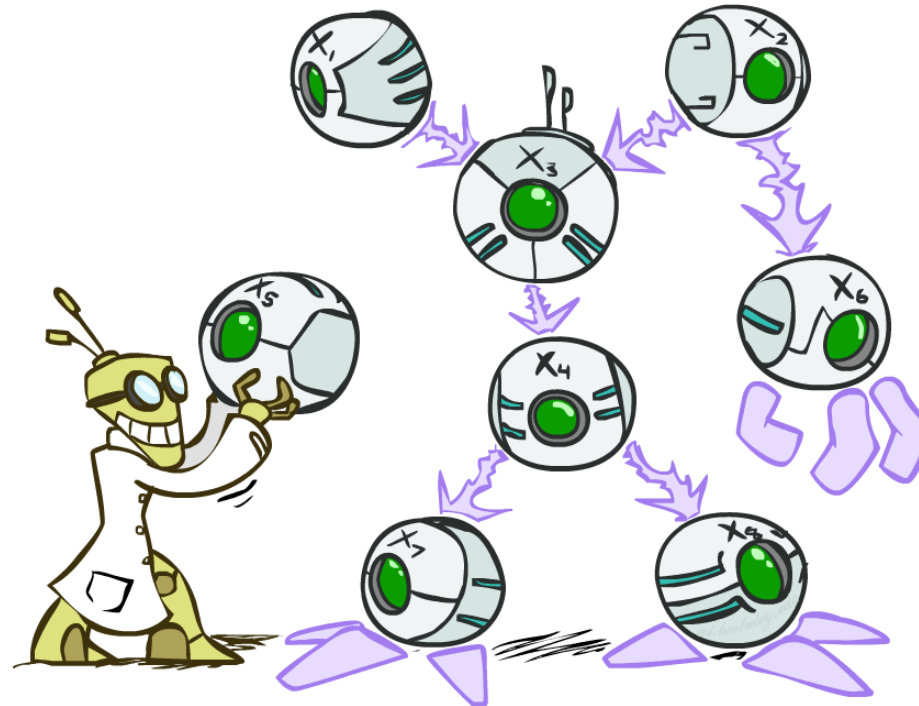


CSE 473: Artificial Intelligence

Bayes' Nets



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Probabilistic Models

- Models describe how (a portion of) the world works
- **Models are always simplifications**
 - May not account for every variable
 - May not account for all interactions between variables
 - “All models are wrong; but some are useful.”
 - George E. P. Box
- What do we do with probabilistic models?
 - We (or our agents) need to reason about unknown variables, given evidence
 - Example: explanation (diagnostic reasoning)
 - Example: prediction (causal reasoning)
 - Example: value of information



Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots$

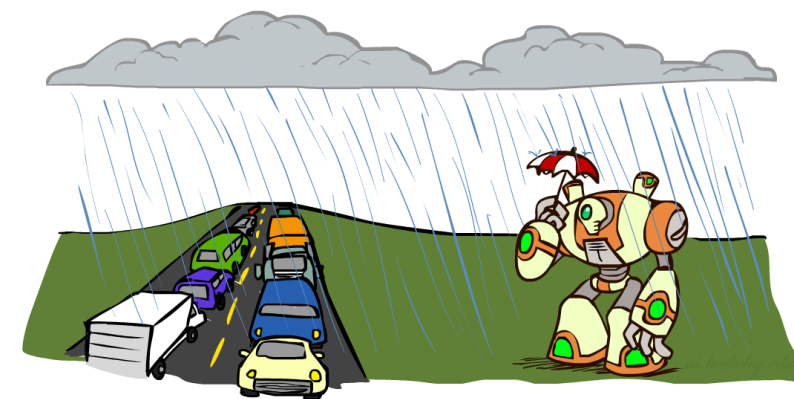
- Trivial decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = \\ P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = \\ P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

- Bayes' nets / graphical models help us express conditional independence assumptions



Ghostbusters Chain Rule

- Each sensor depends only on where the ghost is
- That means, the two sensors are conditionally independent, given the ghost position
- T: Top square is red
B: Bottom square is red
G: Ghost is in the top

- Givens:

$$P(+g) = 0.5$$

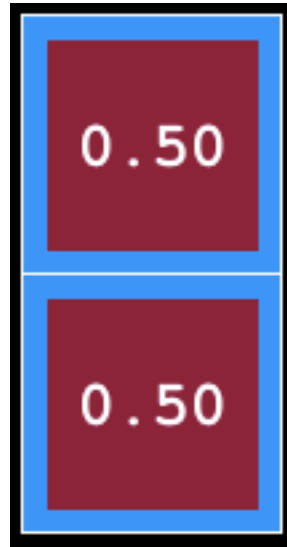
$$P(-g) = 0.5$$

$$P(+t \mid +g) = 0.8$$

$$P(+t \mid -g) = 0.4$$

$$P(+b \mid +g) = 0.4$$

$$P(+b \mid -g) = 0.8$$

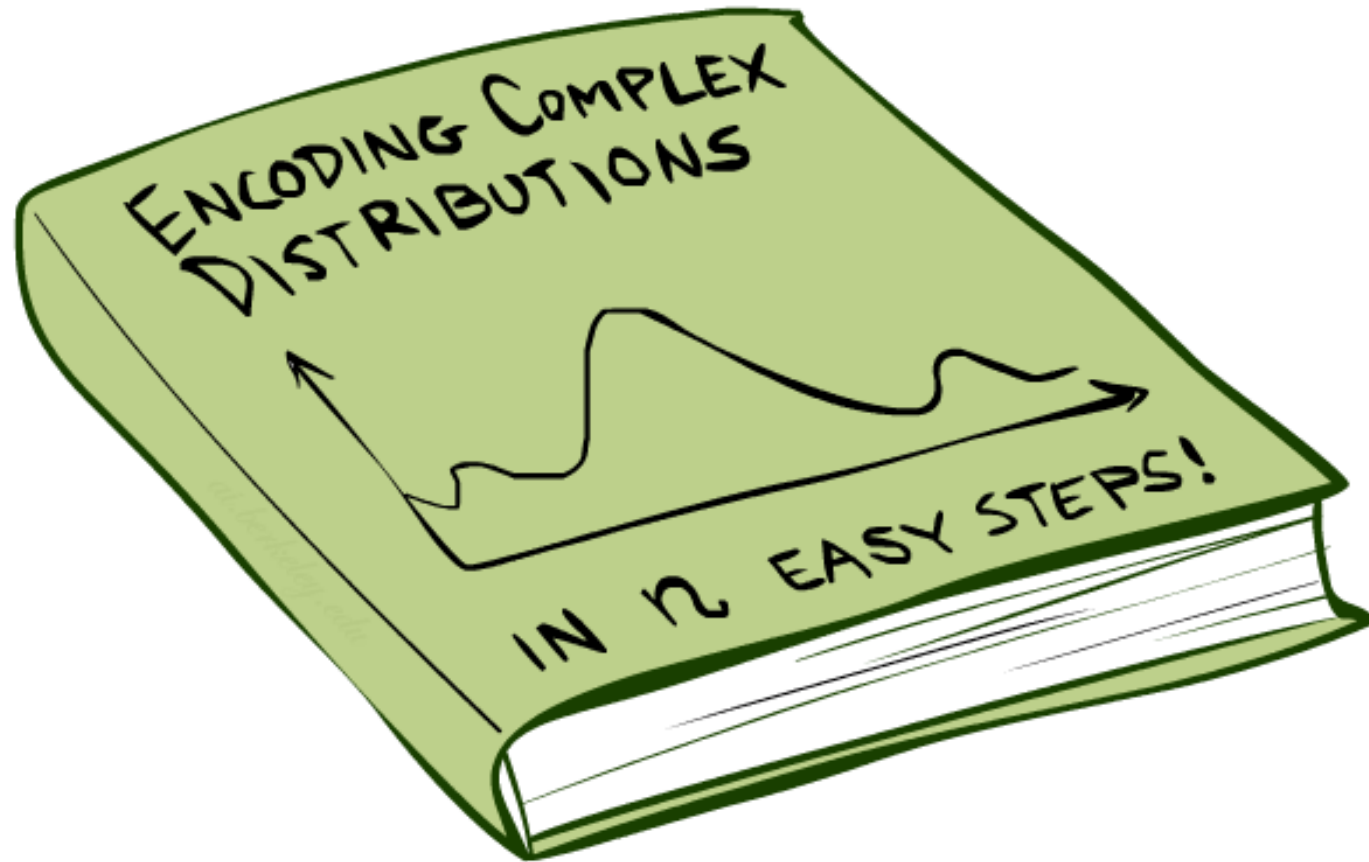


$$P(T,B,G) = P(G) P(T \mid G) P(B \mid G)$$

T	B	G	P(T,B,G)
+t	+b	+g	0.16
+t	+b	-g	0.16
+t	-b	+g	0.24
+t	-b	-g	0.04
-t	+b	+g	0.04
-t	+b	-g	0.24
-t	-b	+g	0.06
-t	-b	-g	0.06

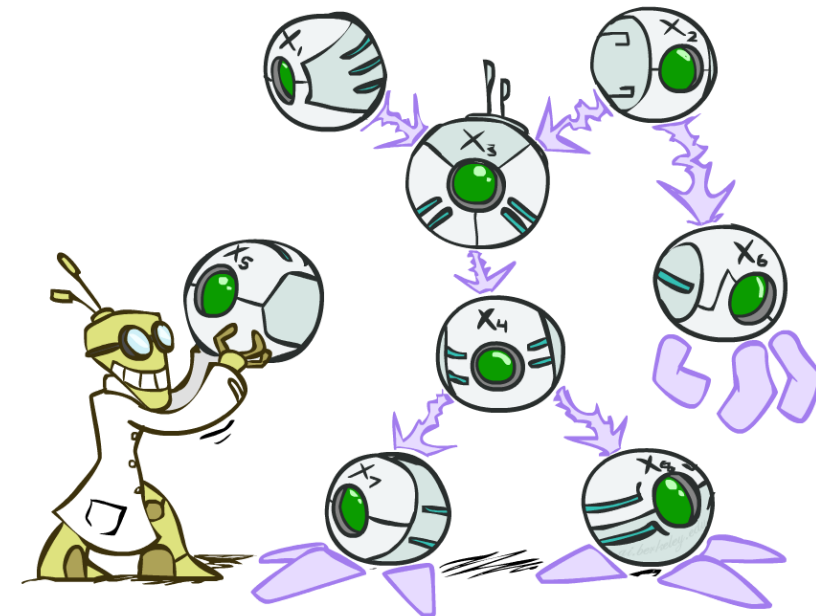


Bayes' Nets: Big Picture

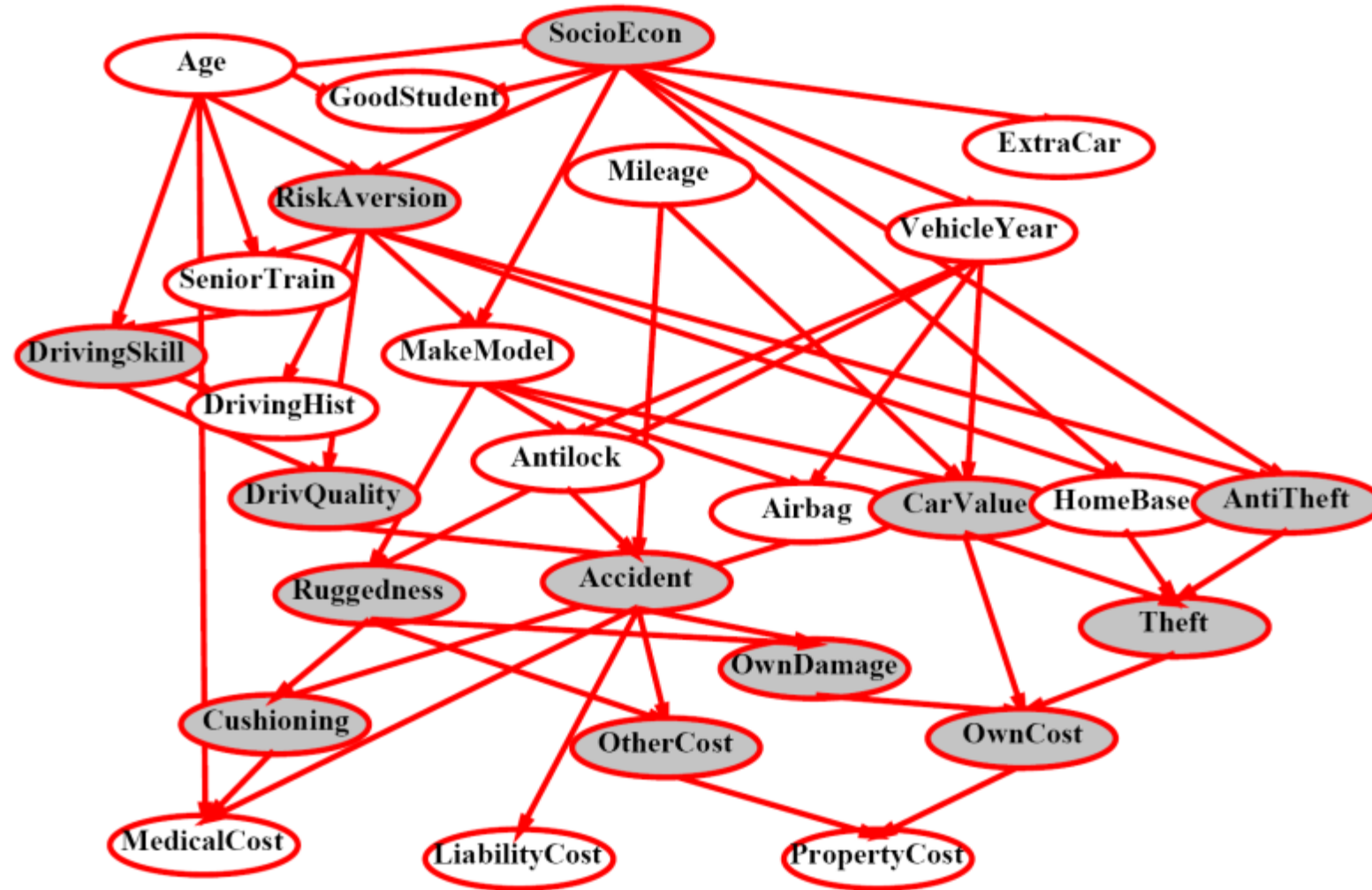


Bayes' Nets: Big Picture

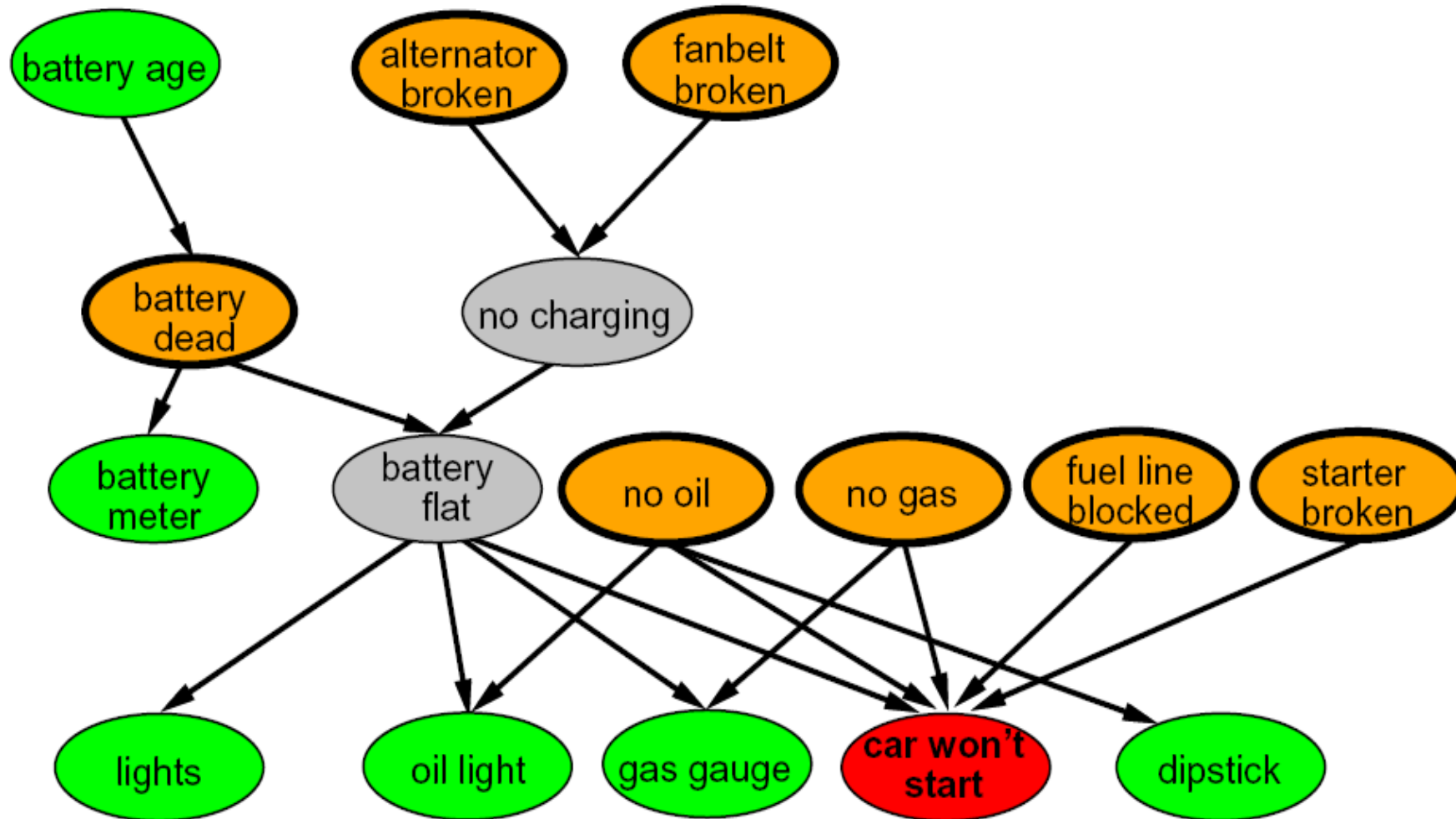
- Two problems with using full joint distribution tables as our probabilistic models:
 - Unless there are only a few variables, the joint is WAY too big to represent explicitly
 - Hard to learn (estimate) anything empirically about more than a few variables at a time
- **Bayes' nets:** a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
 - More properly called **graphical models**
 - We describe how variables locally interact
 - Local interactions chain together to give global, indirect interactions
 - For about 10 min, we'll be vague about how these interactions are specified



Example Bayes' Net: Insurance

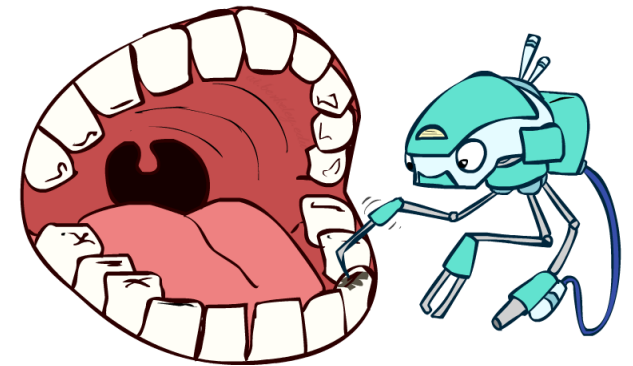
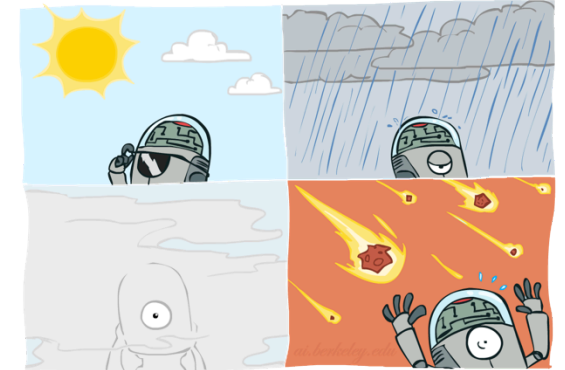
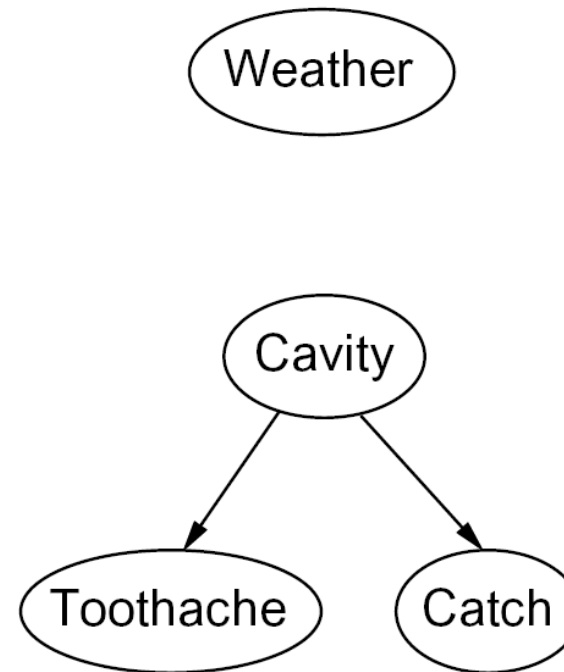


Example Bayes' Net: Car



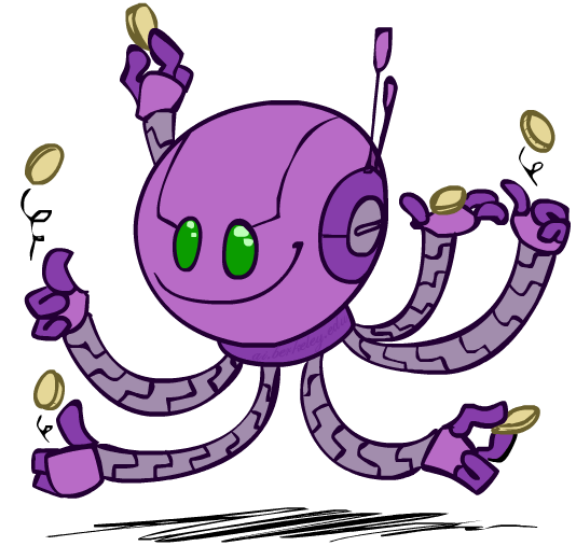
Graphical Model Notation

- **Nodes: variables (with domains)**
 - Can be assigned (observed) or unassigned (unobserved)
- **Arcs: interactions**
 - Indicate “direct influence” between variables
 - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don't!)



Example: Coin Flips

- N independent coin flips



- No interactions between variables: **absolute independence**

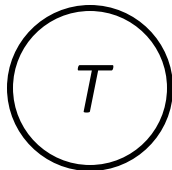
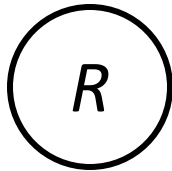
Example: Traffic

- Variables:

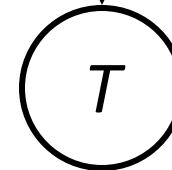
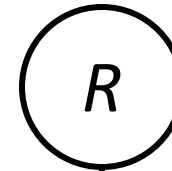
- R: It rains
- T: There is traffic



- Model 1: independence



- Model 2: rain causes traffic

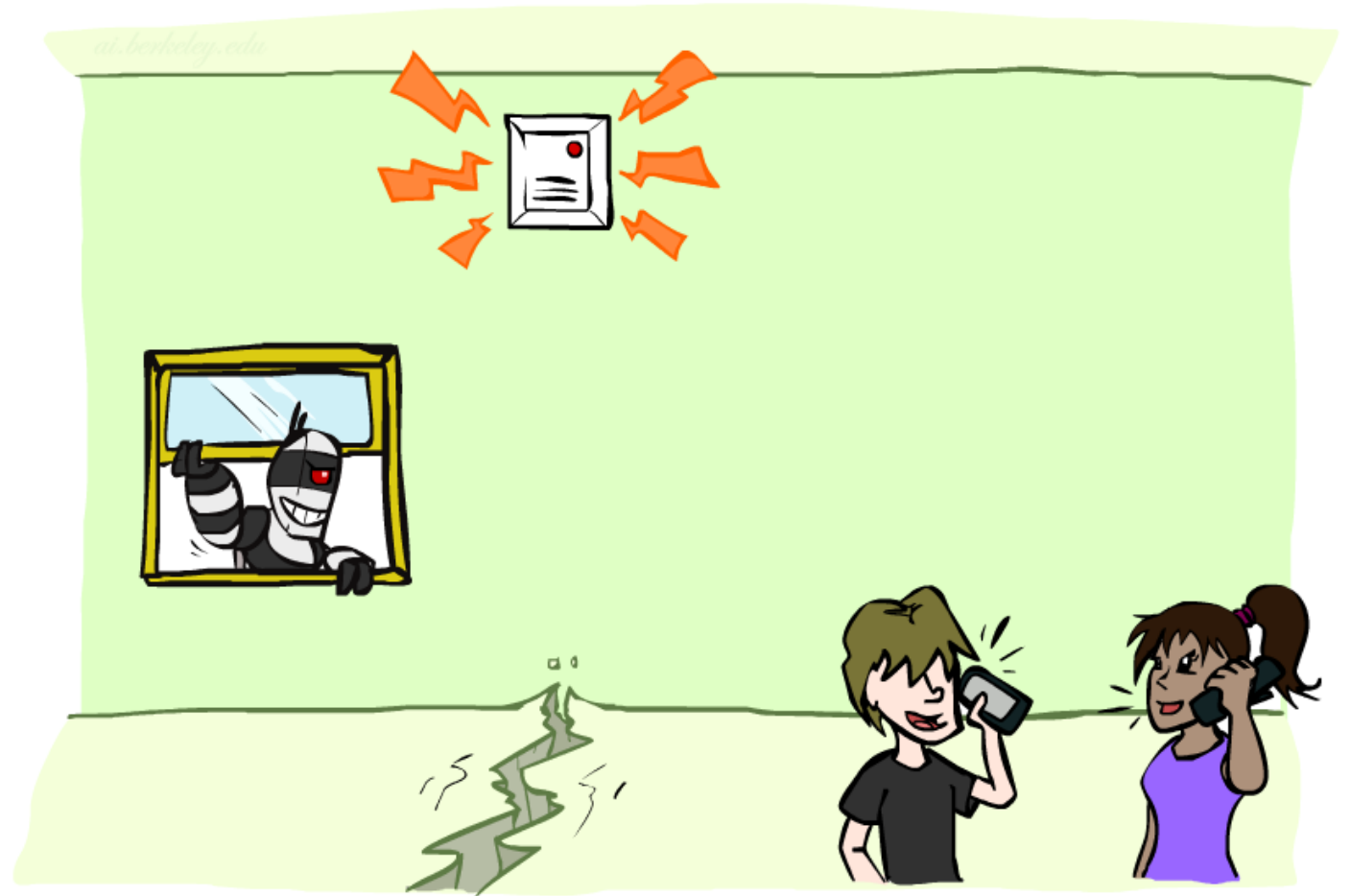


- Why is an agent using model 2 better?

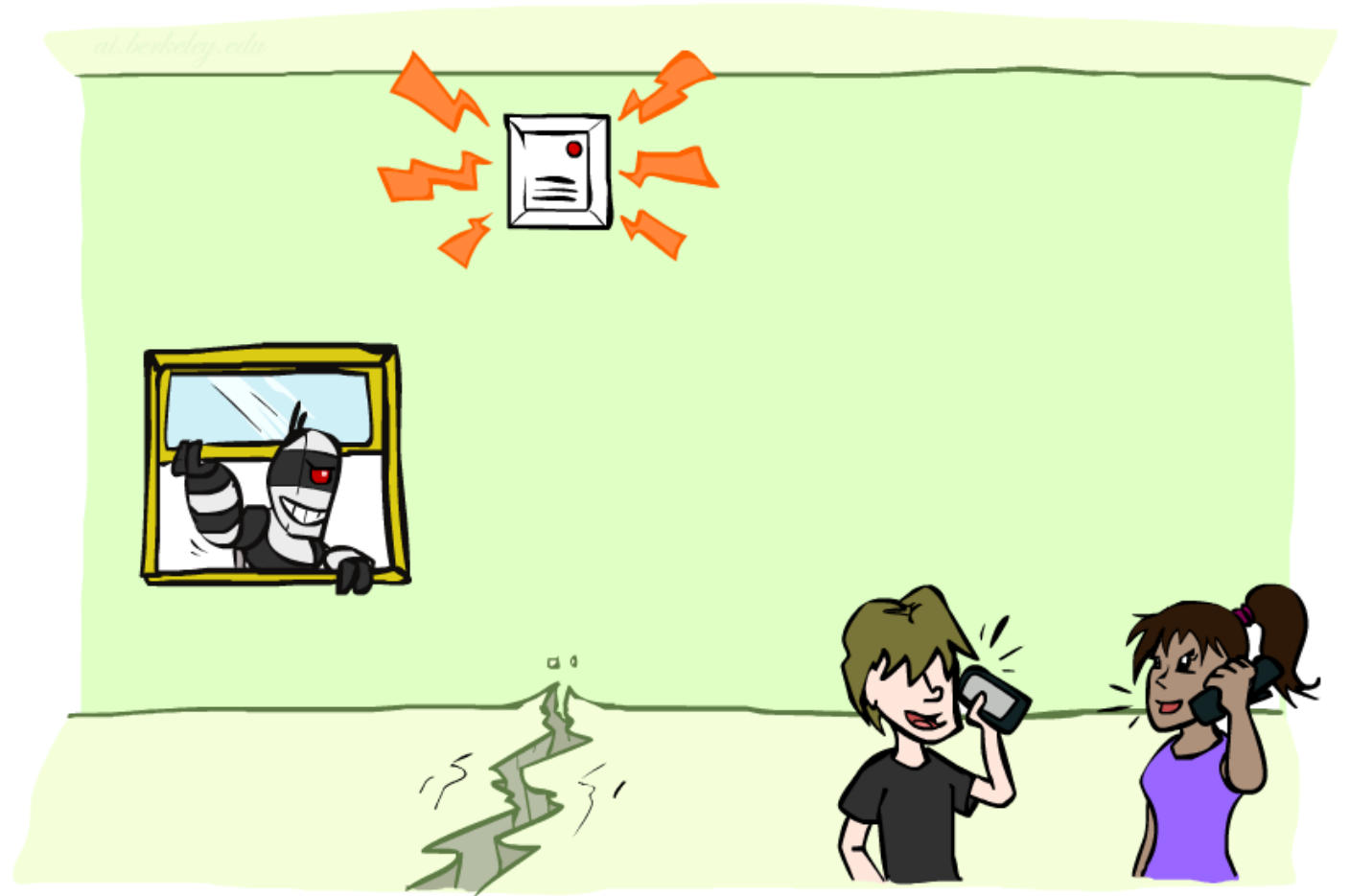
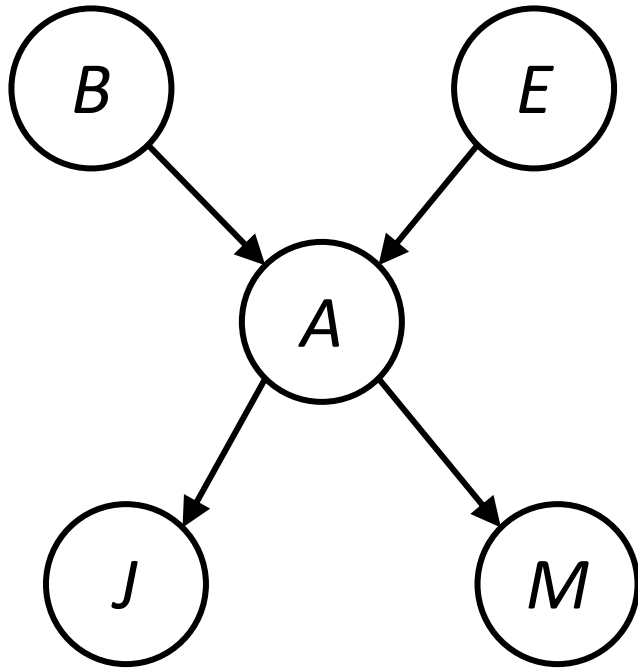
Example: Alarm Network

- Variables

- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!

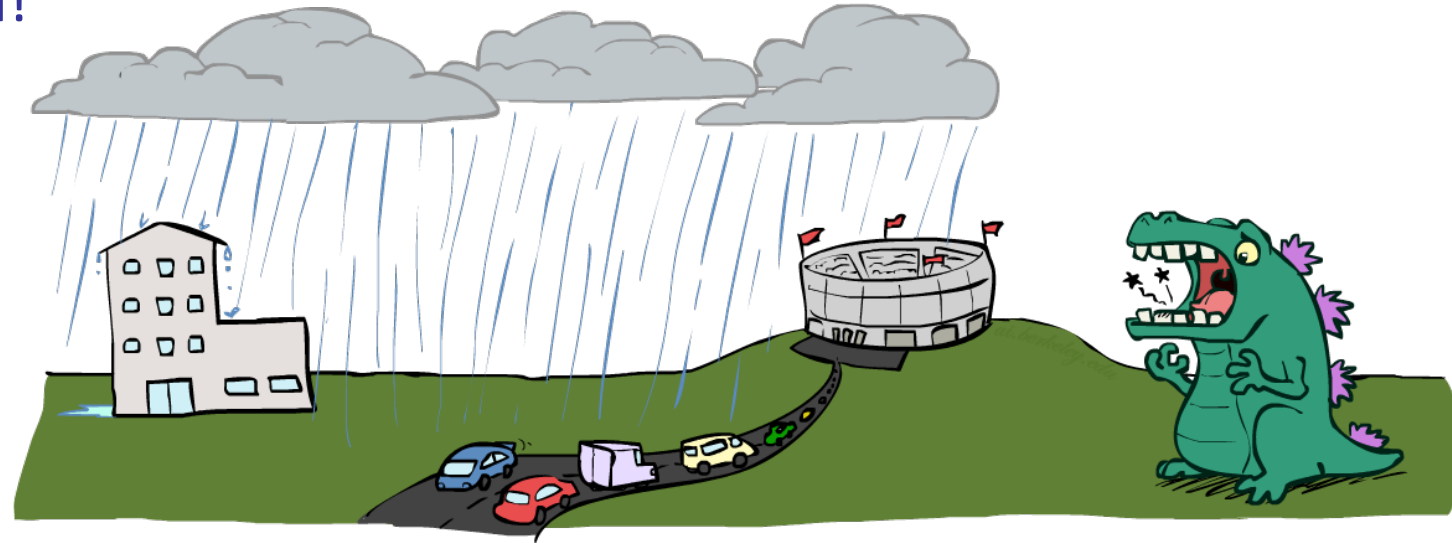


Example: Alarm Network



Example: Traffic II

- Let's build a causal graphical model!
- Variables
 - T: Traffic
 - R: It rains
 - L: Low pressure
 - D: Roof drips
 - B: Ballgame
 - C: Cavity

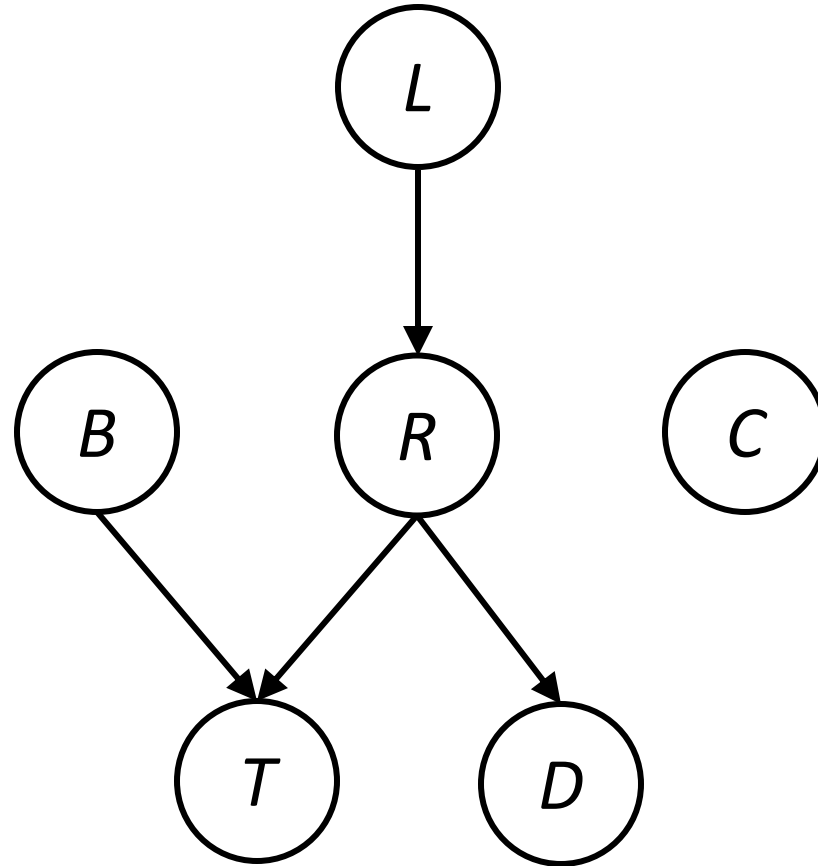


Example: Traffic II

- Let's build a causal graphical model!

- Variables

- T: Traffic
- R: It rains
- L: Low pressure
- D: Roof drips
- B: Ballgame
- C: Cavity



Bayes' Net Semantics



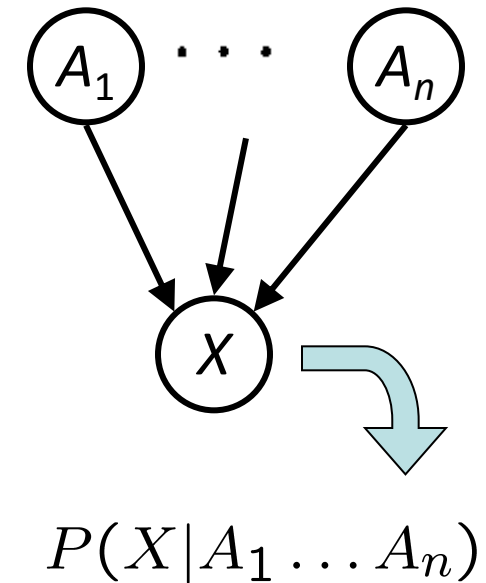
Bayes' Net Semantics



- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
 - A collection of distributions over X , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

- CPT: conditional probability table
- Description of a noisy “causal” process



A Bayes net = Topology (graph) + Local Conditional Probabilities

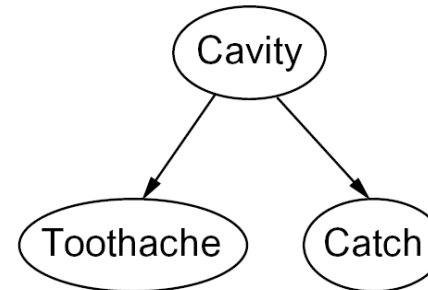
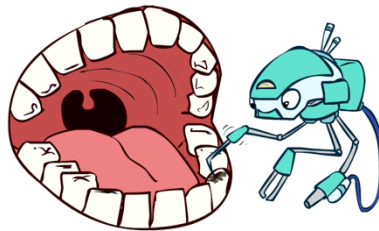
Probabilities in BNs



- Bayes' nets **implicitly** encode joint distributions
 - As a product of local conditional distributions
 - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

- Example:



$$P(+cavity, +catch, -toothache)$$

Probabilities in BNs



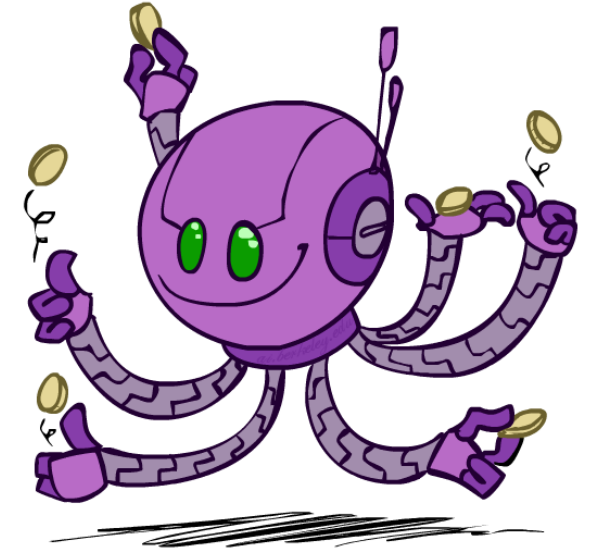
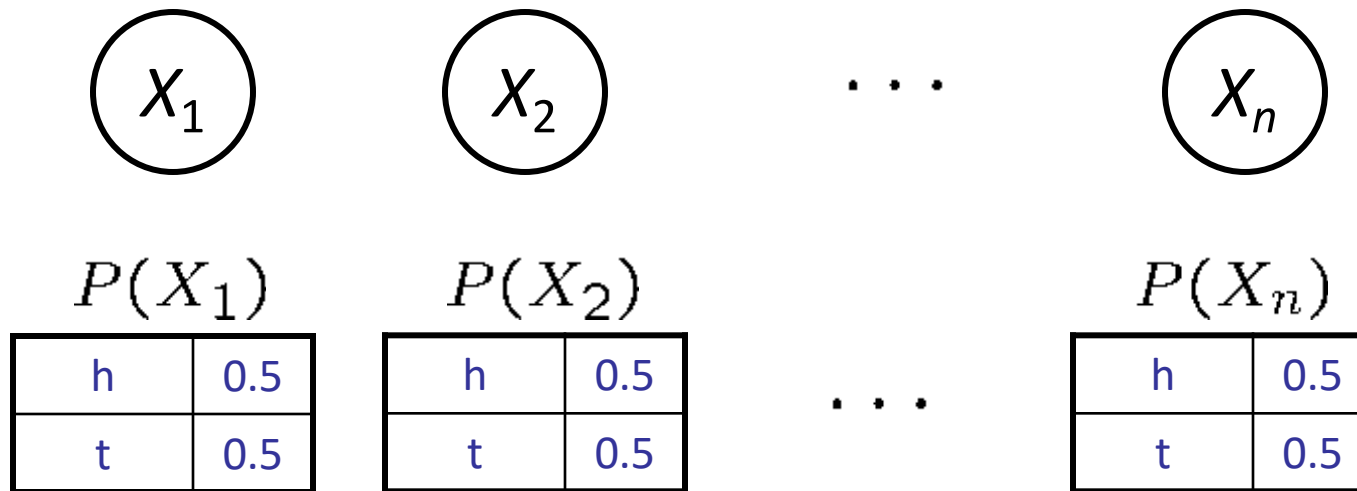
- Why are we guaranteed that setting

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

results in a proper joint distribution?

- Chain rule (valid for all distributions): $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | x_1 \dots x_{i-1})$
- Assume conditional independences: $P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$
 - Consequence: $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$
- Not every BN can represent every joint distribution
 - The topology enforces certain conditional independencies

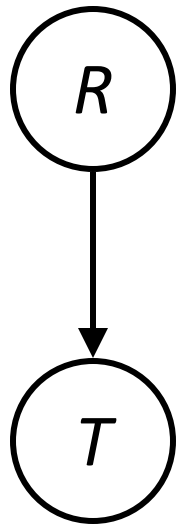
Example: Coin Flips



$$P(h, h, t, h) =$$

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic

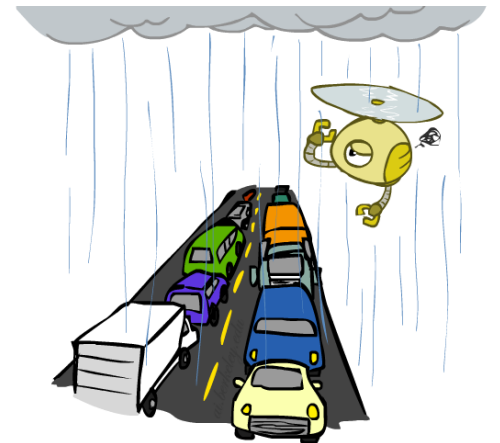
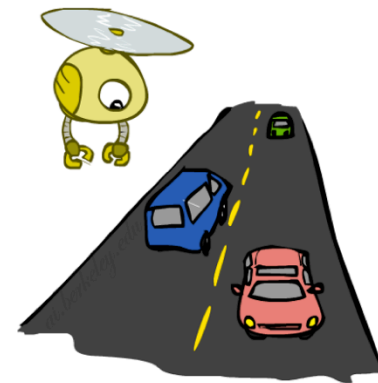

$$P(R)$$

$+r$	$1/4$
$-r$	$3/4$

$$P(T|R)$$

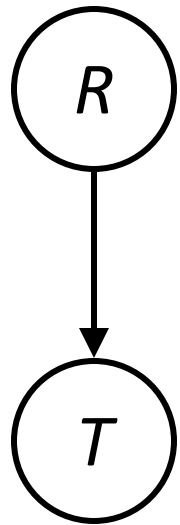
$+r$	$+t$	$3/4$
	$-t$	$1/4$
$-r$	$+t$	$1/2$
	$-t$	$1/2$

$$P(+r, -t) =$$



Example: Traffic

- Causal direction



$P(R)$

+r	1/4
-r	3/4

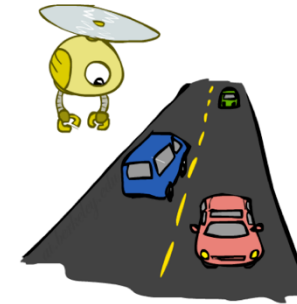
$P(T|R)$

+r	+t	3/4
	-t	1/4

-r	+t	1/2
	-t	1/2

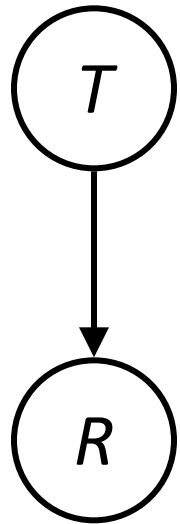
$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16



Example: Reverse Traffic

- Reverse causality?



$P(T)$

+t	9/16
-t	7/16

$P(R|T)$

+t	+r	1/3
	-r	2/3

-t	+r	1/7
	-r	6/7

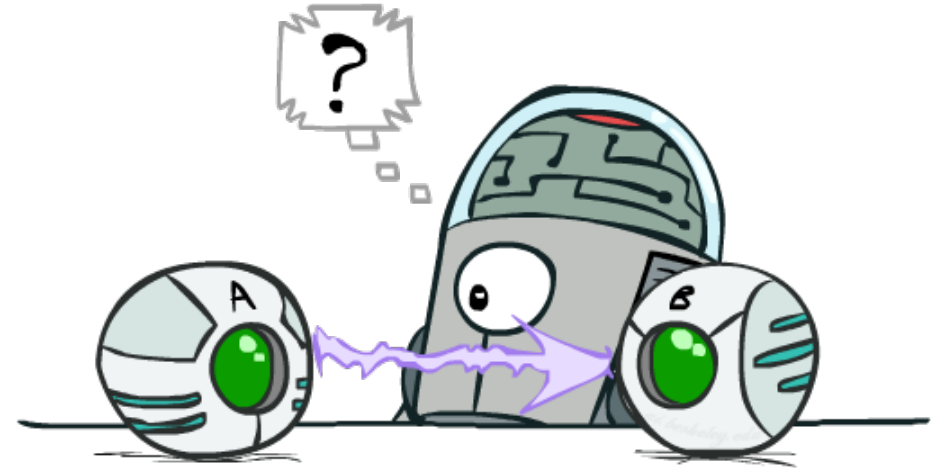


$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Causality?

- When Bayes' nets reflect the true causal patterns:
 - Often simpler (nodes have fewer parents)
 - Often easier to think about
 - Often easier to elicit from experts
- BNs need not actually be causal
 - Sometimes no causal net exists over the domain (especially if variables are missing)
 - E.g. consider the variables *Traffic* and *Drips*
 - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
 - Topology may happen to encode causal structure
 - **Topology really encodes conditional independence**
$$P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$$



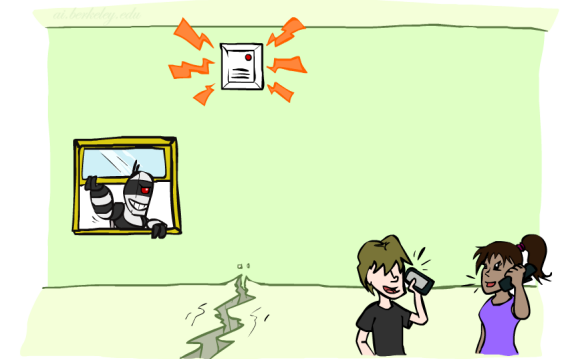
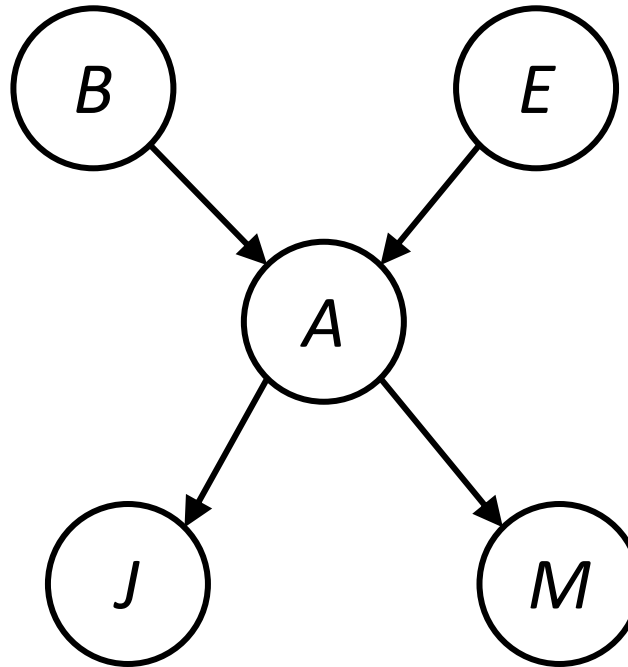
Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999

E	P(E)
+e	0.002
-e	0.998

A	J	P(J A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99



$$P(+b, -e, +a, -j, +m) =$$

B	E	A	P(A B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

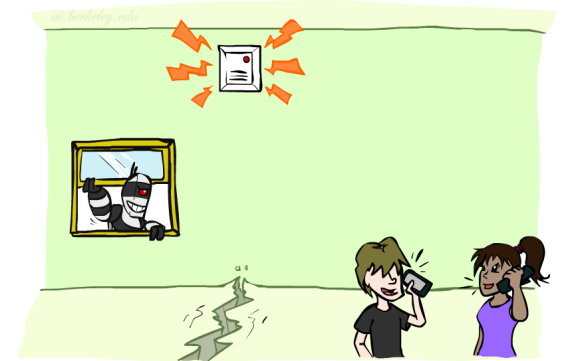
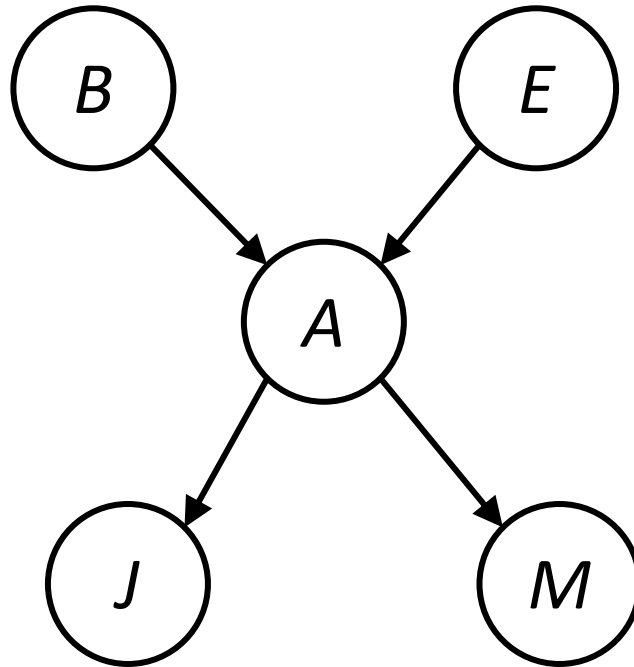
Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999

E	P(E)
+e	0.002
-e	0.998

A	J	P(J A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99



B	E	A	P(A B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$$\begin{aligned}
 P(+b, -e, +a, -j, +m) &= \\
 P(+b)P(-e)P(+a|+b, -e)P(-j|+a)P(+m|+a) &= \\
 0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7 &
 \end{aligned}$$

Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?

$$2^N$$

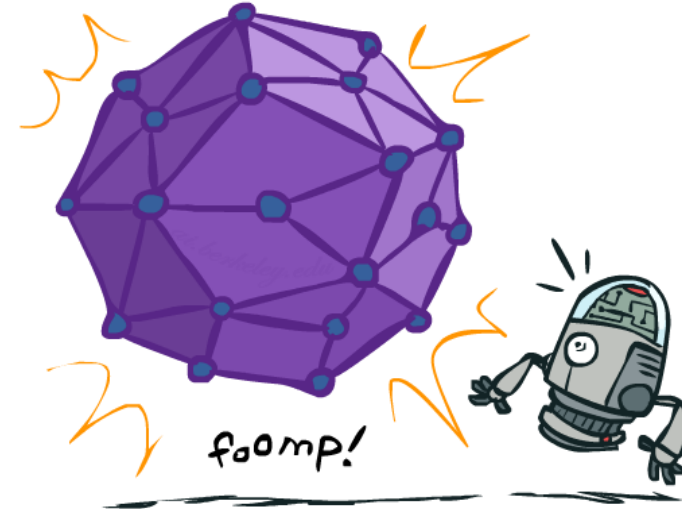
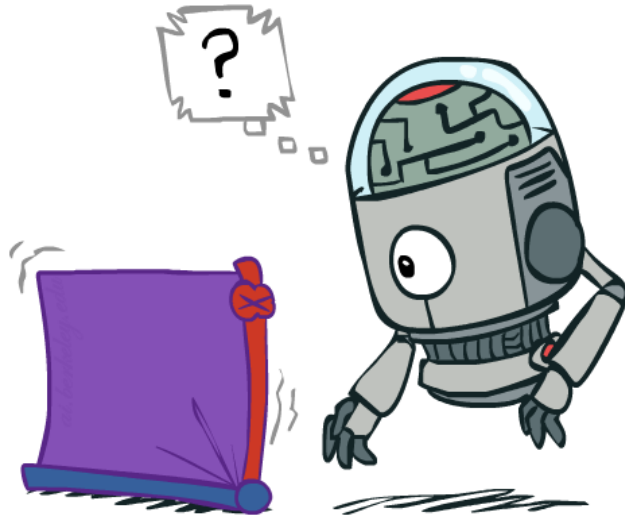
- How big is an N -node net if nodes have up to k parents?

$$O(N * 2^{k+1})$$

- Both give you the power to calculate

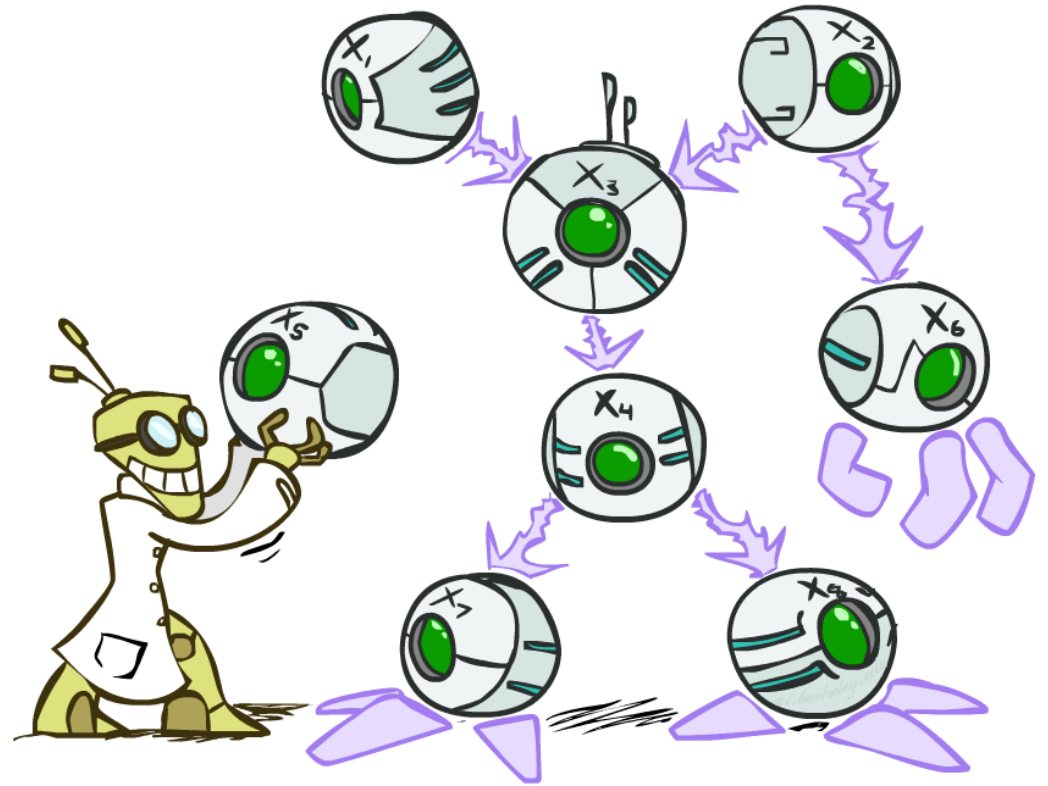
$$P(X_1, X_2, \dots, X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)



Bayes' Nets

- So far: how a Bayes' net encodes a joint distribution
- Next: how to answer queries about that distribution
 - Today:
 - First assembled BNs using an intuitive notion of conditional independence as causality
 - Then saw that key property is conditional independence
 - Main goal: answer queries about conditional independence and influence
- After that: how to answer numerical queries (inference)



Bayes' Nets

Representation

- Conditional Independences
- Probabilistic Inference
- Learning Bayes' Nets from Data

Conditional Independence

- X and Y are **independent** if

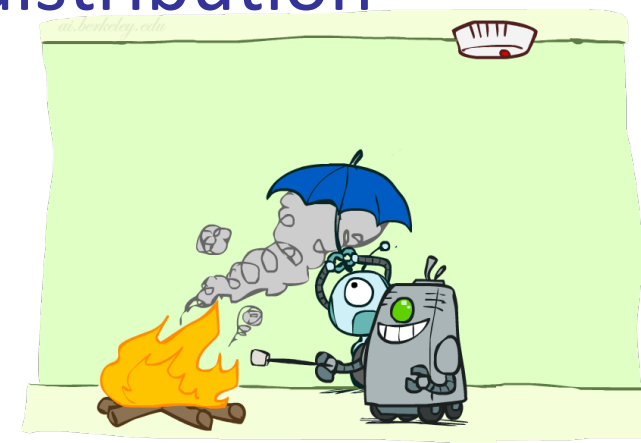
$$\forall x, y \quad P(x, y) = P(x)P(y) \quad \text{---} \rightarrow \quad X \perp Y$$

- X and Y are **conditionally independent** given Z

$$\forall x, y, z \quad P(x, y|z) = P(x|z)P(y|z) \quad \text{---} \rightarrow \quad X \perp Y|Z$$

- (Conditional) independence is a property of a distribution

- Example: $Alarm \perp Fire|Smoke$



Bayes Nets: Assumptions

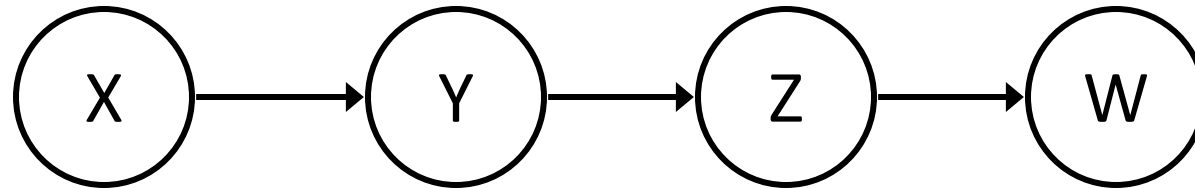
- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i | x_1 \cdots x_{i-1}) = P(x_i | \text{parents}(X_i))$$

- Beyond above “chain rule → Bayes net” conditional independence assumptions
 - Often additional conditional independences
 - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



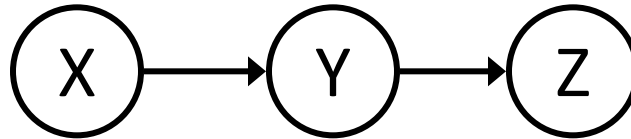
Example



- Conditional independence assumptions directly from simplifications in chain rule:
- Additional implied conditional independence assumptions?

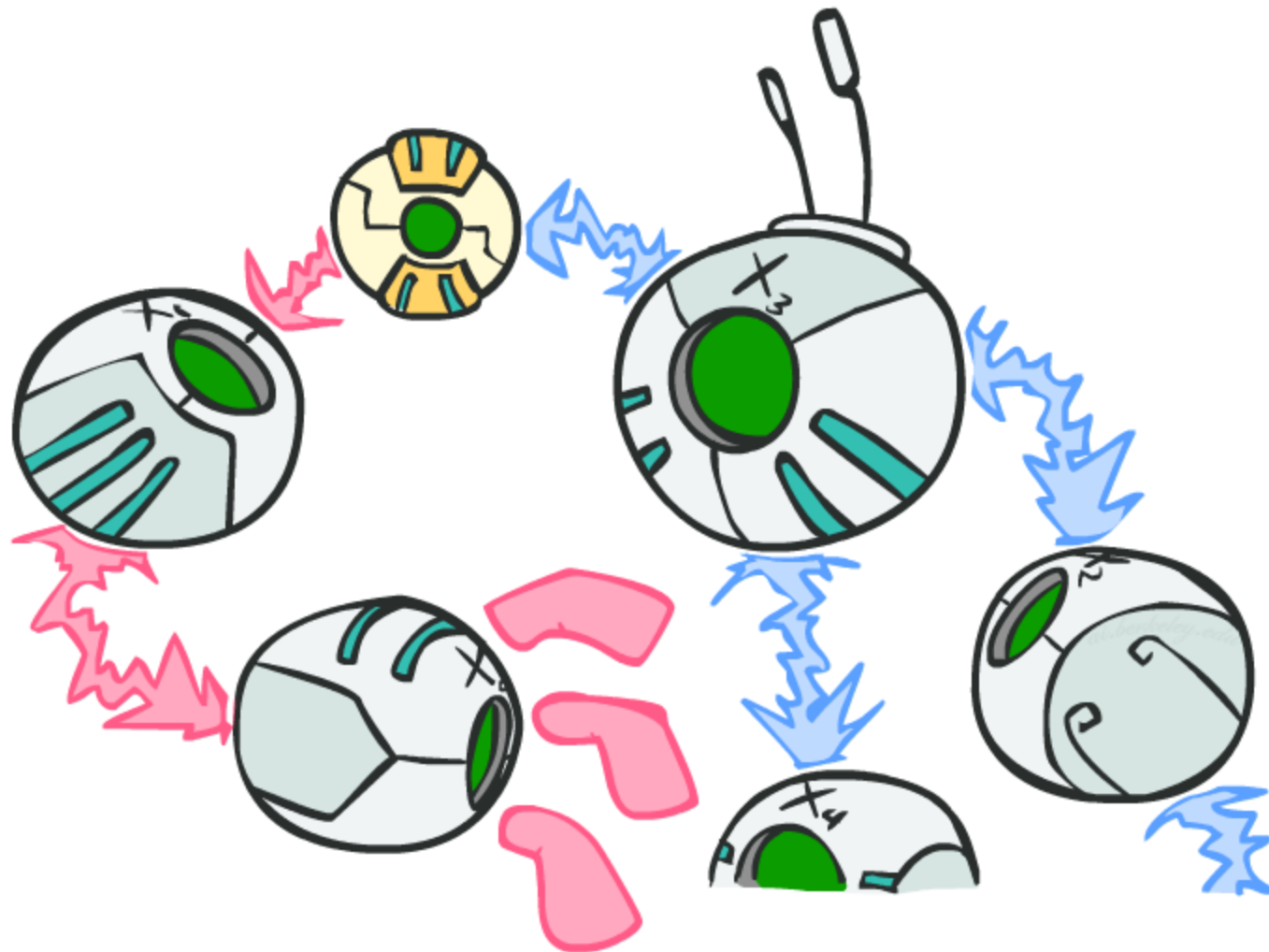
Independence in a BN

- Important question about a BN:
 - Are two nodes independent given certain evidence?
 - If yes, can prove using algebra (tedious in general)
 - If no, can prove with a counter example
 - Example:



- Question: are X and Z necessarily independent?
 - Answer: no. Example: low pressure causes rain, which causes traffic.
 - X can influence Z, Z can influence X (via Y)
 - Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

Causal Chains

- This configuration is a “causal chain”



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

- Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic

- In numbers:

$$P(+y \mid +x) = 1, P(-y \mid -x) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

Causal Chains

- This configuration is a “causal chain”

- Guaranteed X independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

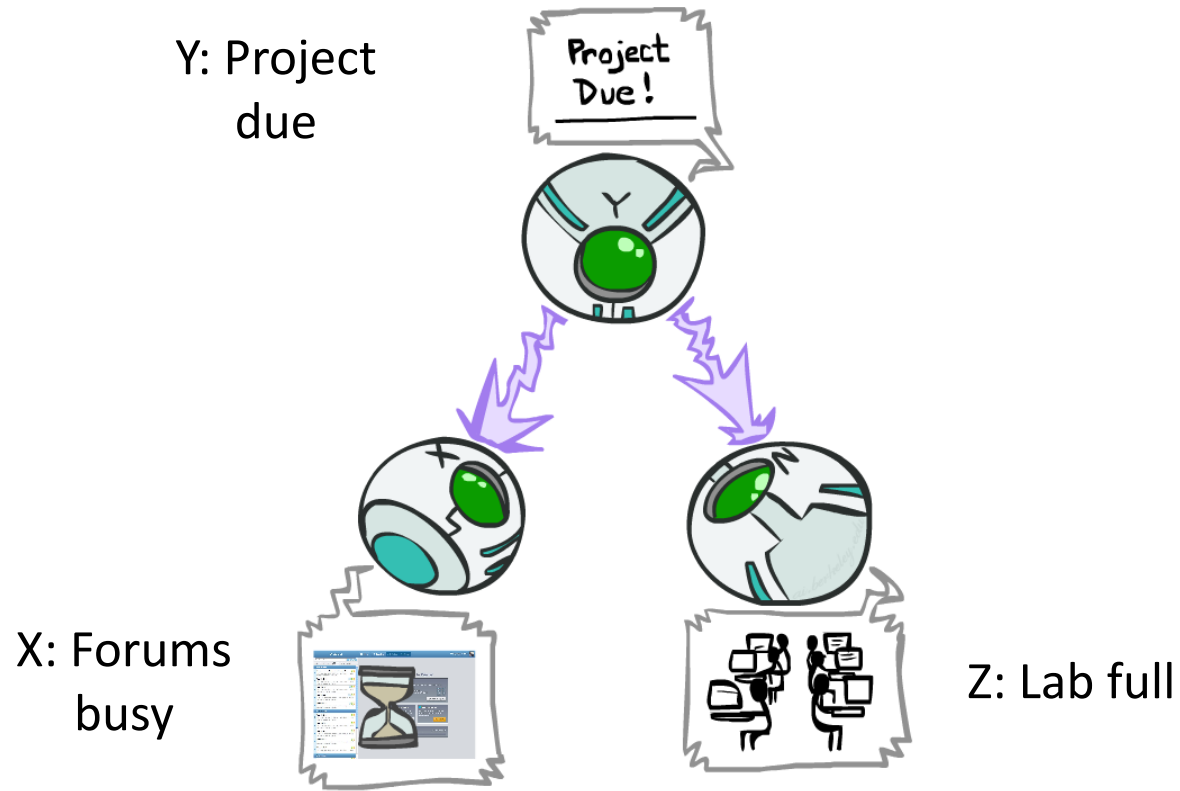
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Evidence along the chain “blocks” the influence

Common Cause

- This configuration is a “common cause”



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

- Project due causes both forums busy and lab full

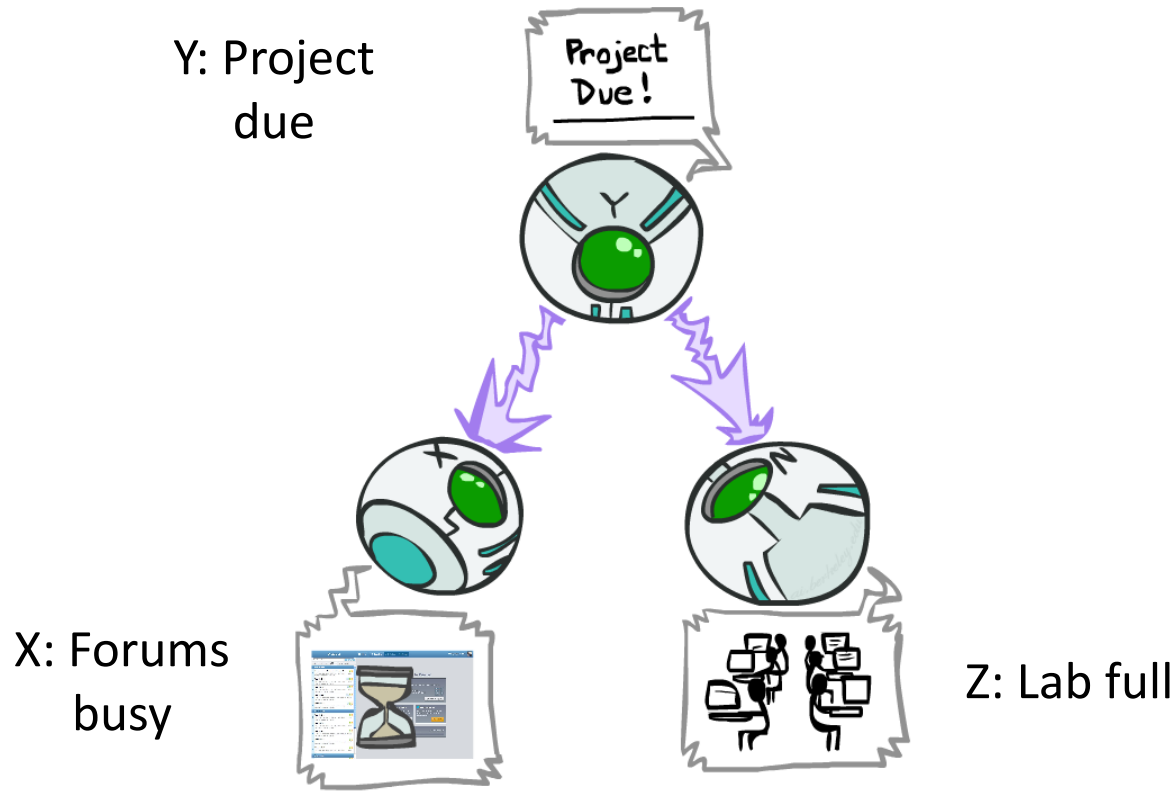
- In numbers:

$$P(+x \mid +y) = 1, P(-x \mid -y) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

Common Cause

- This configuration is a “common cause”

- Guaranteed X and Z independent given Y?



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

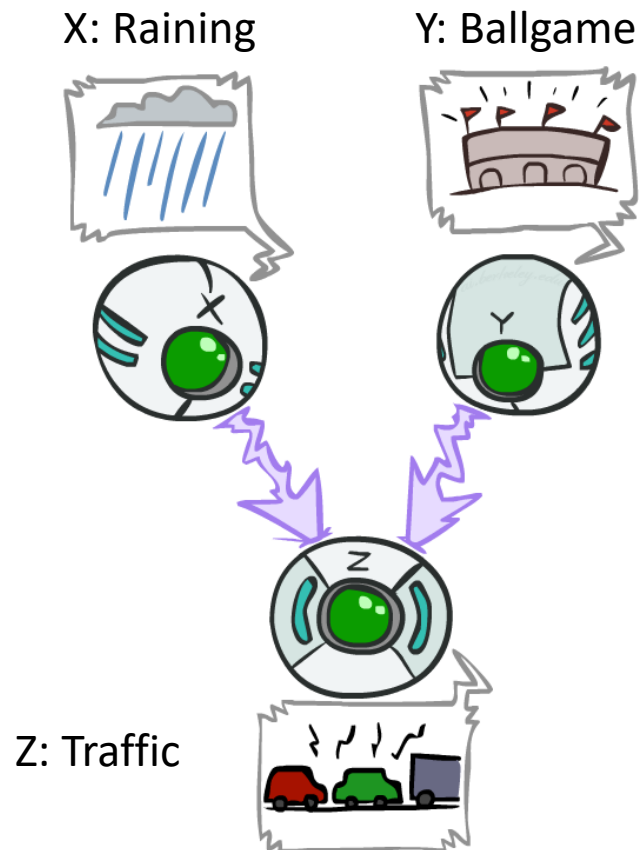
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Observing the cause blocks influence between effects.

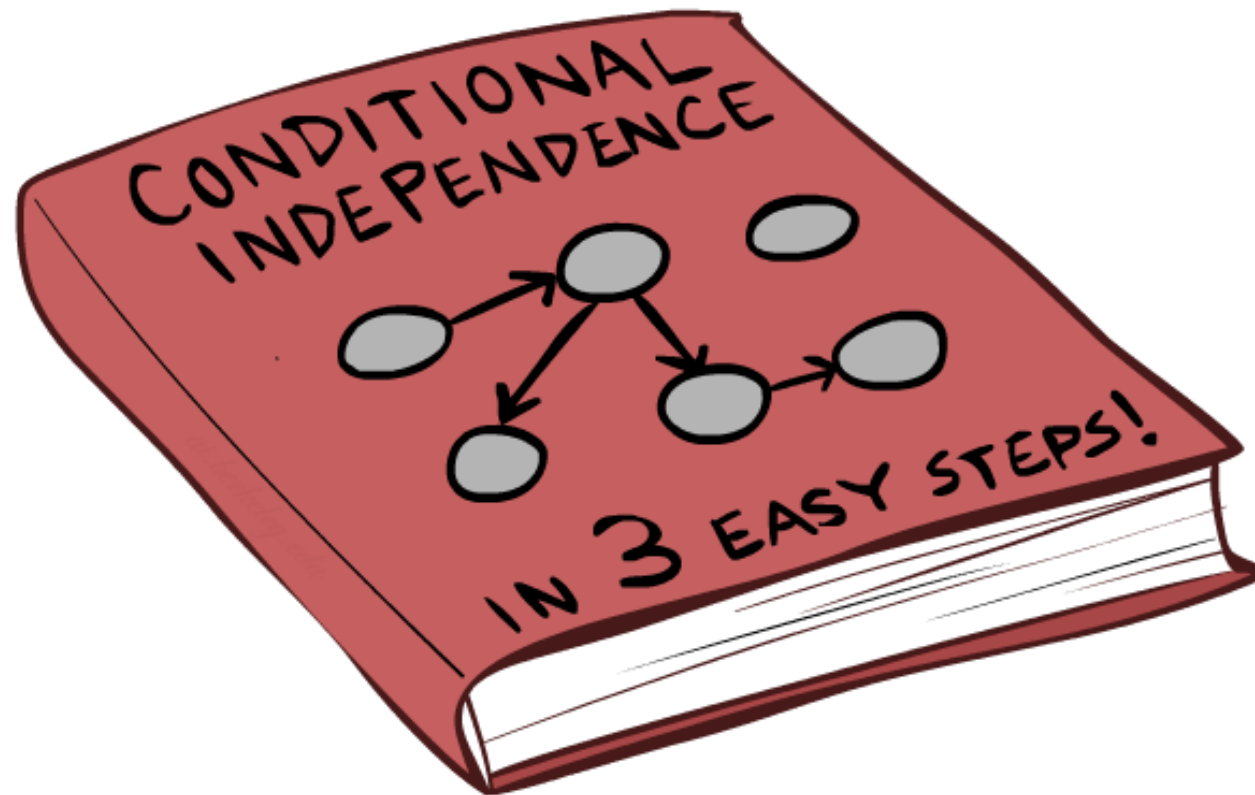
Common Effect

- Last configuration: two causes of one effect (v-structures)



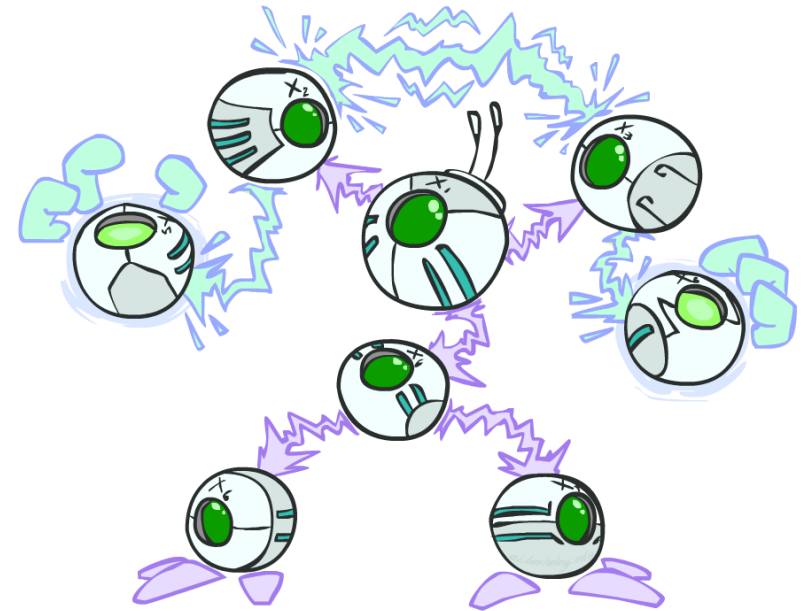
- Are X and Y independent?
 - **Yes**: the ballgame and the rain cause traffic, but they are not correlated
 - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
 - **No**: seeing traffic puts the rain and the ballgame in competition as explanation.
- **This is backwards from the other cases**
 - Observing an effect **activates** influence between possible causes.

The General Case



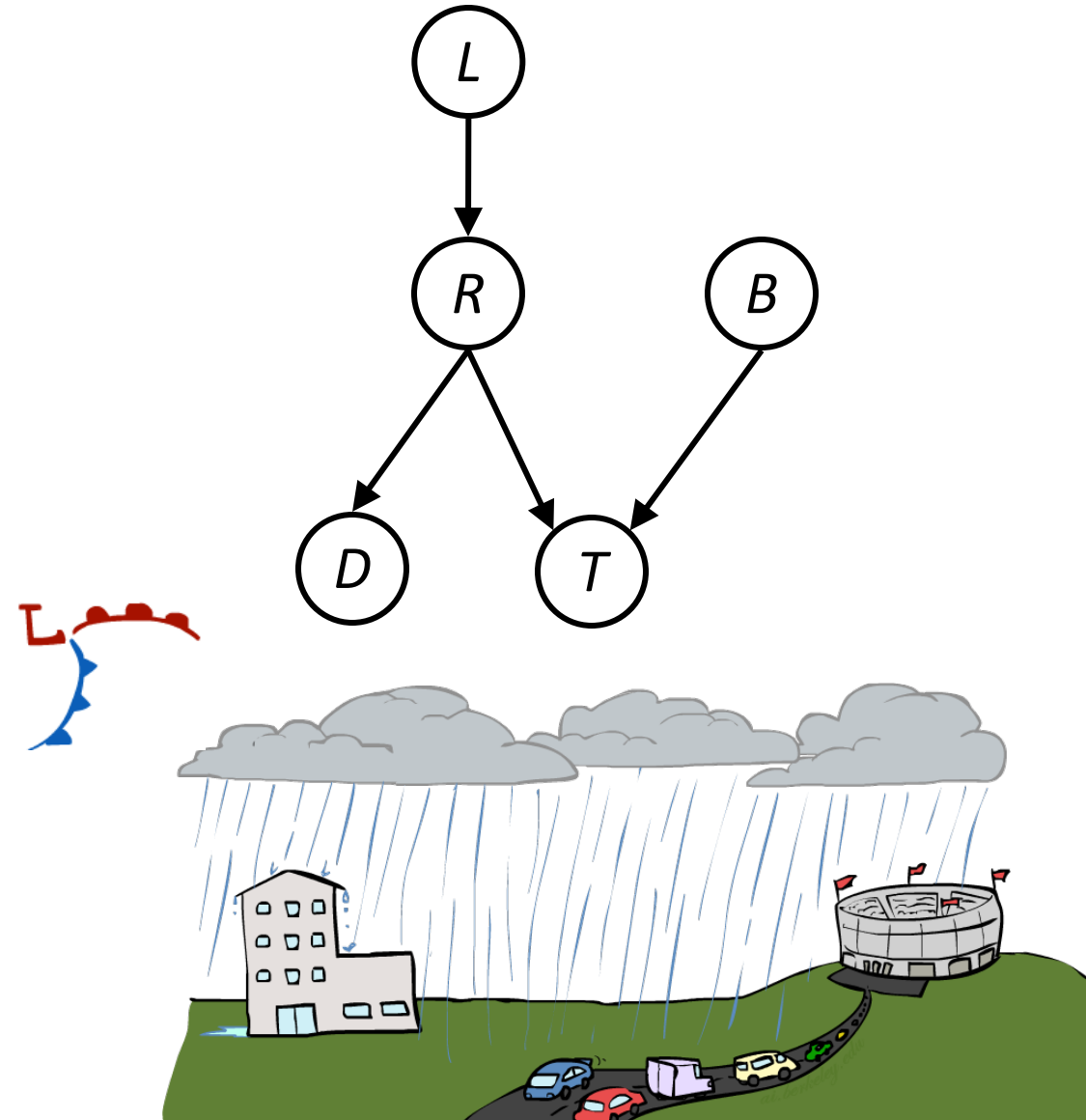
The General Case

- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases



Reachability

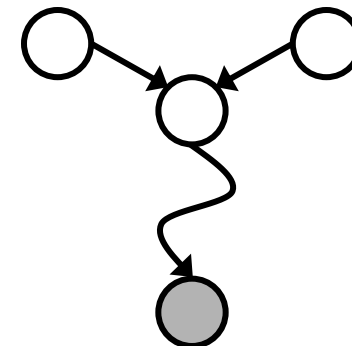
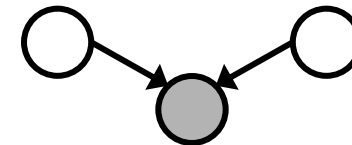
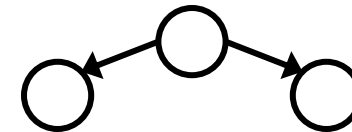
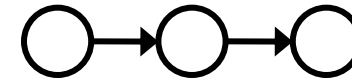
- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
 - Where does it break?
 - Answer: the v-structure at T doesn't count as a link in a path unless "active"



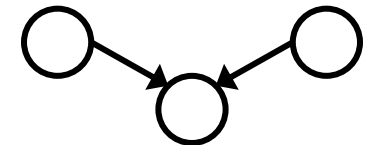
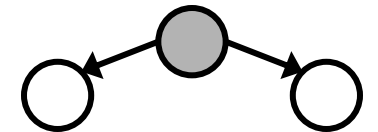
Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
 - Yes, if X and Y “d-separated” by Z
 - Consider all (undirected) paths from X to Y
 - No active paths = independence!
- A path is active if each triple is active:
 - Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
 - Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
 - Common effect (aka v-structure)
 $A \rightarrow B \leftarrow C$ where B or one of its descendants is observed
- All it takes to block a path is a single inactive segment

Active Triples



Inactive Triples



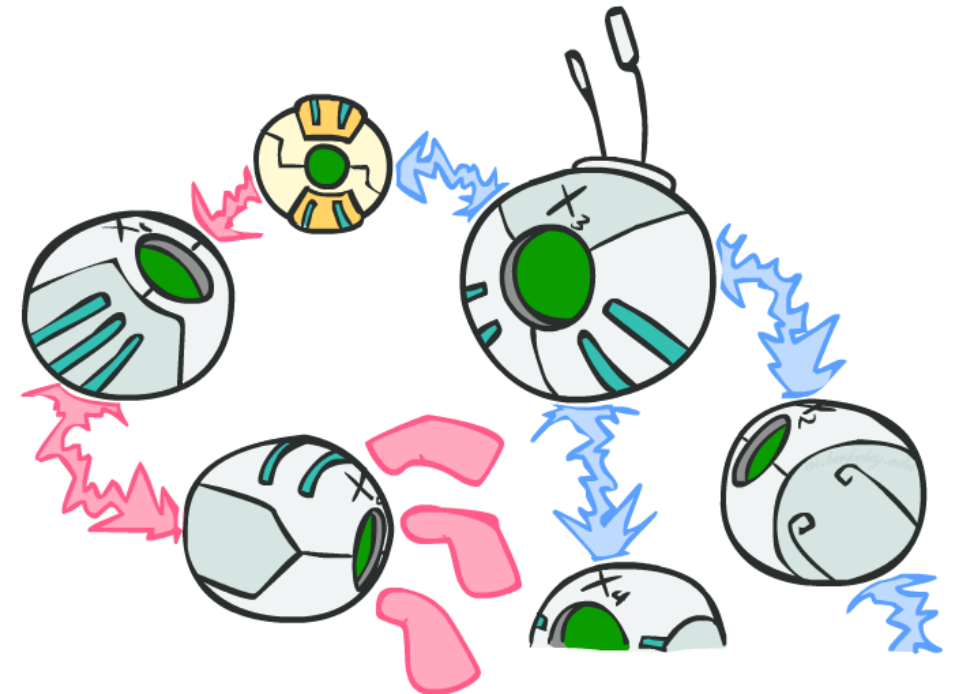
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\} ?$
- Check all (undirected!) paths between X_i and X_j
 - If one or more active, then independence not guaranteed

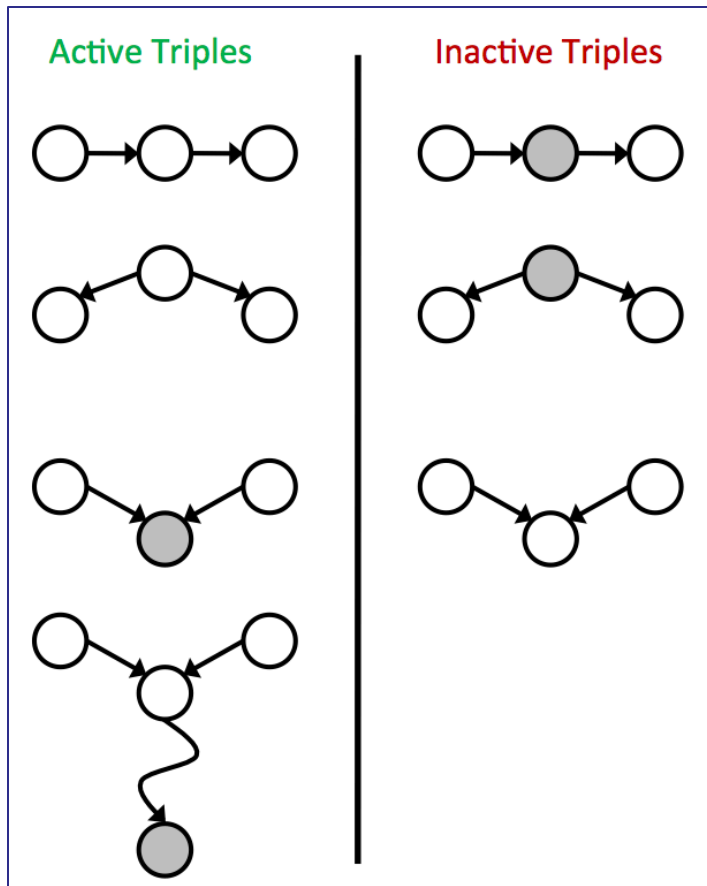
$$X_i \not\perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$

- Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$



Example

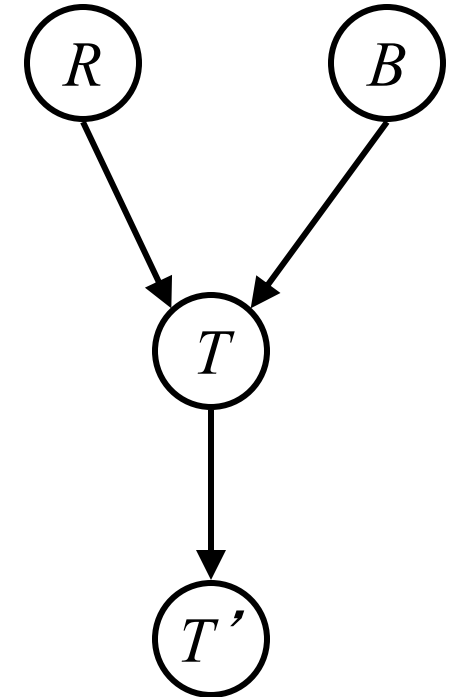


$$R \perp\!\!\!\perp B$$

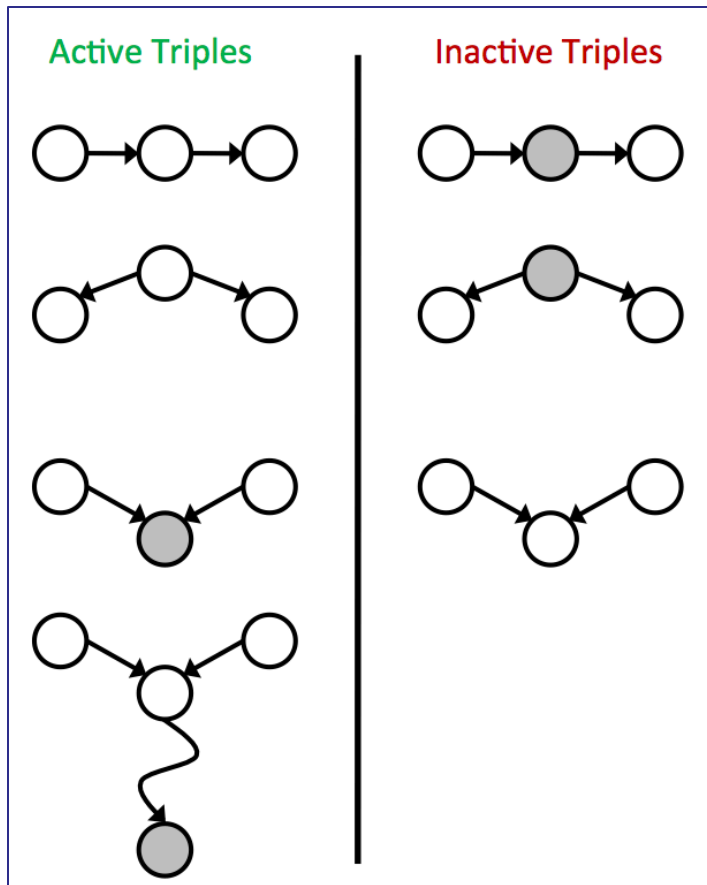
Yes

$$R \perp\!\!\!\perp B | T$$

$$R \perp\!\!\!\perp B | T'$$



Example



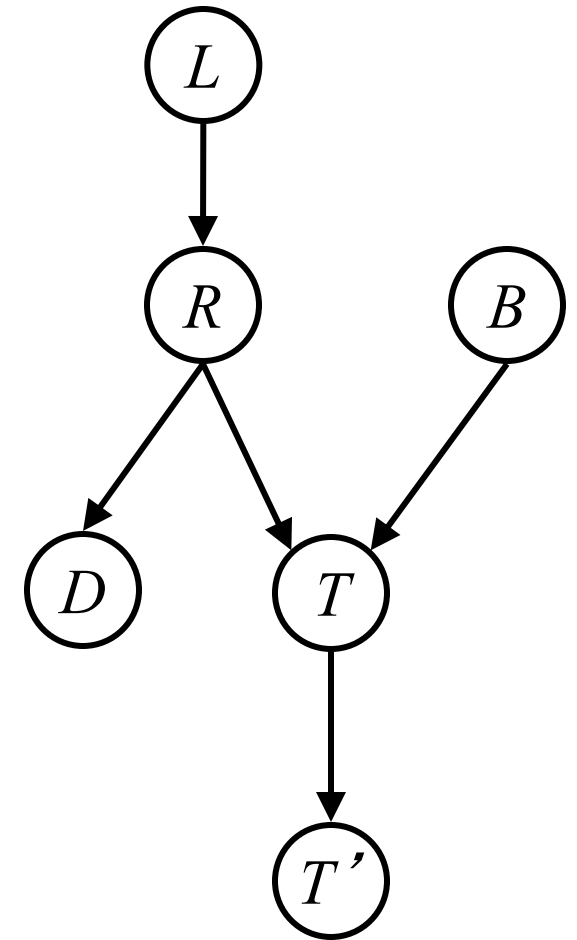
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

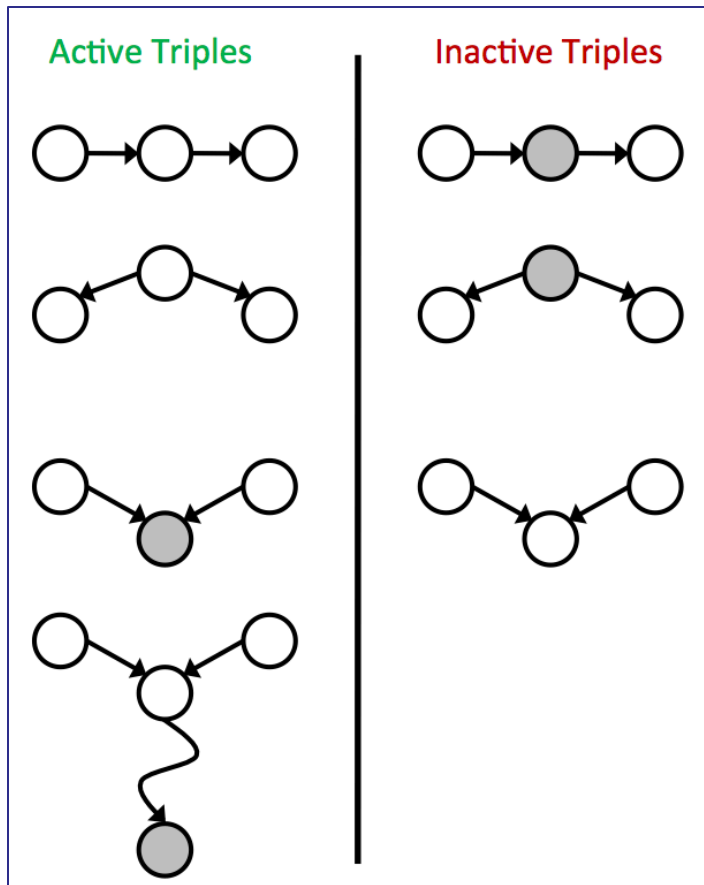
$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$ *Yes*



Example



■ Variables:

- R: Raining
- T: Traffic
- D: Roof drips
- S: I'm sad

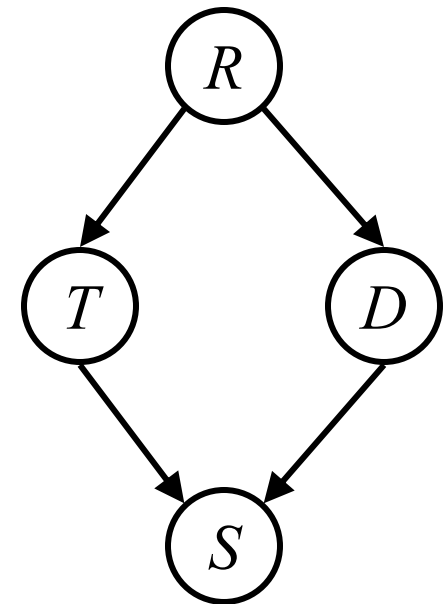
■ Questions:

$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R$$

Yes

$$T \perp\!\!\!\perp D | R, S$$

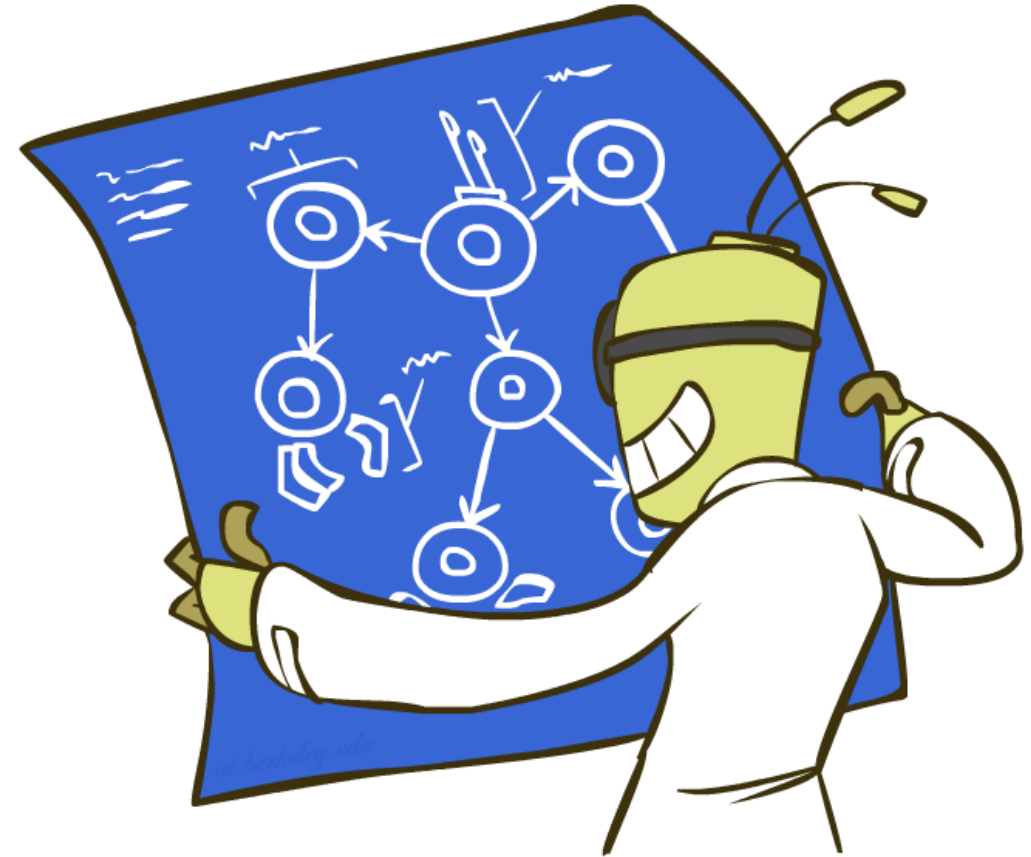


Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

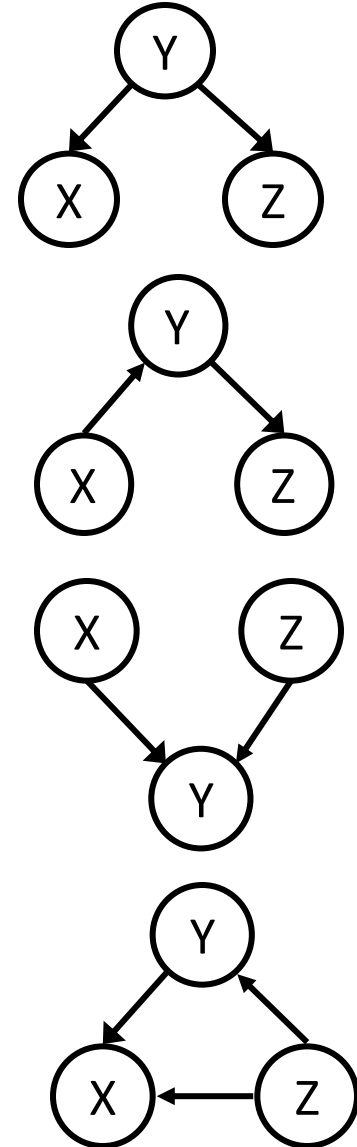
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented



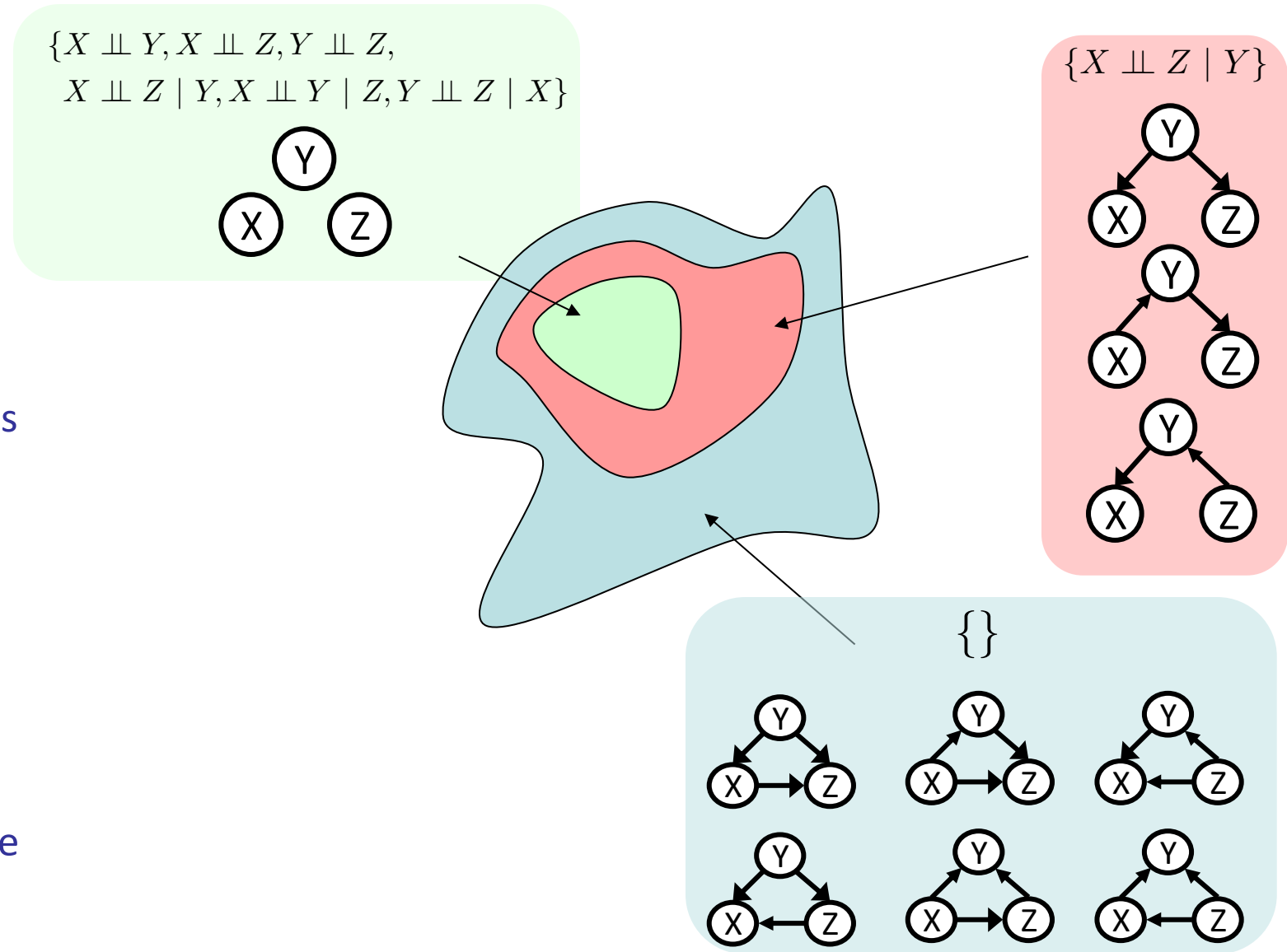
Computing All Independences

COMPUTE ALL THE
INDEPENDENCES!



Topology Limits Distributions

- Given some graph topology G , only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

Bayes' Nets

✓ Representation

✓ Conditional Independences

- Probabilistic Inference

- Enumeration (exact, exponential complexity)
- Variable elimination (exact, worst-case exponential complexity, often better)
- Probabilistic inference is NP-complete
- Sampling (approximate)

- Learning Bayes' Nets from Data