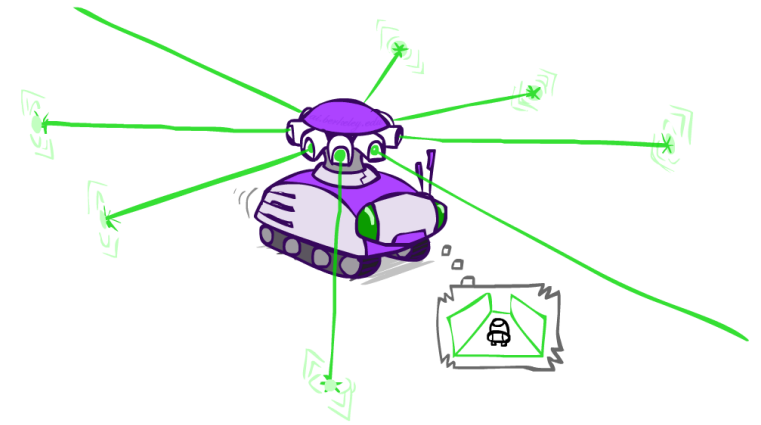
*Example from  
Michael Pfeiffer*



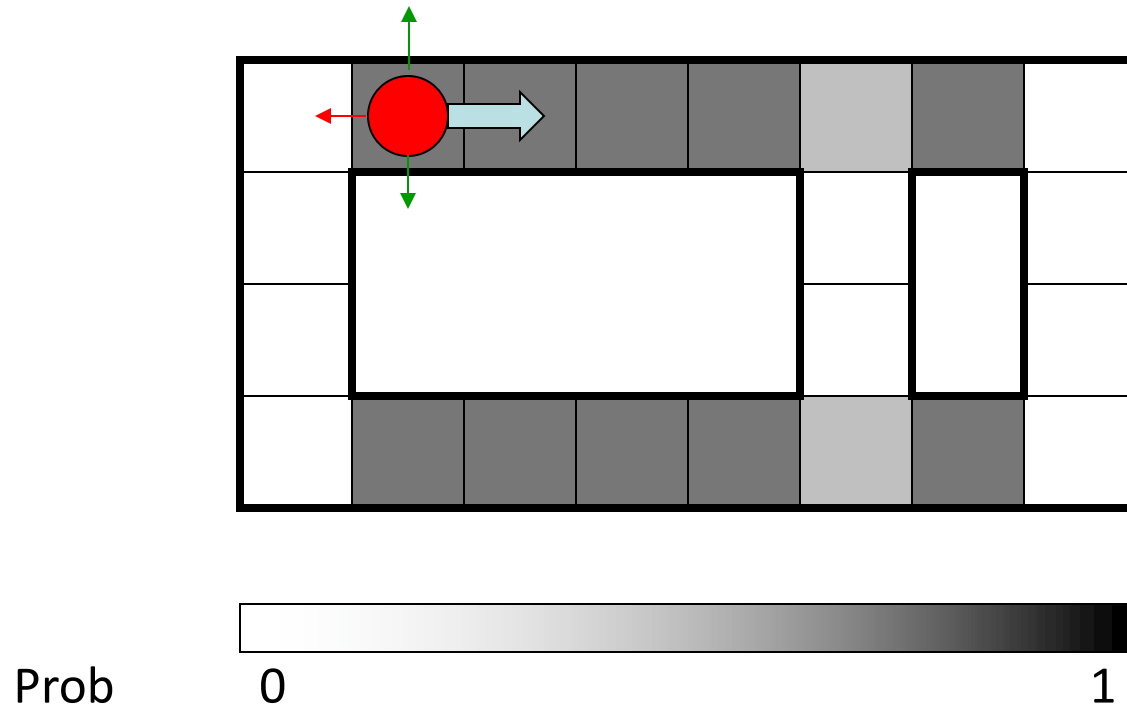
$t=0$

Sensor model: can read in which directions there is a wall,  
never more than 1 mistake

Motion model: may not execute action with small prob.

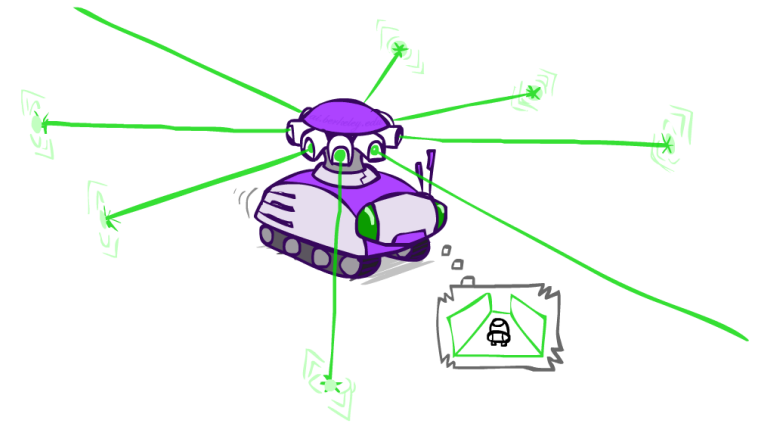


# Example: Robot Localization

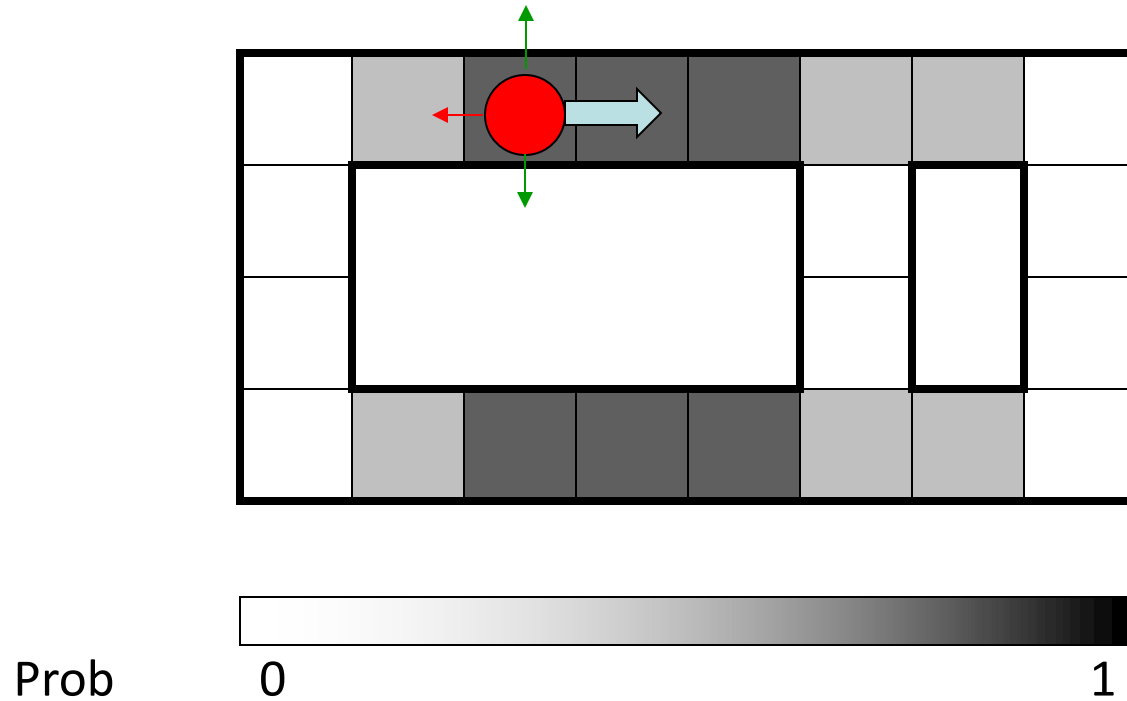


$t=1$

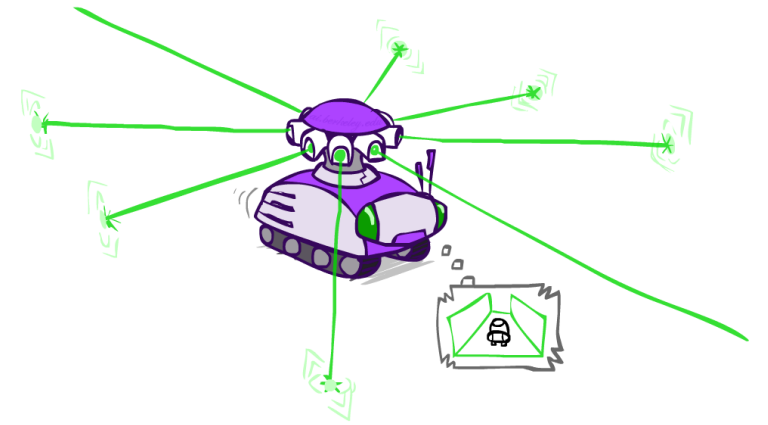
Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake



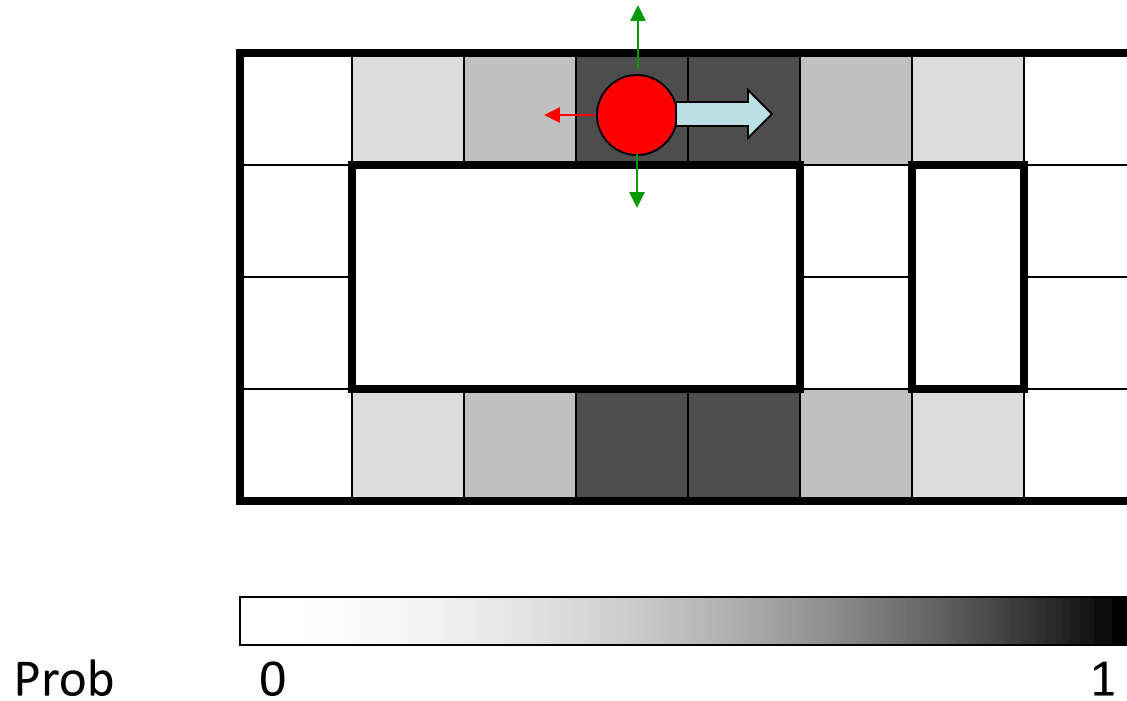
# Example: Robot Localization



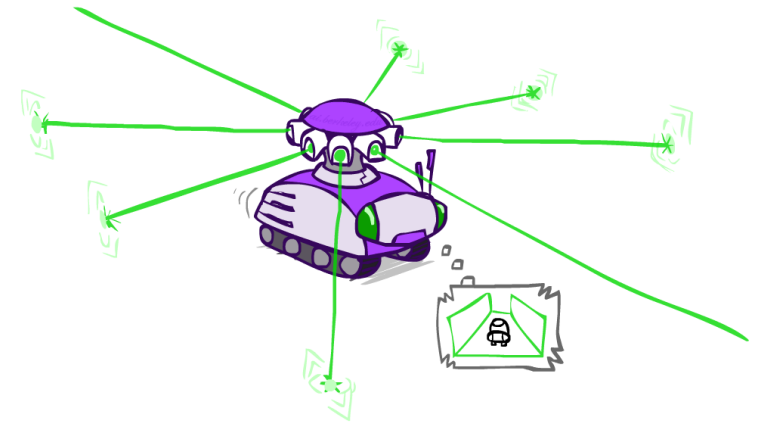
t=2



# Example: Robot Localization

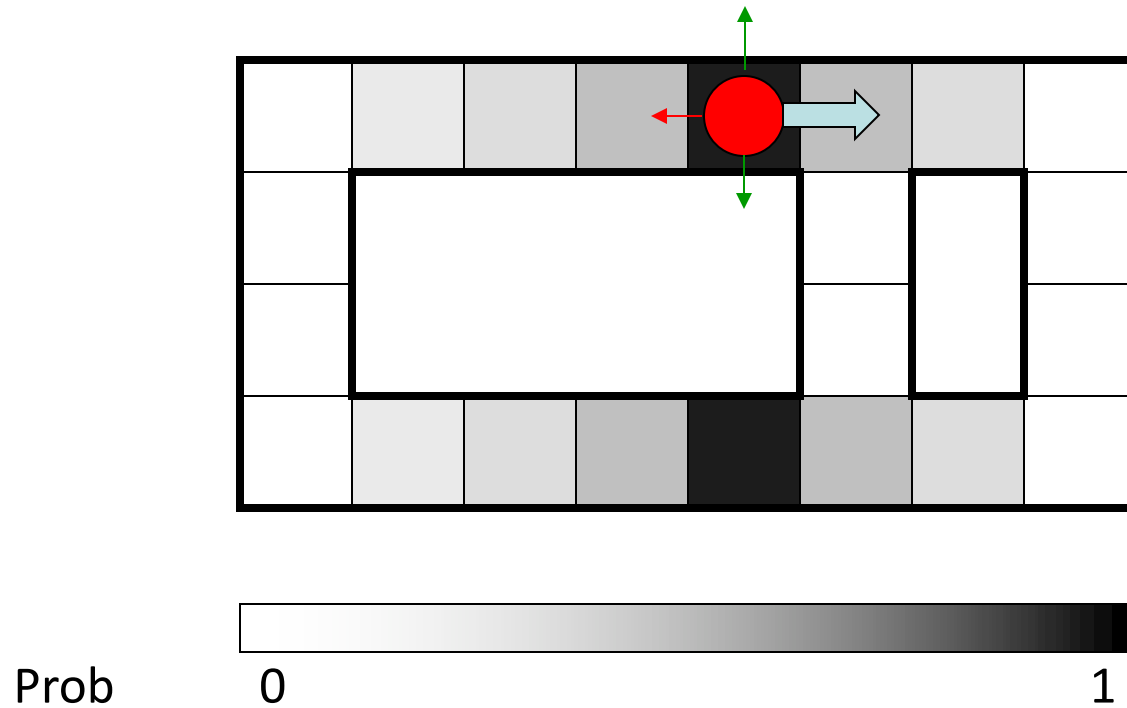


$t=3$

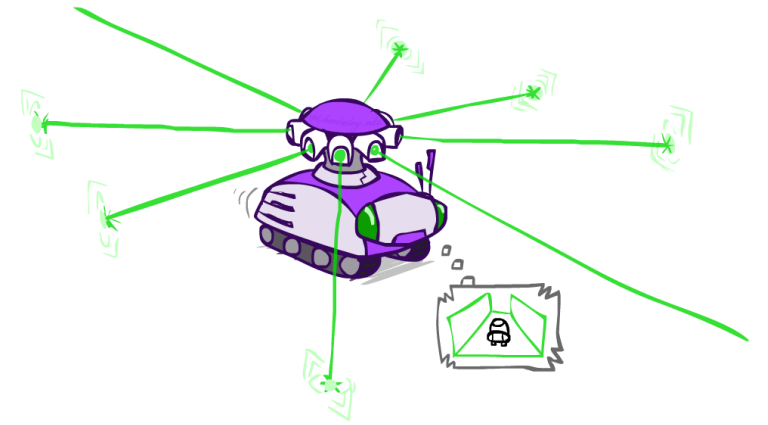




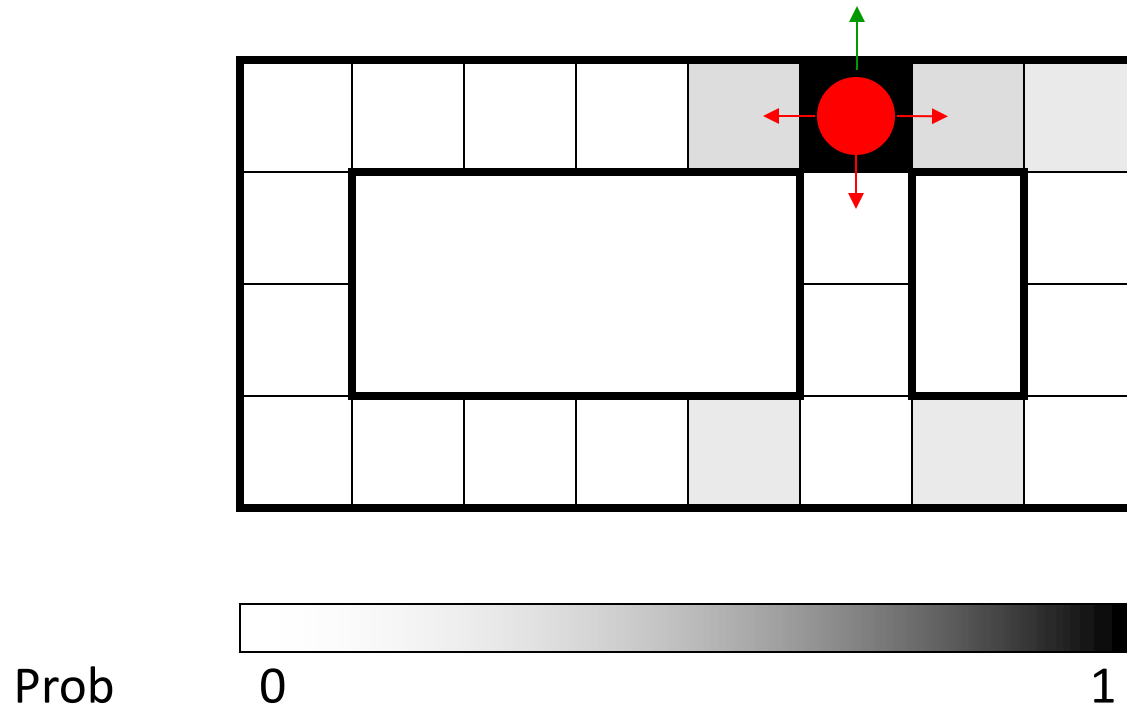
# Example: Robot Localization



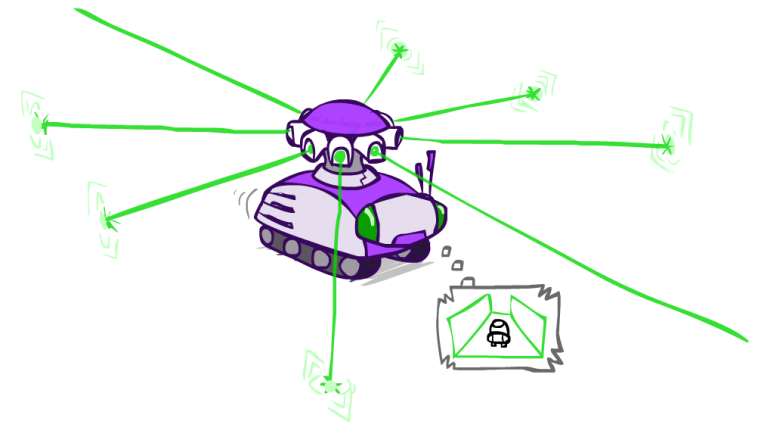
$t=4$



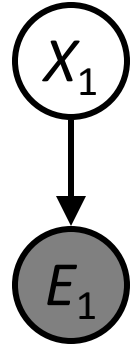
# Example: Robot Localization



t=5

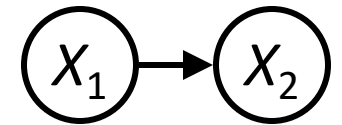
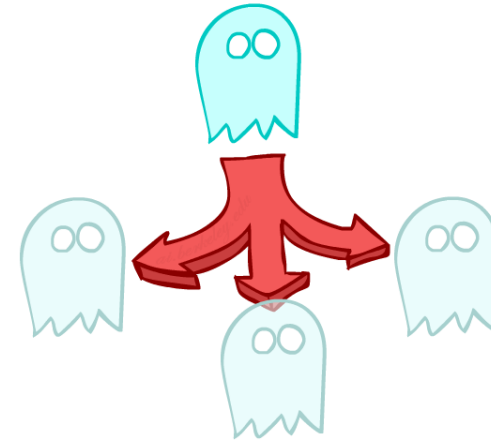


# Inference: Base Cases



$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$



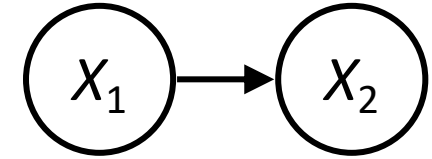
$$P(X_2)$$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$

# Passage of Time

- Assume we have current belief  $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$



- Then, after one time step passes:

$$\begin{aligned} P(X_{t+1} | e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t}) \end{aligned}$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X_{t+1} | x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step  $t$  the belief is about, and what evidence it includes

# Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

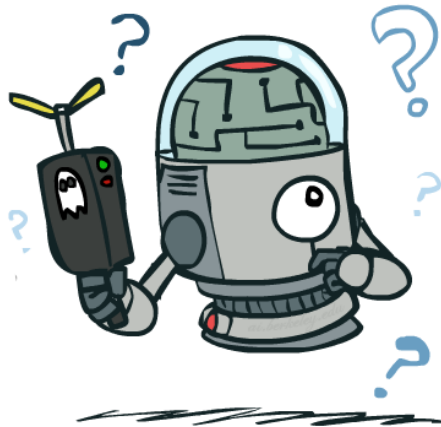
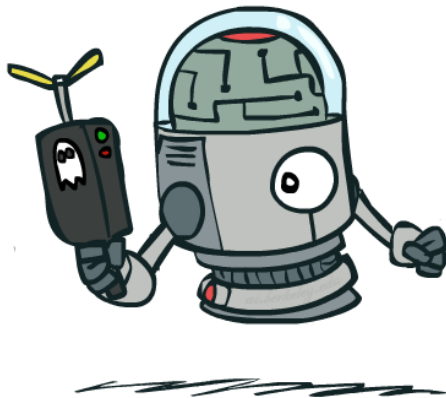
T = 1

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

T = 2

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5



# Observation

- Assume we have current belief  $P(X \mid \text{previous evidence})$ :

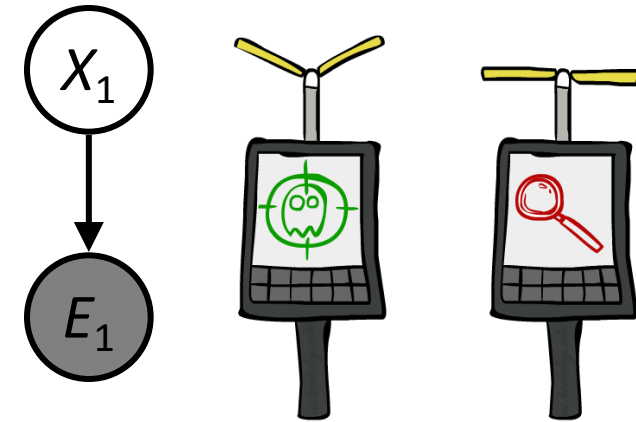
$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then, after evidence comes in:

$$\begin{aligned} P(X_{t+1} | e_{1:t+1}) &= P(X_{t+1}, e_{t+1} | e_{1:t}) / P(e_{t+1} | e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | e_{1:t}, X_{t+1}) P(X_{t+1} | e_{1:t}) \\ &= P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t}) \end{aligned}$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} | X_{t+1}) B'(X_{t+1})$$



- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

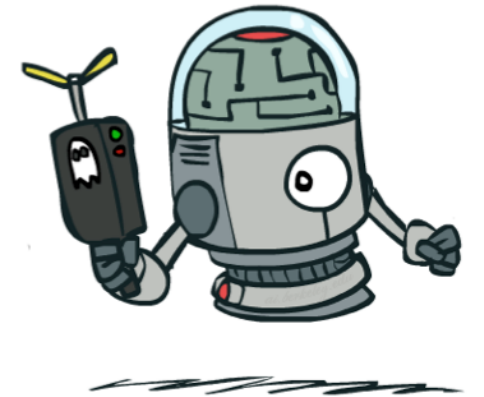
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

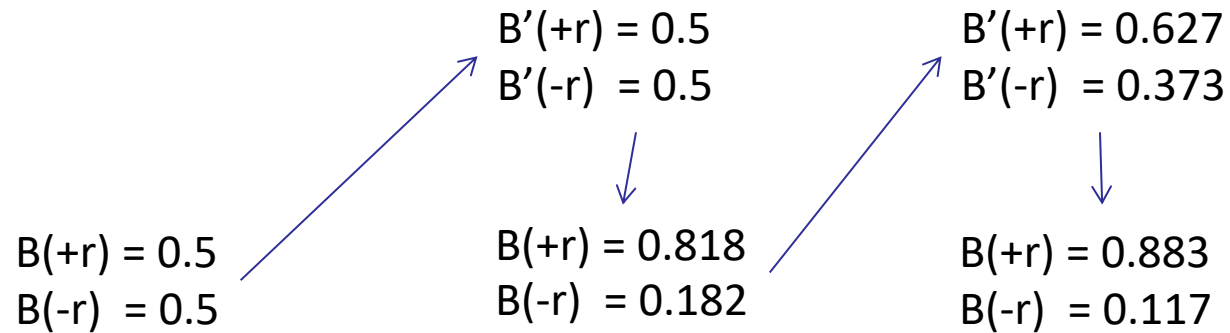
$$B(X) \propto P(e|X)B'(X)$$



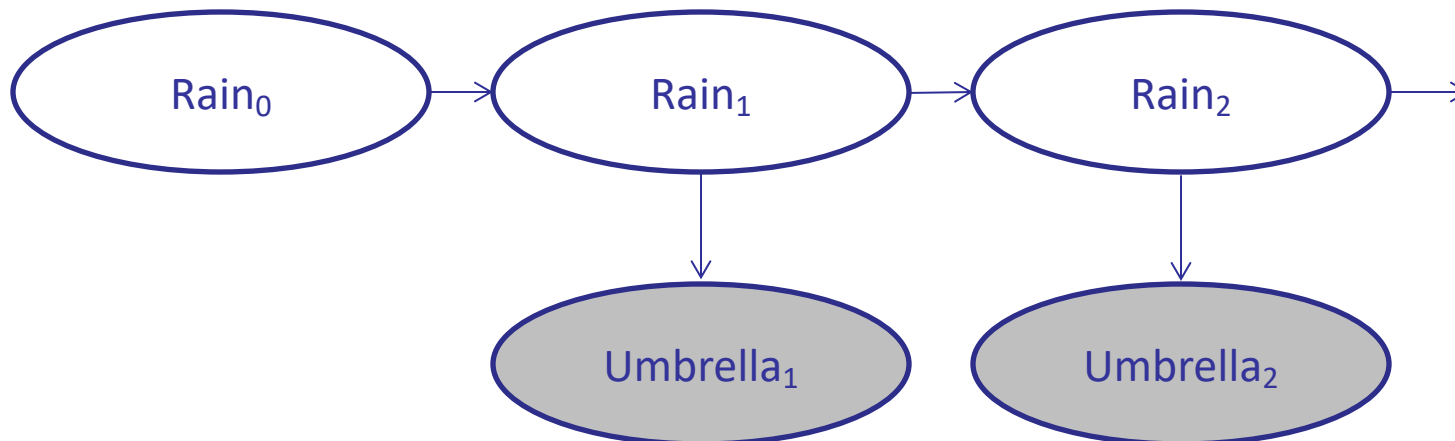
# Example: Weather HMM



$$B'(X_{t+1}) = \sum_{x_t} P(X_{t+1}|x_t) B(x_t) \quad B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1}) B'(X_{t+1})$$



$R_t$	$R_{t+1}$	$P(R_{t+1} R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7



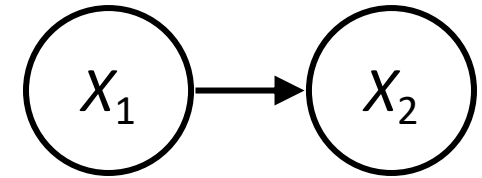
$R_t$	$U_t$	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8



# Online Belief Updates

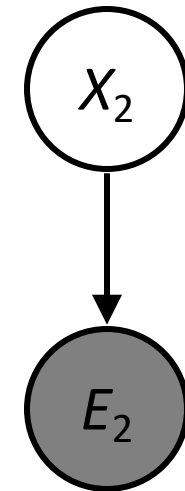
- Every time step, we start with current  $P(X \mid \text{evidence})$
- We update for time:

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$



- We update for evidence:

$$P(x_t | e_{1:t}) \propto_X P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$



- The forward algorithm does both at once (and doesn't normalize)

# Proof of Forward Algorithm

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

- Question: What's  $P(X_T|e_1, \dots, e_T)$ ?

$$P(x_T, e_1, \dots, e_T) = \sum_{x_1, \dots, x_{T-1}} P(x_1, e_1, \dots, x_T, e_T) \quad \text{[Inference by enumeration]}$$

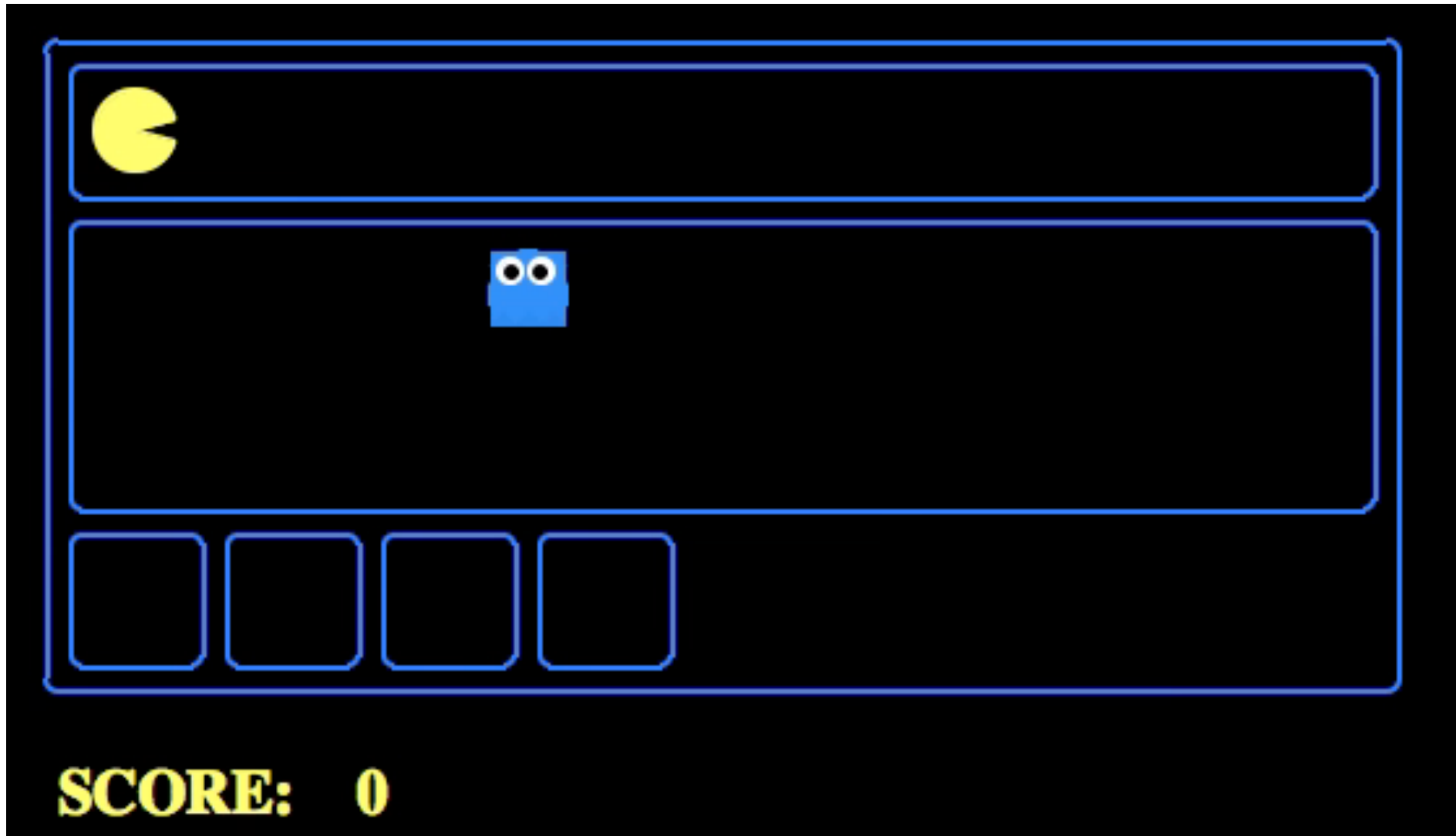
$$= \sum_{x_1, \dots, x_{T-1}} P(x_1)P(e_1|x_1) \prod_{t=2}^T P(x_t|x_{t-1})P(e_t|x_t) \quad \text{[Def. of HMM]}$$

$$= P(e_T|x_T) \sum_{x_{T-1}} P(x_T|x_{T-1}) \sum_{x_1, \dots, x_{T-2}} P(x_1)P(e_1|x_1) \prod_{t=2}^{T-1} P(x_t|x_{t-1})P(e_t|x_t) \quad \text{[Factoring: basic algebra]}$$

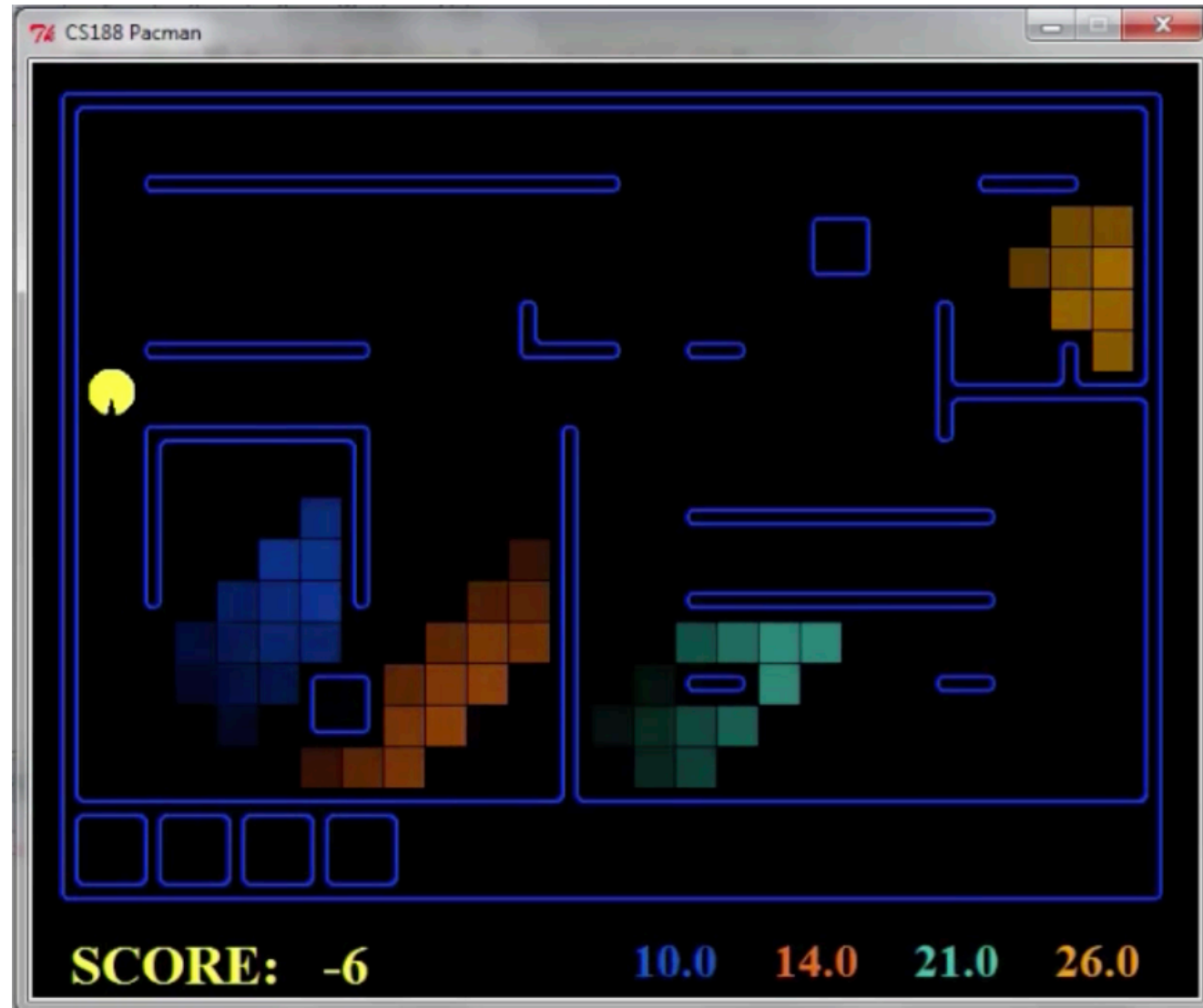
$$= P(e_T|x_T) \sum_{x_{T-1}} P(x_T|x_{T-1})P(x_{T-1}, e_1, \dots, e_{T-1}) \quad \text{[Def. of HMM]}$$

Final step: normalize entries in  $P(X_T, e_1, \dots, e_T)$  to get  $P(X_T|e_1, \dots, e_T)$

# Forward Algorithm

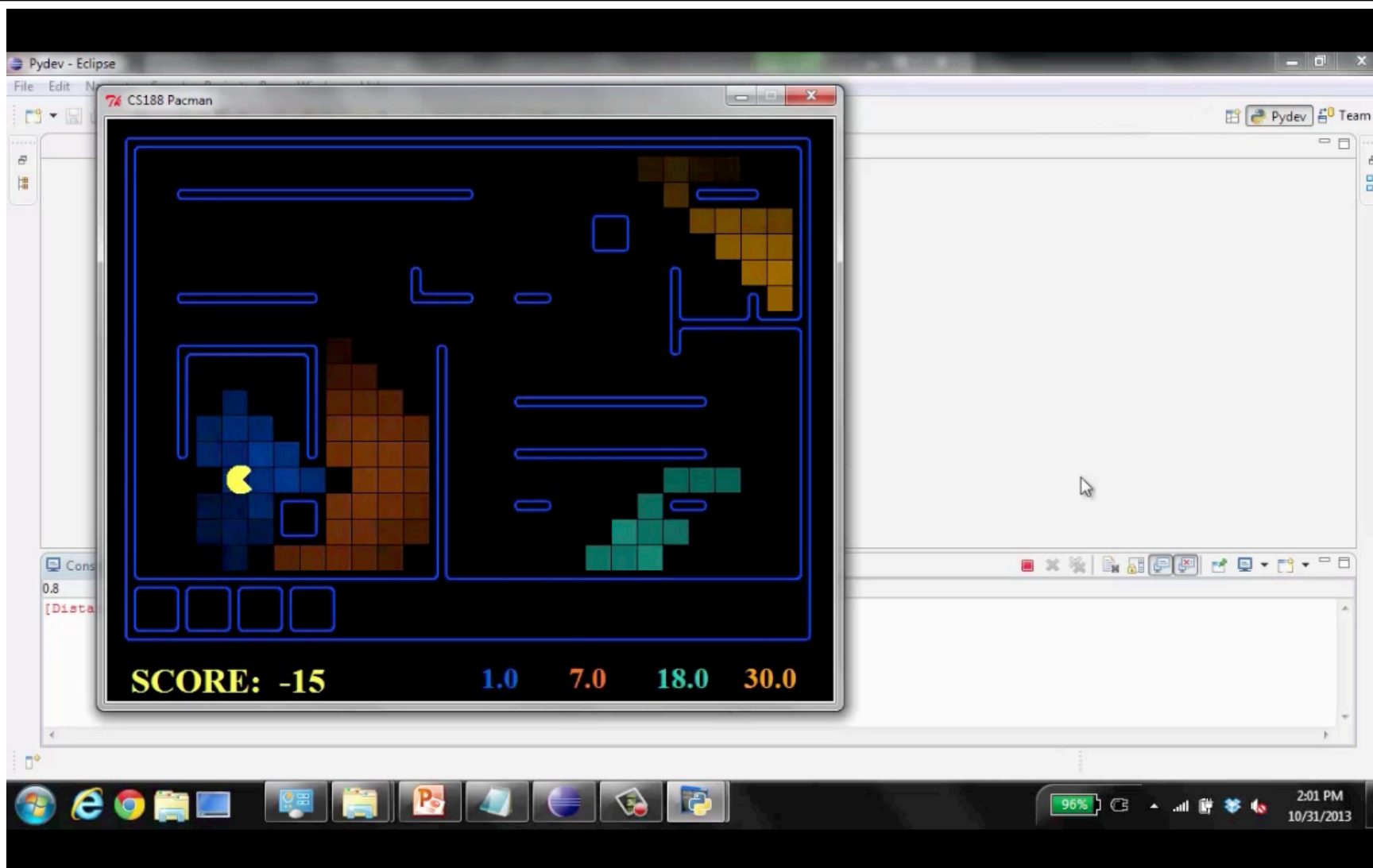


# Pacman – Sonar (P4)

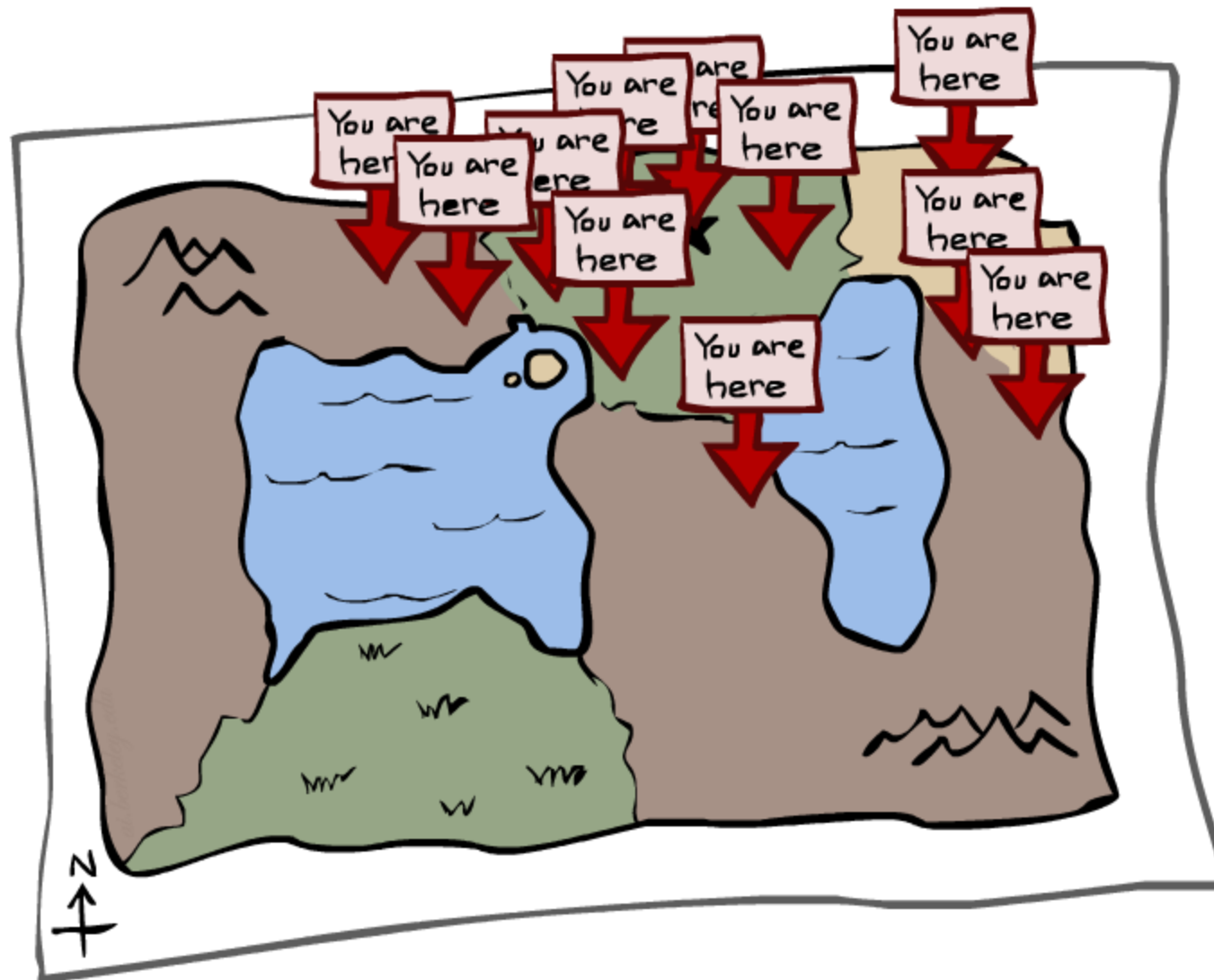


[Demo: Pacman – Sonar – No Beliefs(L14D1)]

# Video of Demo Pacman – Sonar (with beliefs)



# Particle Filtering



# Particle Filtering

- Filtering: approximate solution
- Sometimes  $|X|$  is too big to use exact inference
  - $|X|$  may be too big to even store  $B(X)$
  - E.g.  $X$  is continuous
- Solution: approximate inference
  - Track samples of  $X$ , not all values
  - Samples are called particles
  - Time per step is linear in the number of samples
  - But: number needed may be large
  - In memory: list of particles, not states
- This is how robot localization works in practice
- Particle is just new name for sample

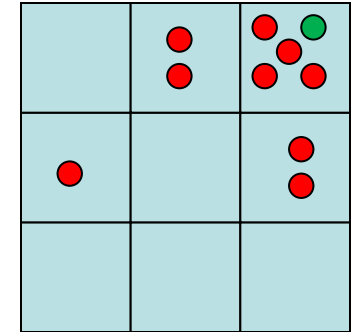
0.0	0.1	0.0
0.0	0.0	0.2
0.0	0.2	0.5



	●	
		● ●
	● ●	● ● ● ●

# Representation: Particles

- Our representation of  $P(X)$  is now a list of  $N$  particles (samples)
  - Generally,  $N \ll |X|$
  - Storing map from  $X$  to counts would defeat the point
- $P(x)$  approximated by number of particles with value  $x$ 
  - So, many  $x$  may have  $P(x) = 0$ !
  - More particles, more accuracy
- For now, all particles have a weight of 1



Particles:

(3,3)  
(2,3)  
(3,3)  
(3,2)  
(3,3)  
(3,2)  
(1,2)  
(3,3)  
(3,3)  
(2,3)



# Particle Filtering: Elapse Time

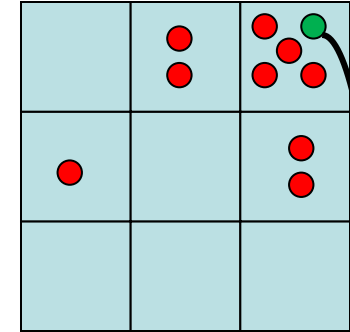
- Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

- This is like prior sampling – samples' frequencies reflect the transition probabilities
  - Here, most samples move clockwise, but some move in another direction or stay in place
- This captures the passage of time
    - If enough samples, close to exact values before and after (consistent)

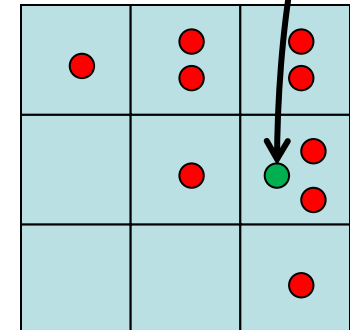
Particles:

(3,3)  
(2,3)  
(3,3)  
(3,2)  
(3,3)  
(3,2)  
(1,2)  
(3,3)  
(3,3)  
(2,3)



Particles:

(3,2)  
(2,3)  
(3,2)  
(3,1)  
(3,3)  
(3,2)  
(1,3)  
(2,3)  
(3,2)  
(2,2)



# Particle Filtering: Observe

- Slightly trickier:

- Don't sample observation, fix it
- Similar to likelihood weighting, downweight samples based on the evidence

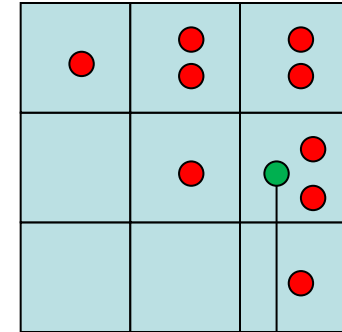
$$w(x) = P(e|x)$$

$$B(X) \propto P(e|X)B'(X)$$

- As before, the probabilities don't sum to one, since all have been downweighted (in fact they now sum to (N times) an approximation of  $P(e)$ )

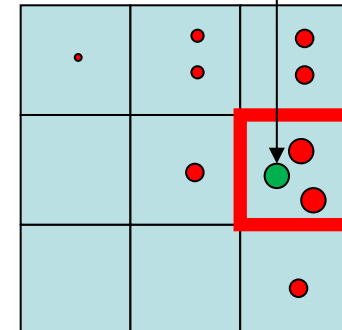
Particles:

(3,2)  
(2,3)  
(3,2)  
(3,1)  
(3,3)  
(3,2)  
(1,3)  
(2,3)  
(3,2)  
(2,2)



Particles:

(3,2) w=.9  
(2,3) w=.2  
(3,2) w=.9  
(3,1) w=.4  
(3,3) w=.4  
(3,2) w=.9  
(1,3) w=.1  
(2,3) w=.2  
(3,2) w=.9  
(2,2) w=.4



# Particle Filtering: Resample

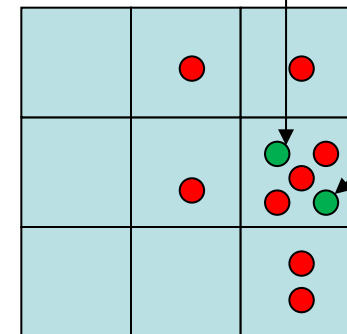
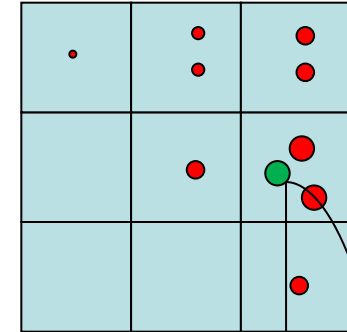
- Rather than tracking weighted samples, we resample
- N times, we choose from our weighted sample distribution (i.e. draw with replacement)
- This is equivalent to renormalizing the distribution
- Now the update is complete for this time step, continue with the next one

Particles:

(3,2) w=.9  
(2,3) w=.2  
(3,2) w=.9  
(3,1) w=.4  
(3,3) w=.4  
(3,2) w=.9  
(1,3) w=.1  
(2,3) w=.2  
(3,2) w=.9  
(2,2) w=.4

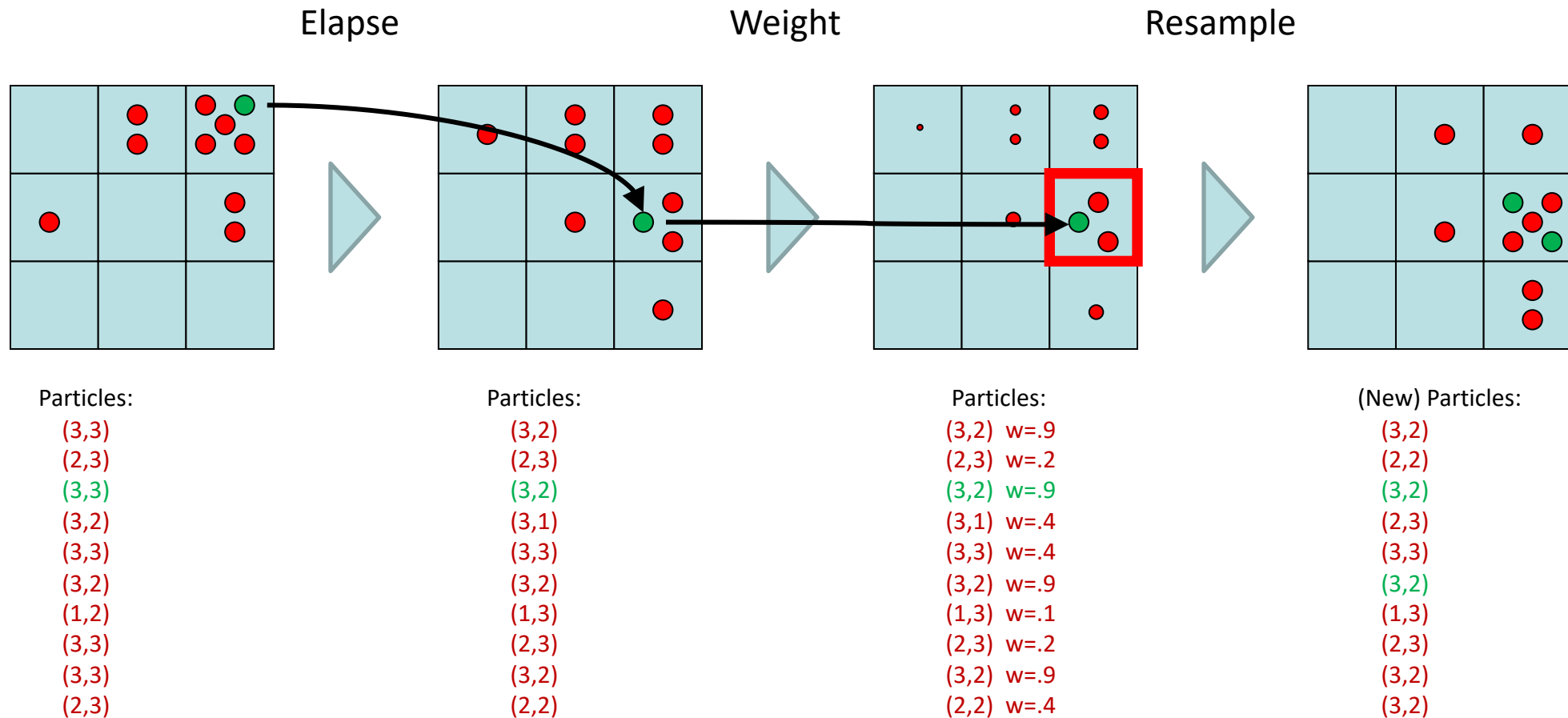
(New) Particles:

(3,2)  
(2,2)  
(3,2)  
(2,3)  
(3,3)  
(3,2)  
(1,3)  
(2,3)  
(3,2)  
(3,2)



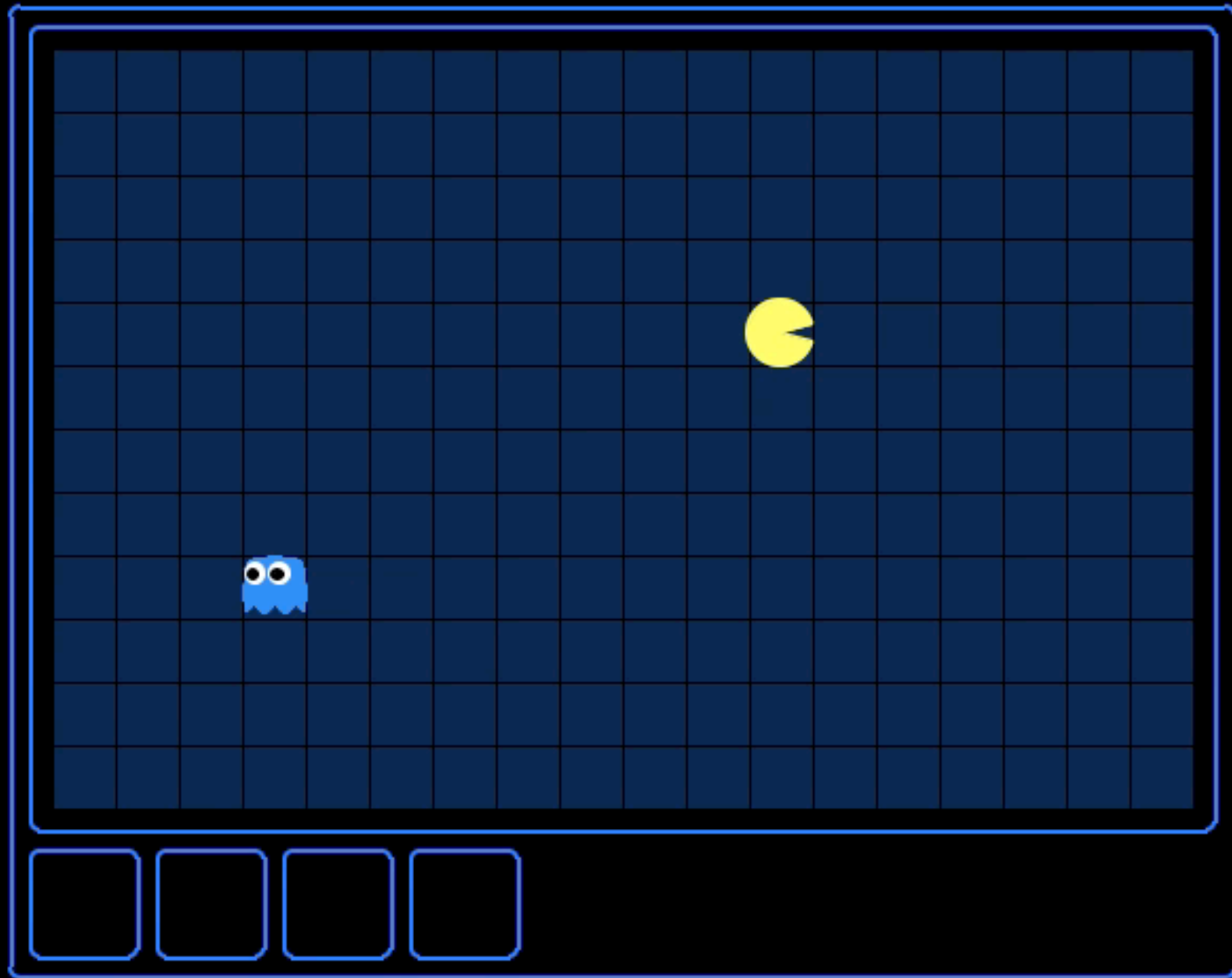
# Recap: Particle Filtering

- Particles: track samples of states rather than an explicit distribution



# Which Algorithm?

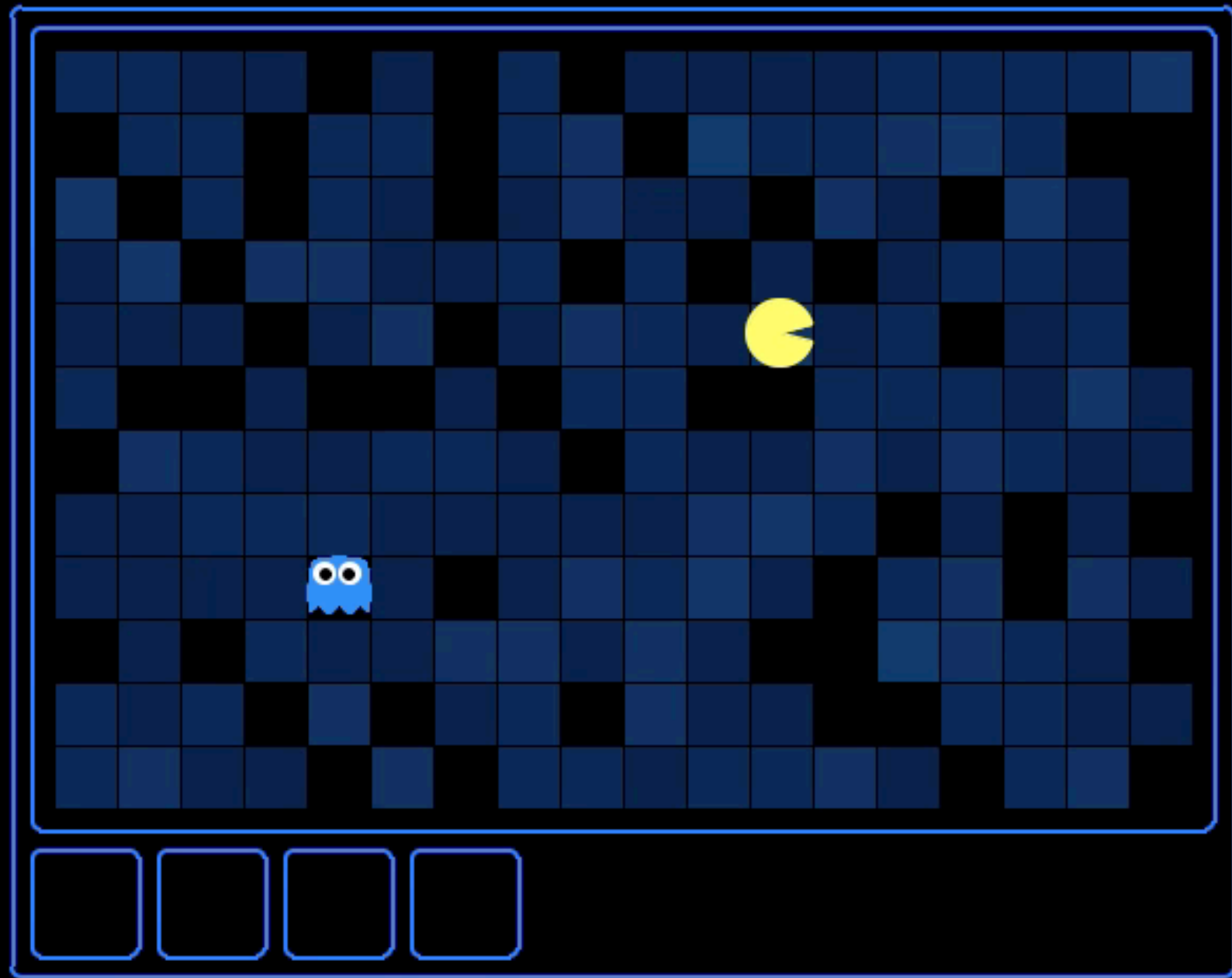
Exact filter,  
uniform initial  
beliefs



**SCORE: -1**

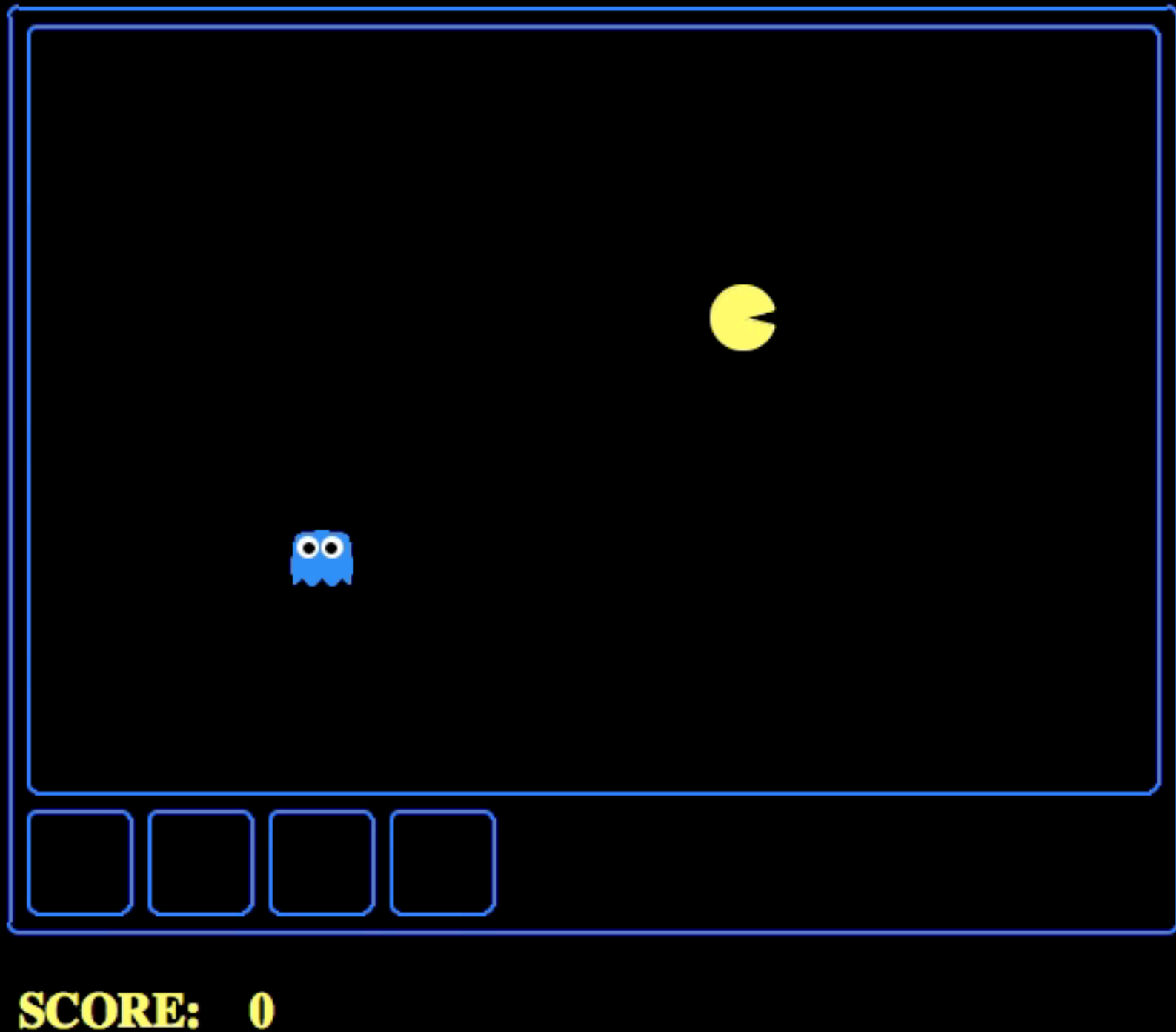
# Which Algorithm?

Particle filter,  
uniform initial  
beliefs, 300  
particles



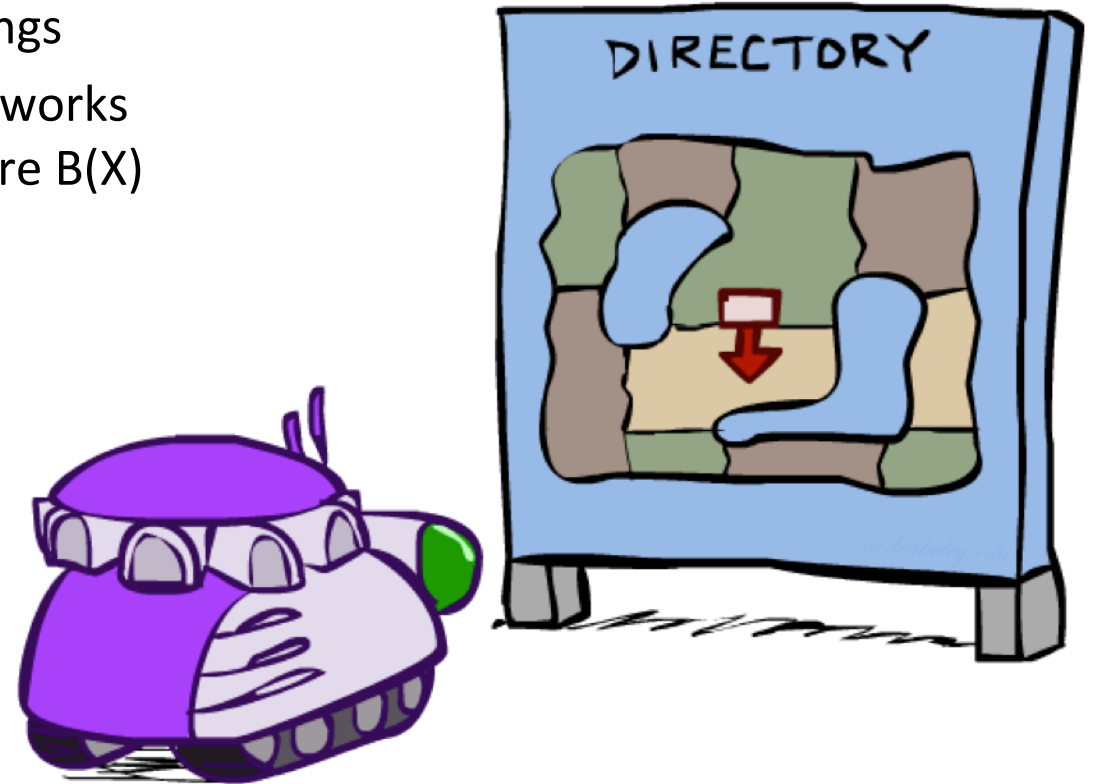
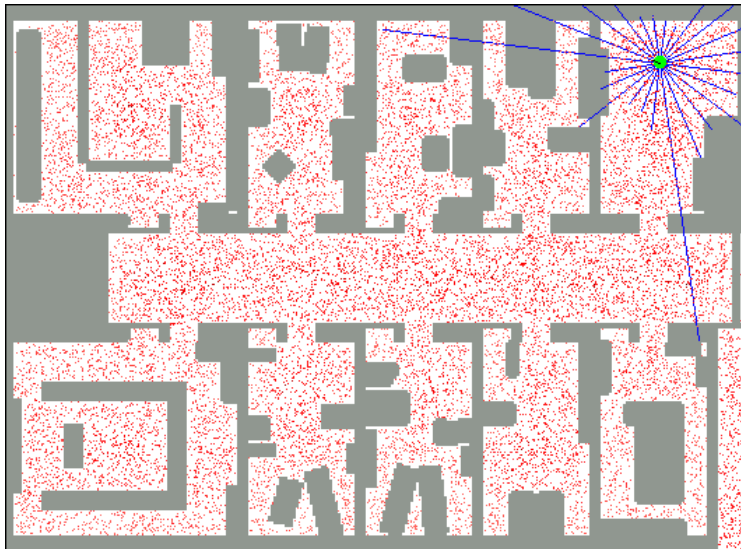
# Which Algorithm?

Particle filter,  
uniform initial  
beliefs, 25  
particles



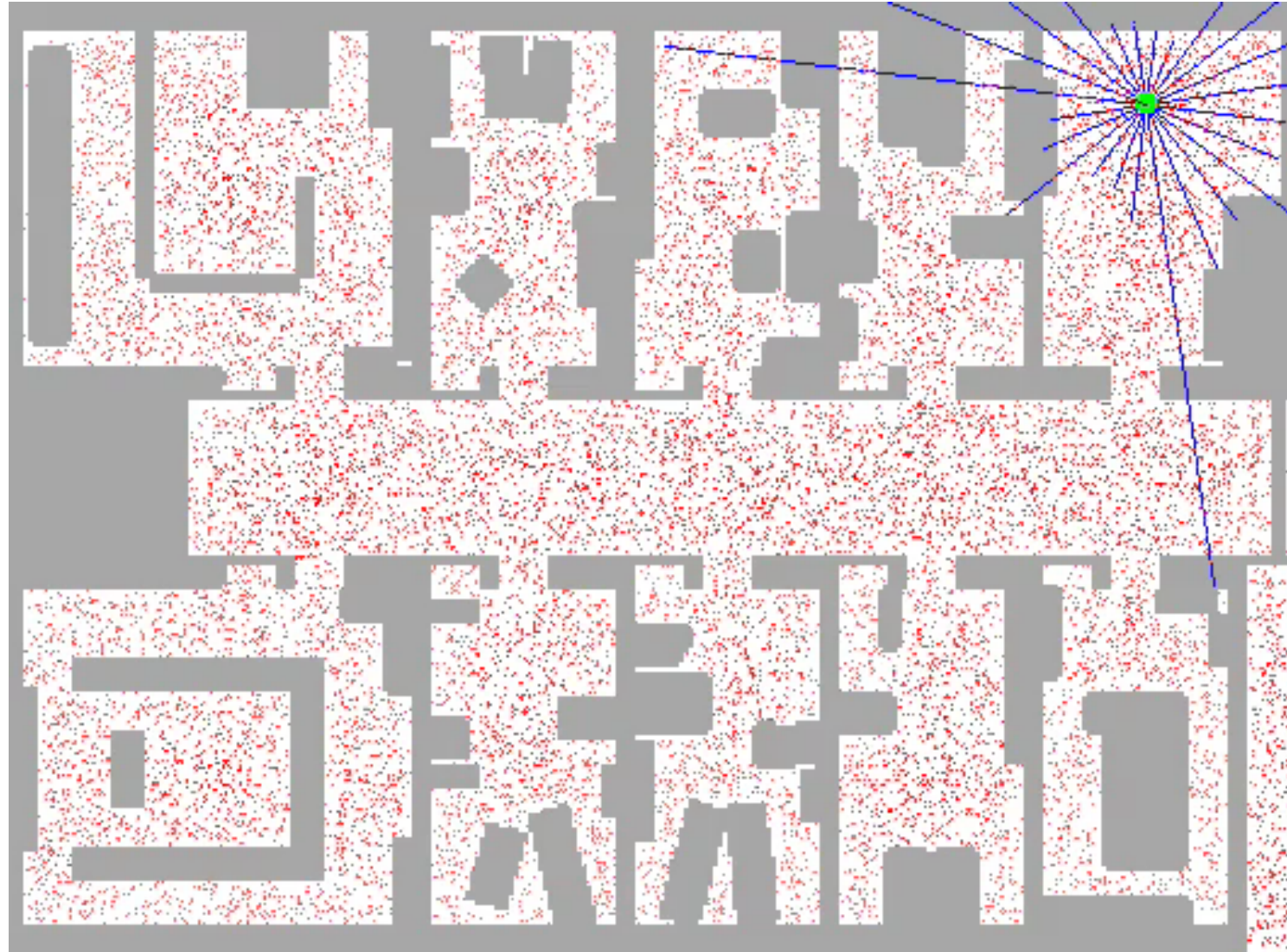
# Robot Localization

- In robot localization:
  - We know the map, but not the robot's position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store  $B(X)$
  - Particle filtering is a main technique

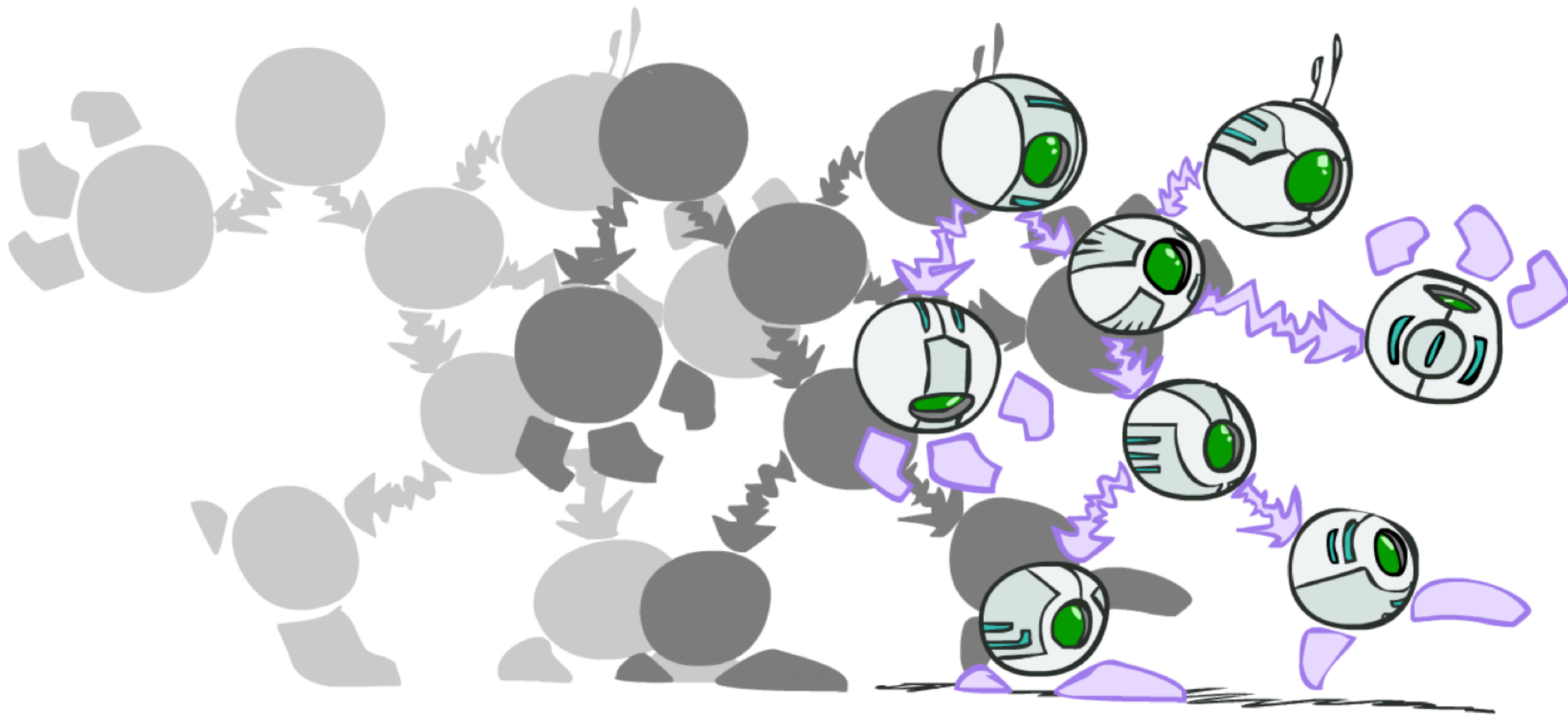




# Particle Filter Localization

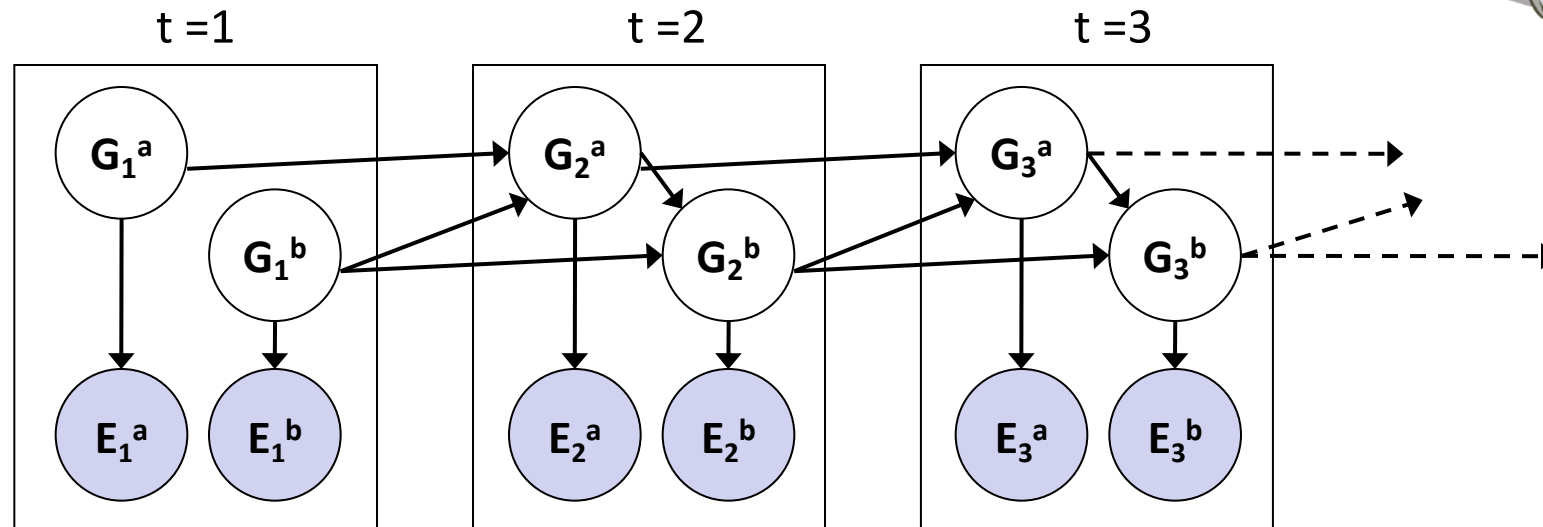
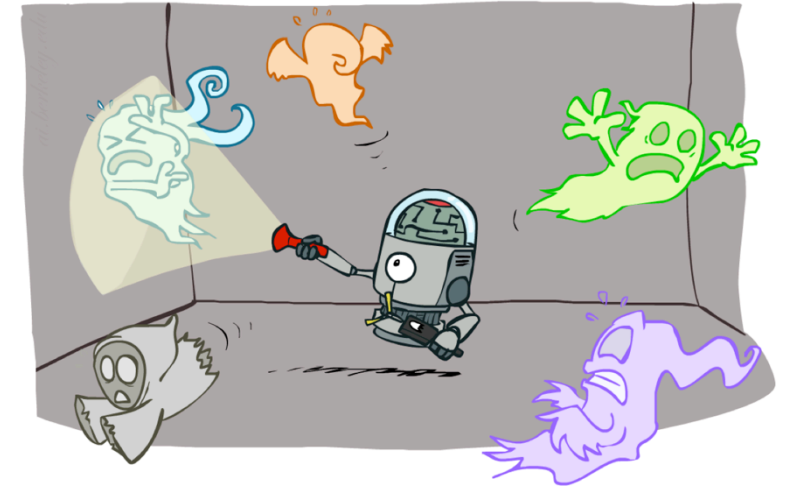


# Dynamic Bayes Nets



# Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net structure at each time
- Variables from time  $t$  can condition on those from  $t-1$



- Dynamic Bayes nets are a generalization of HMMs

# DBN Particle Filters

---

- A particle is a complete sample for a time step
- **Initialize:** Generate prior samples for the  $t=1$  Bayes net
  - Example particle:  $\mathbf{G}_1^a = (3,3)$   $\mathbf{G}_1^b = (5,3)$
- **Elapse time:** Sample a successor for each particle
  - Example successor:  $\mathbf{G}_2^a = (2,3)$   $\mathbf{G}_2^b = (6,3)$
- **Observe:** Weight each entire sample by the likelihood of the evidence conditioned on the sample
  - Likelihood:  $P(\mathbf{E}_1^a | \mathbf{G}_1^a) * P(\mathbf{E}_1^b | \mathbf{G}_1^b)$
- **Resample:** Select prior samples (tuples of values) in proportion to their likelihood