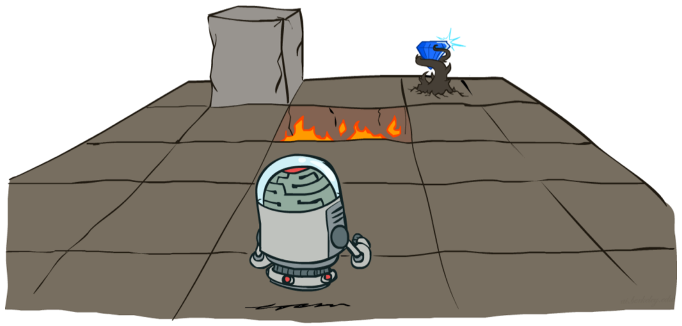
# CSE 473: Introduction to Artificial Intelligence

Hanna Hajishirzi

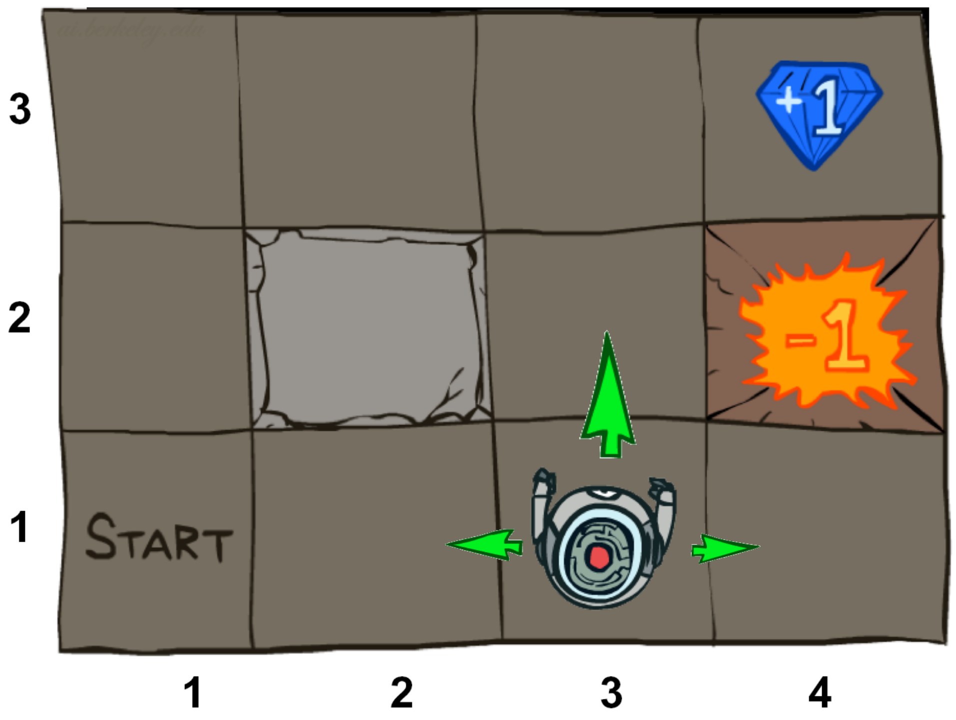
## Markov Decision Processes

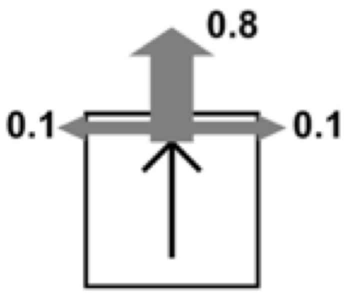


slides adapted from
Dan Klein, Pieter Abbeel ai.berkeley.edu
And Dan Weld, Luke Zettelmoyer

# Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North
    (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards



# Recap: Defining MDPs

- Markov decision processes:
  - Set of states S
  - Start state $s_0$
  - Set of actions A
  - Transitions P(s' | s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount $\gamma$)

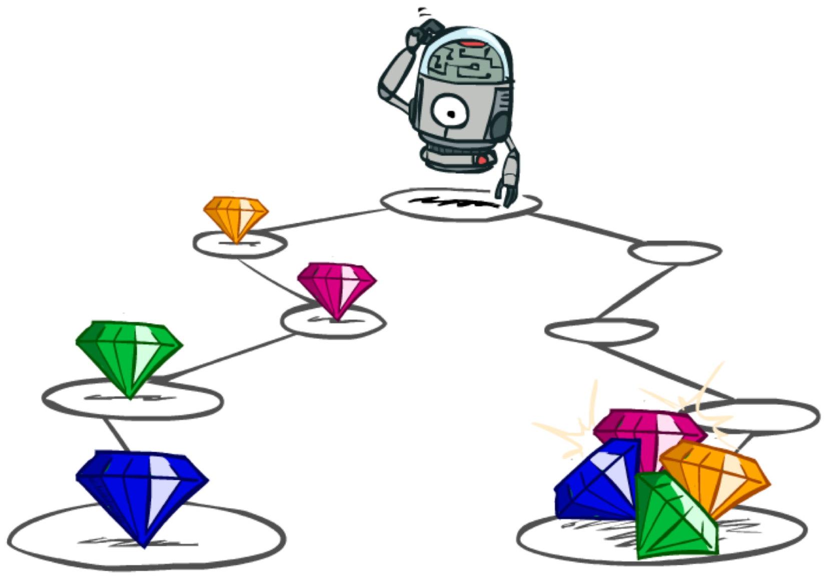
- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility = sum of (discounted) rewards

# Utilities of Sequences

- What preferences should an agent have over reward sequences?

- More or less? [1, 2, 2] or [2, 3, 4]

- Now or later? [0, 0, 1] or [1, 0, 0]



# Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially

$1$

Worth Now

$\gamma$

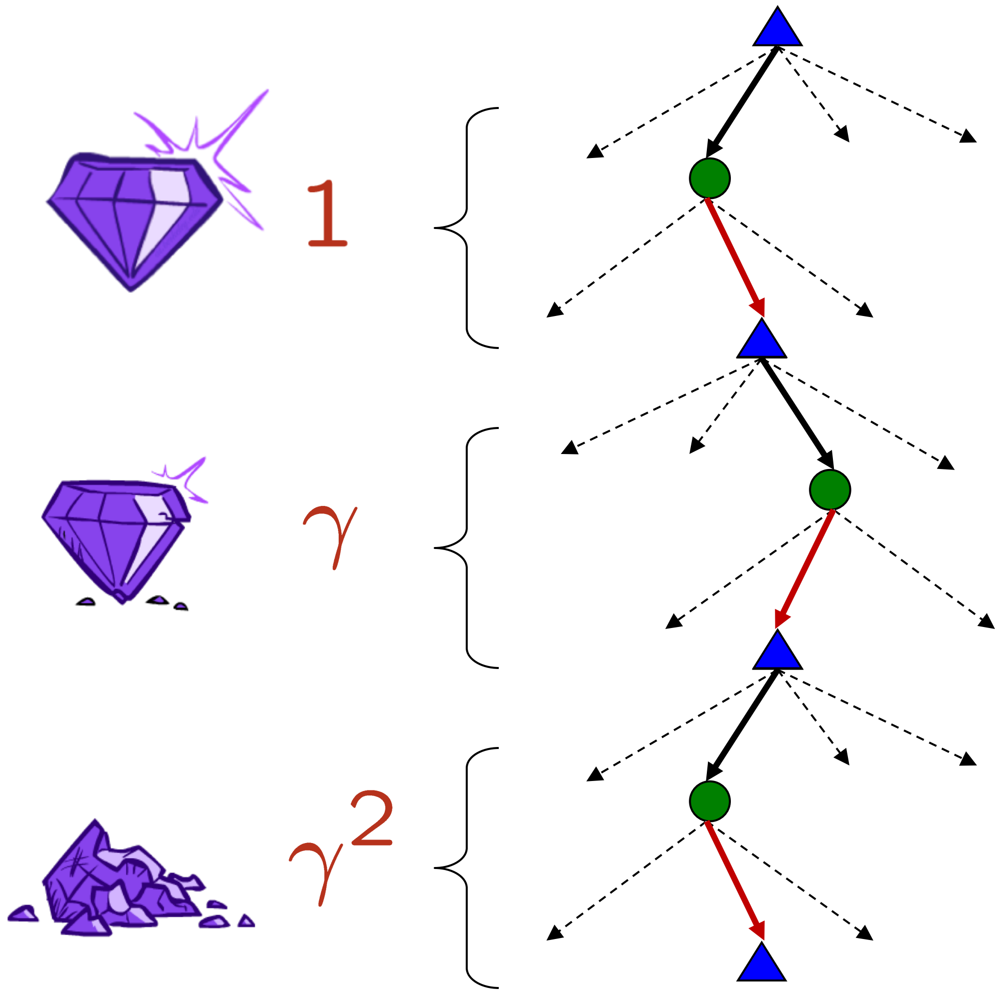
Worth Next Step

$\gamma^2$

Worth In Two Steps

# Discounting

- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Think of it as a gamma chance of ending the process at every step
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])



# Quiz: Discounting

- Given:

| a | b | c | d | e |
|---|---|---|---|---|
| 10 | | | | 1 |

  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic

- Quiz 1: For $\gamma = 1$, what is the optimal policy?

| | | | | |
|---|---|---|---|---|
| 10 | <- | <- | <- | 1 |

- Quiz 2: For $\gamma = 0.1$, what is the optimal policy?

| | | | | |
|---|---|---|---|---|
| 10 | <- | <- | -> | 1 |

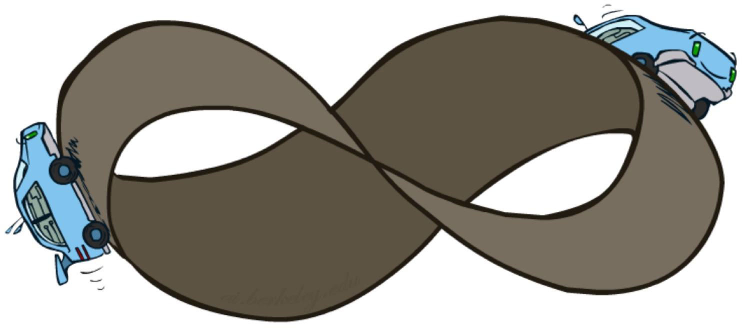
- Quiz 3: For which $\gamma$ are West and East equally good when in state d?

$1\gamma = 10\gamma^3$

# Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?

- Solutions:
  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed T steps (e.g. life)
    - Policy $\pi$ depends on time left



  - Discounting: use $0 < \gamma < 1$

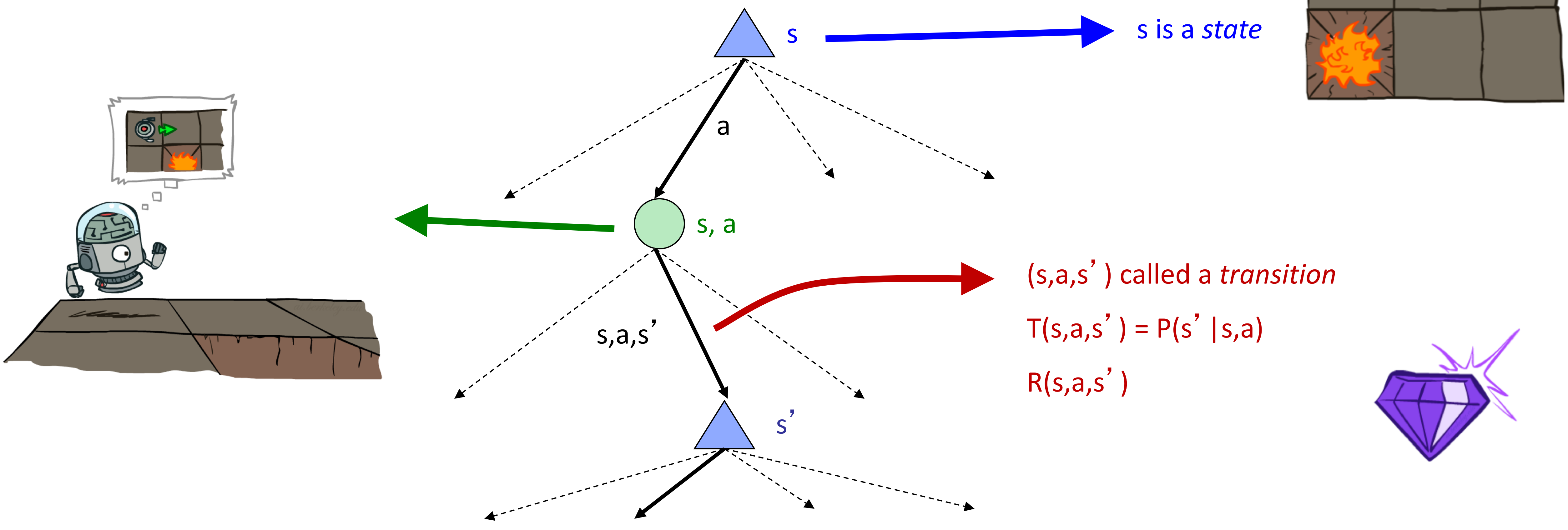
$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller $\gamma$ means smaller "horizon" – shorter term focus

  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)

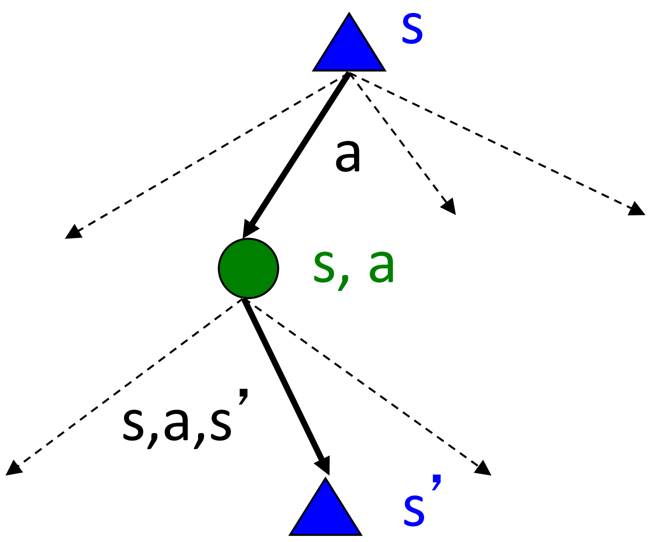
# MDP Search Trees

- Each MDP state projects an expectimax-like search tree



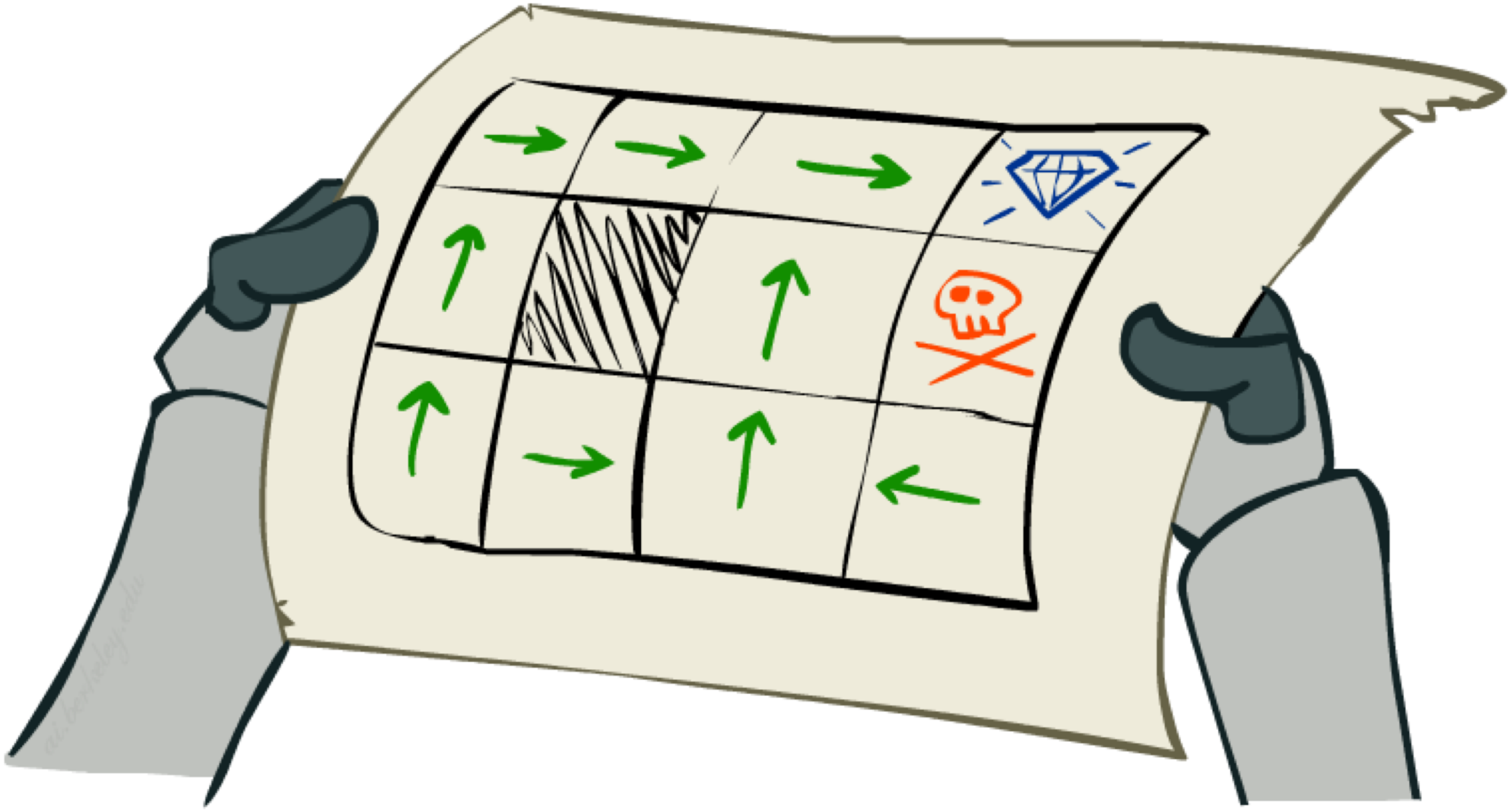
# Recap: Defining MDPs

- Markov decision processes:
  - Set of states S
  - Start state $s_0$
  - Set of actions A
  - Transitions P(s' | s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount $\gamma$)



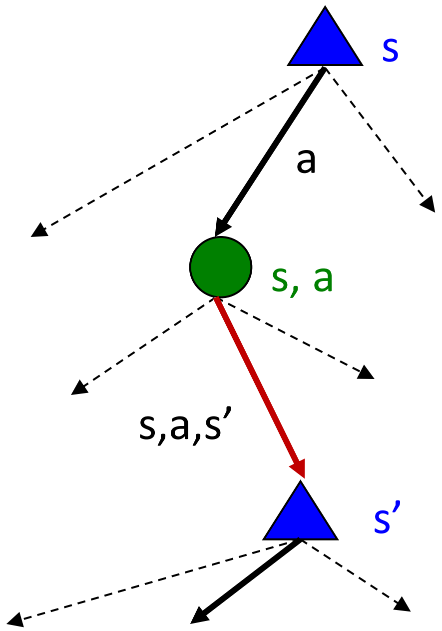

- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility = sum of (discounted) rewards

# Solving MDPs



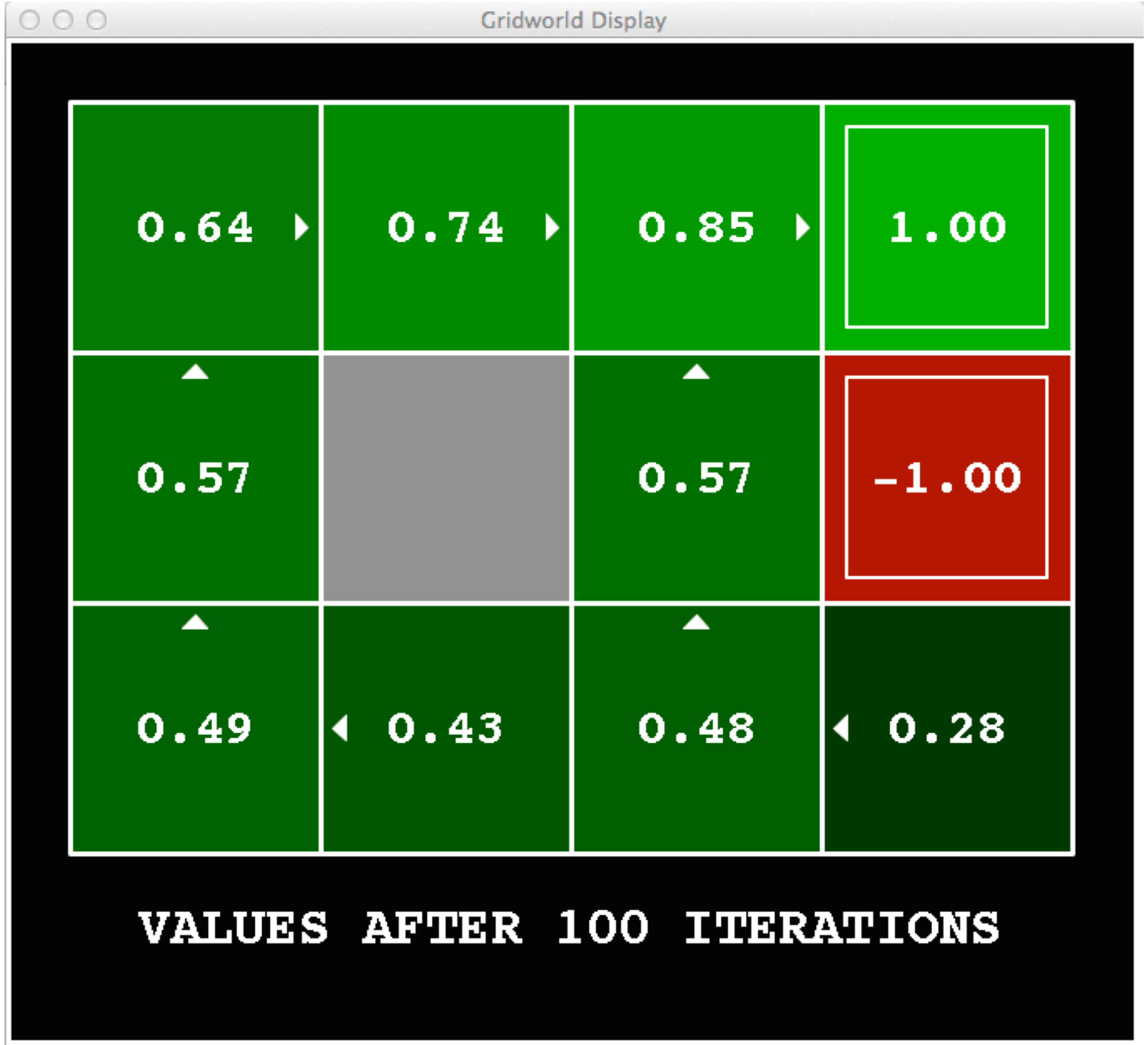
# Optimal Quantities

- The value (utility) of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
  $\pi^*(s)$ = optimal action from state s



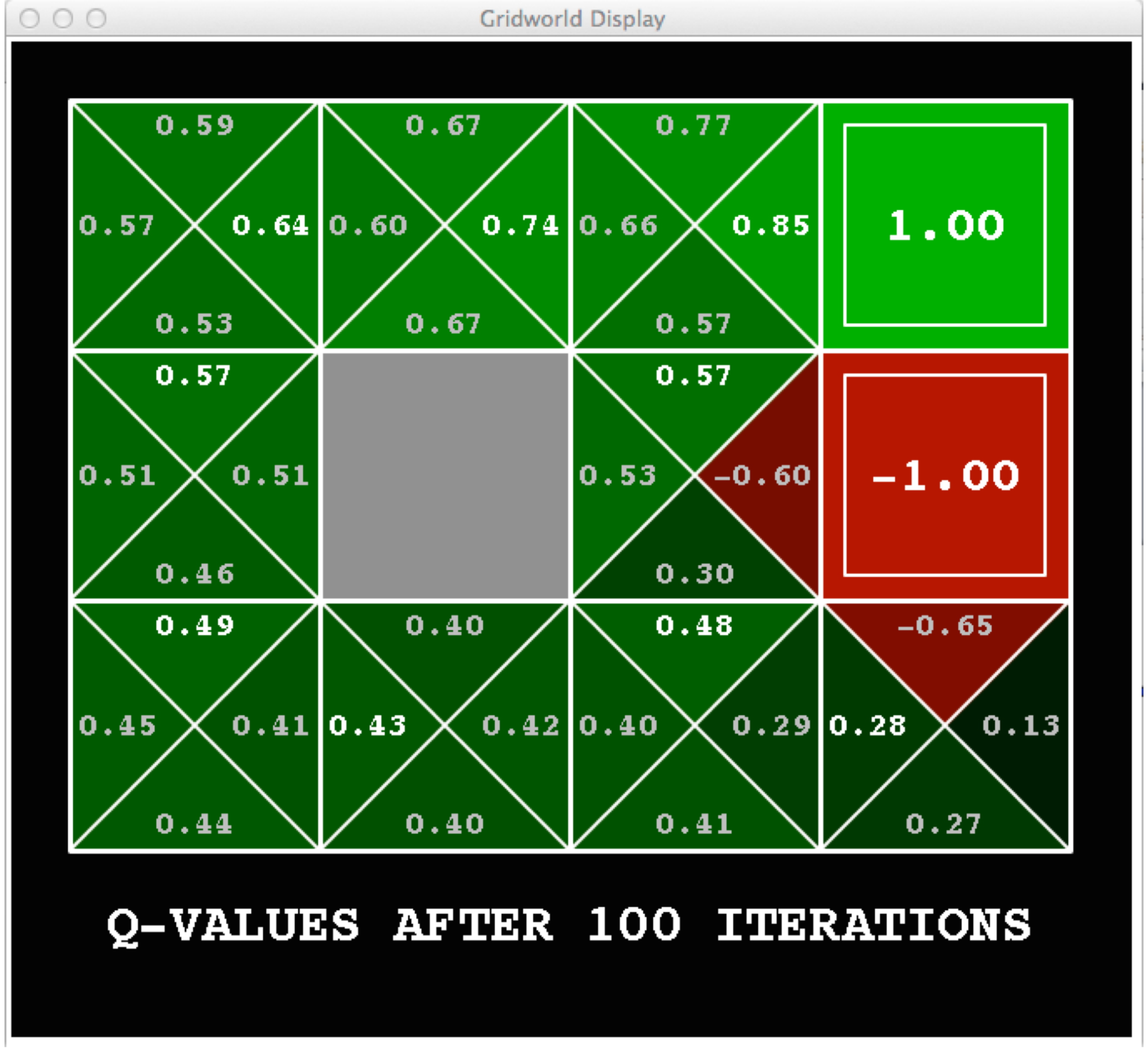

[Demo – gridworld values (L8D4)]

# Snapshot of Demo – Gridworld V Values




Noise = 0.2
Discount = 0.9
Living reward = 0

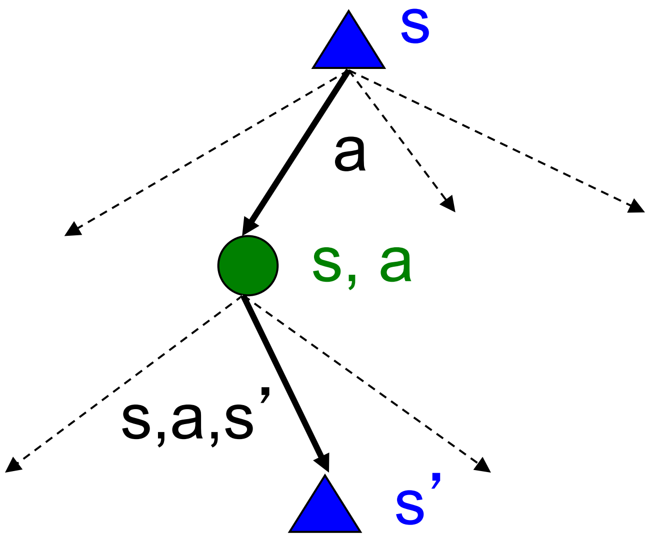
# Snapshot of Demo – Gridworld Q Values




Noise = 0.2
Discount = 0.9
Living reward = 0

# Values of States (Bellman Equations)

- Fundamental operation: compute the (expectimax) value of a state
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is just what expectimax computed!



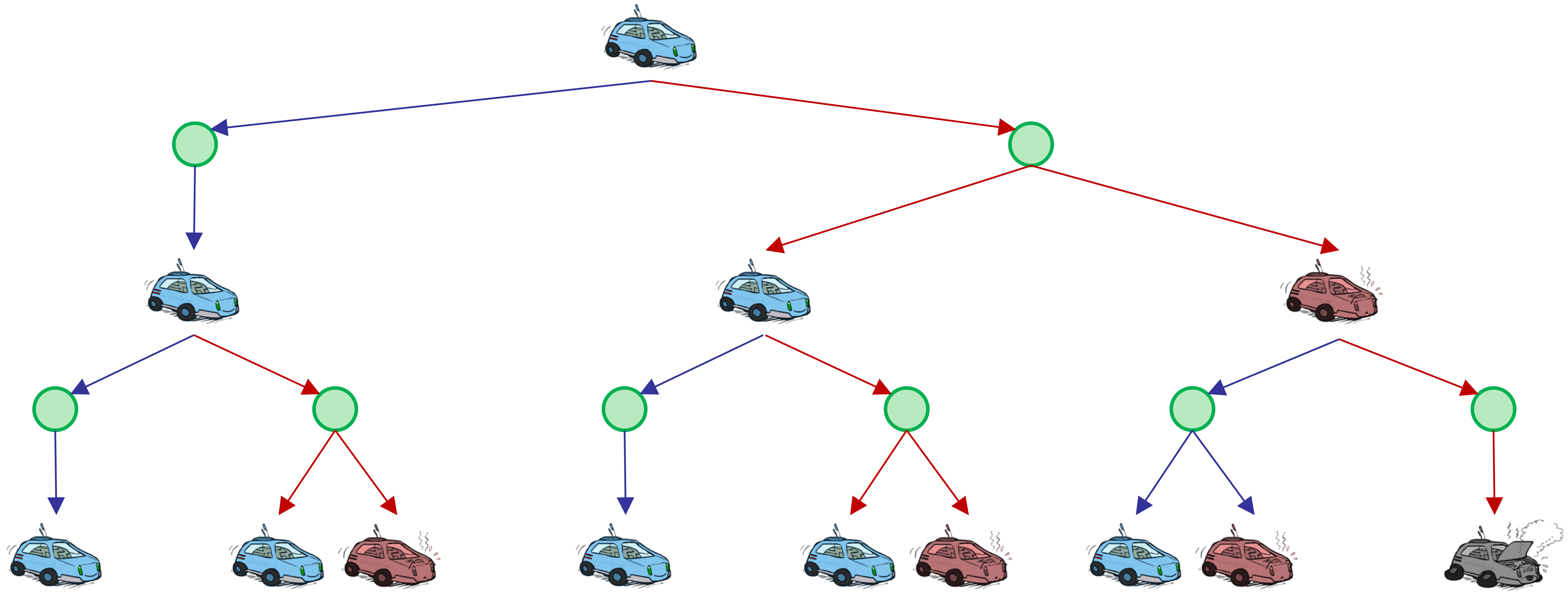

- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

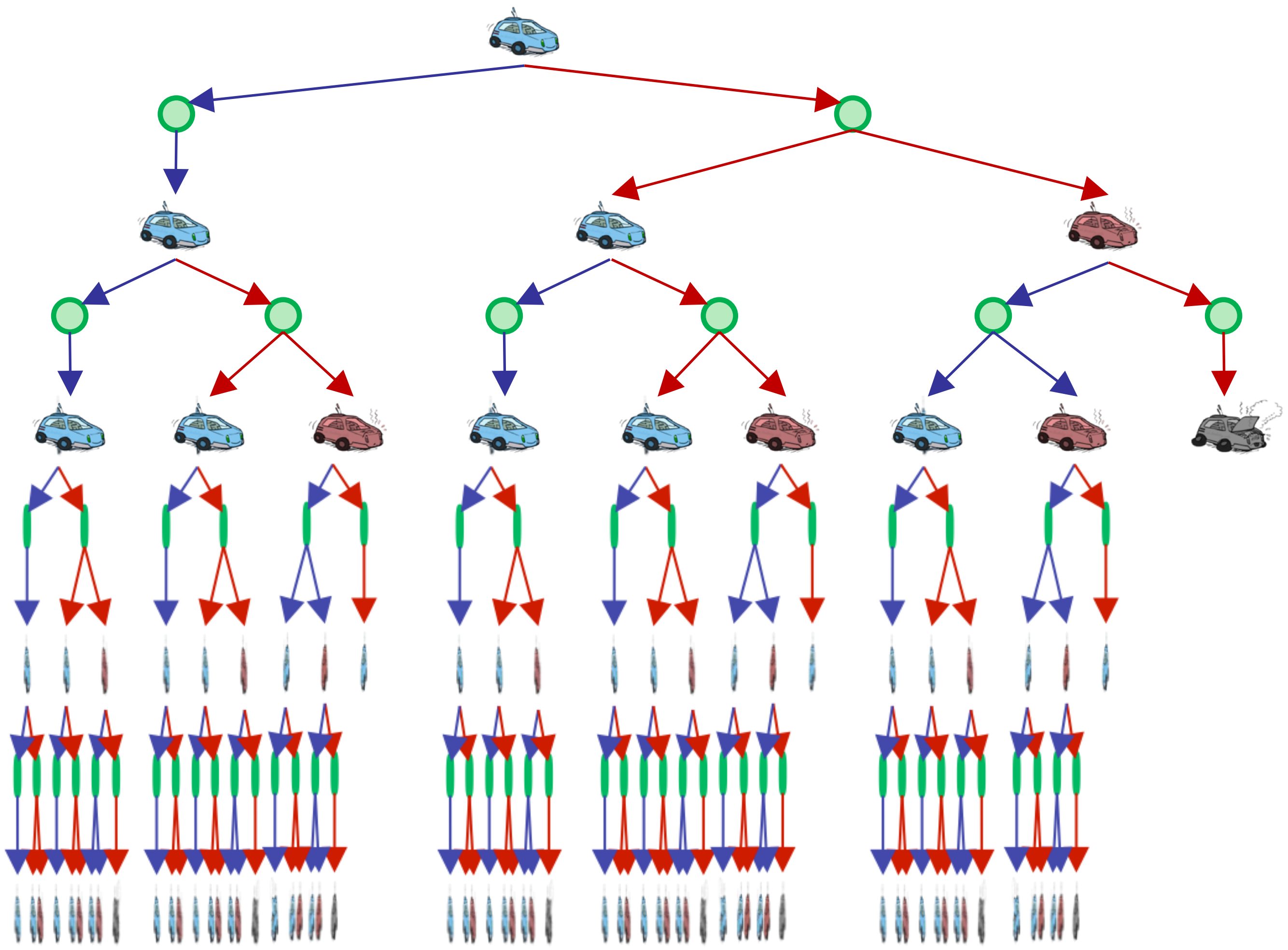
$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

# Racing Search Tree

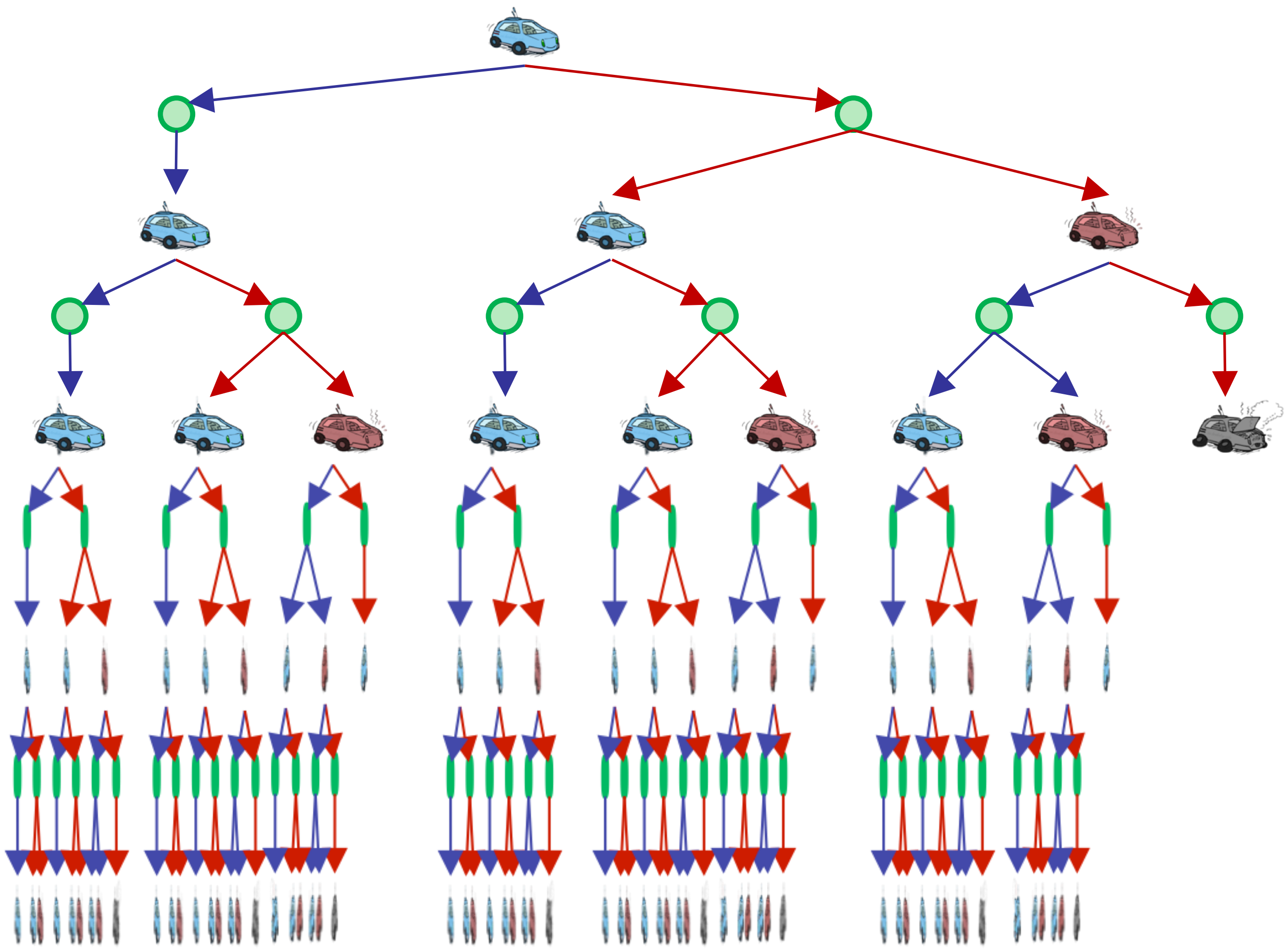


# Racing Search Tree



# Racing Search Tree

- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$

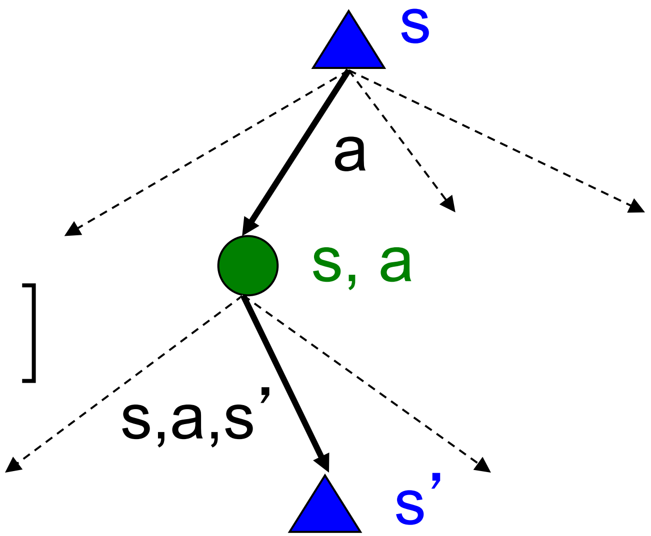


# Values of States

- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

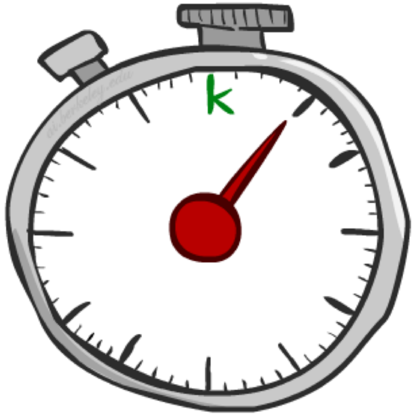
$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[ R(s,a,s') + \gamma V^*(s') \right]$$

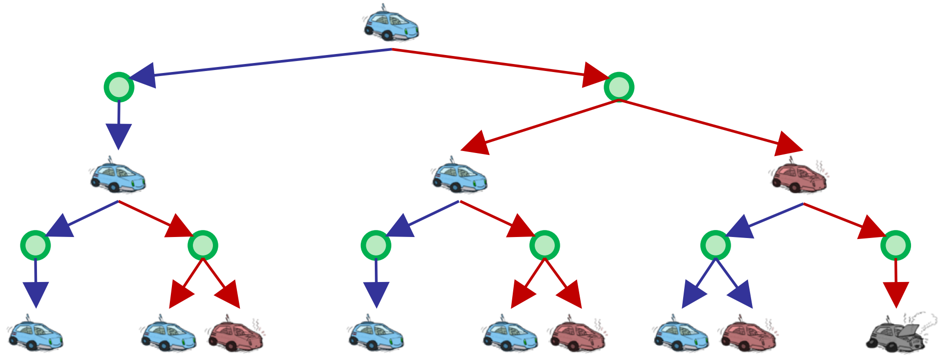


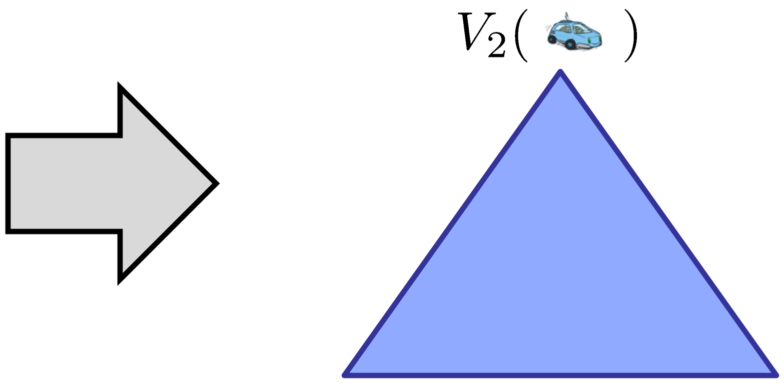

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

# Time-Limited Values

- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s

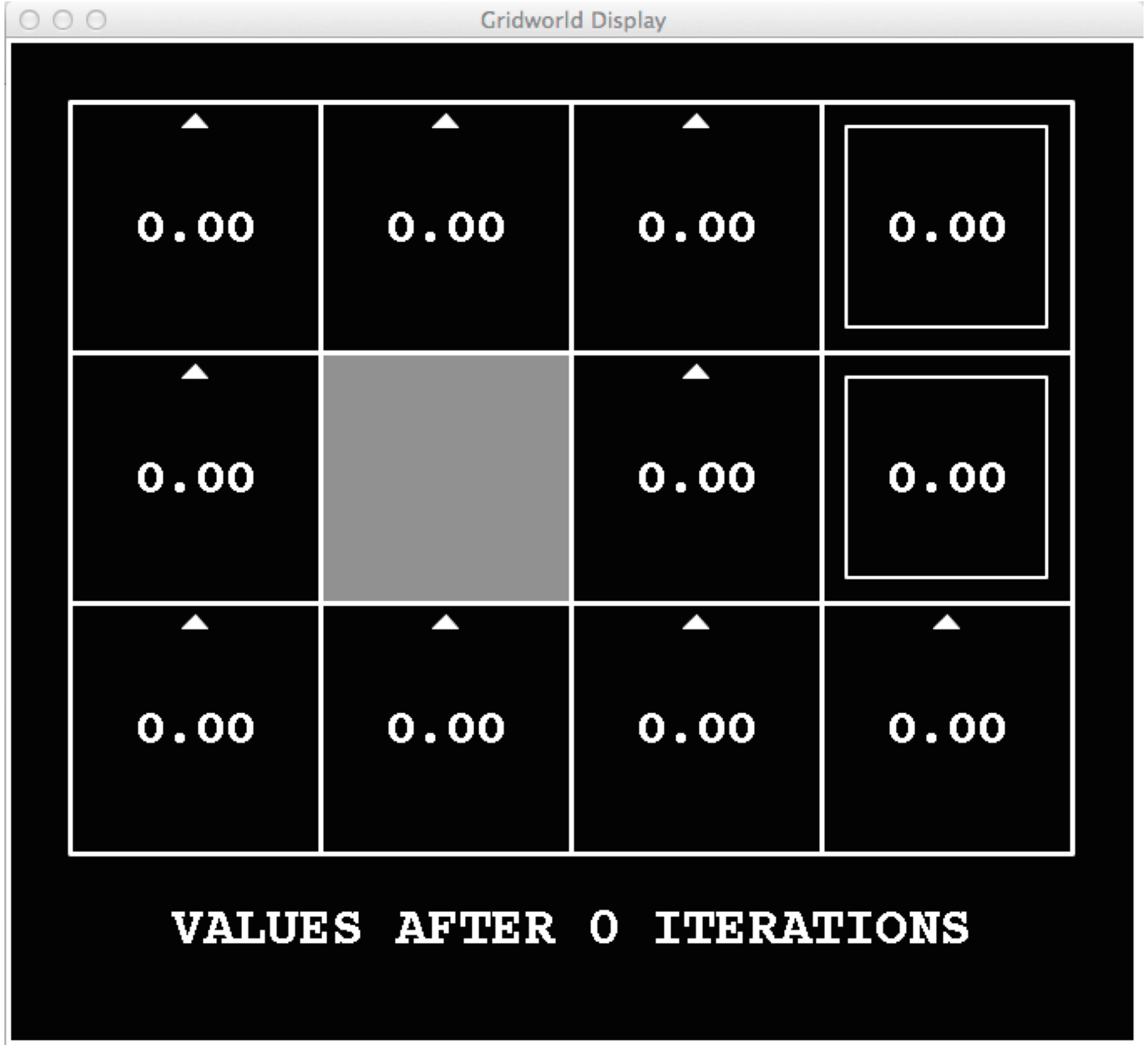




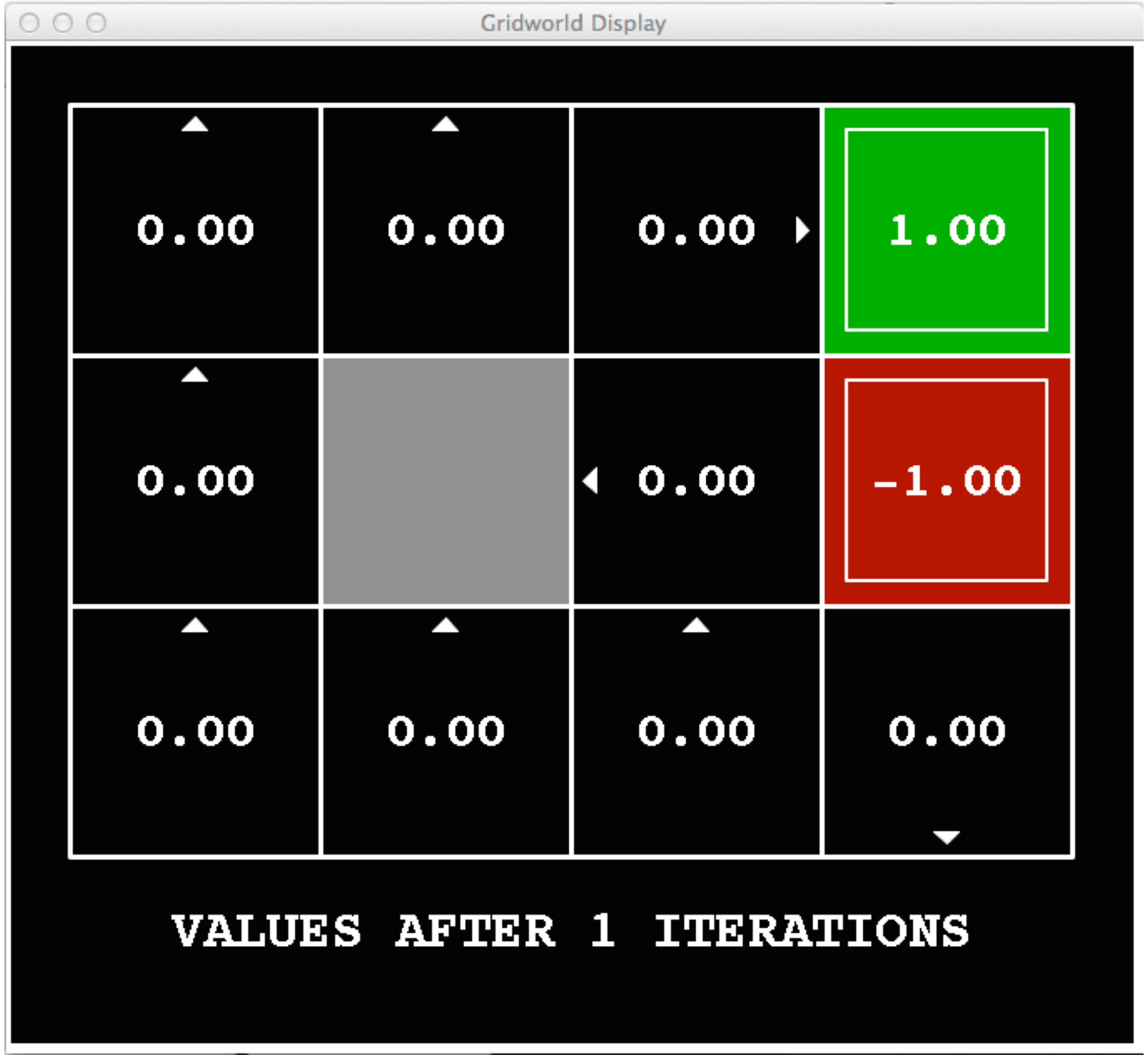

[Demo – time-limited values (L8D6)]

# k=0




Noise = 0.2
Discount = 0.9
Living reward = 0

# k=1




Noise = 0.2
Discount = 0.9
Living reward = 0

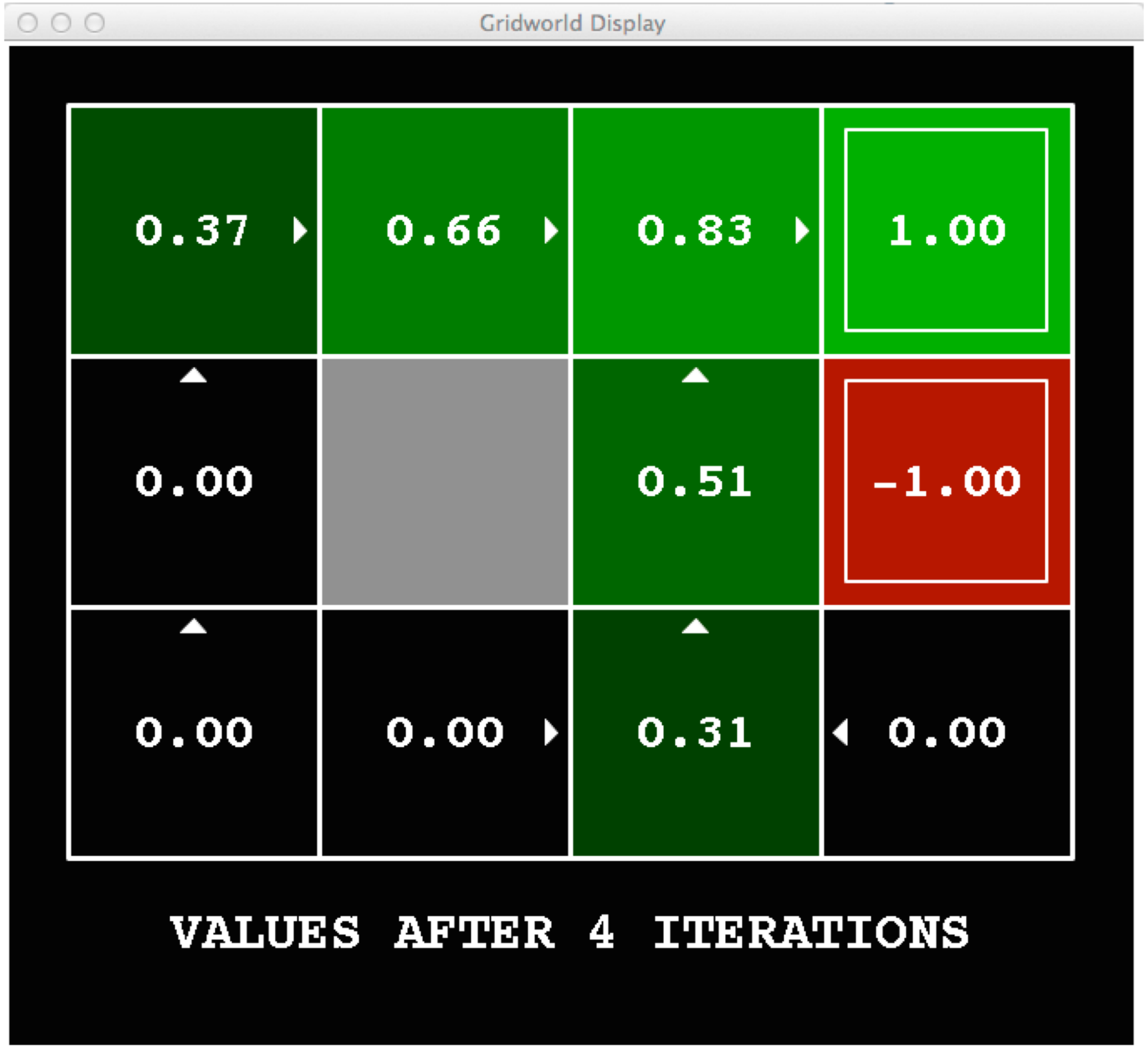
# k=2


Noise = 0.2
Discount = 0.9
Living reward = 0

# k=3

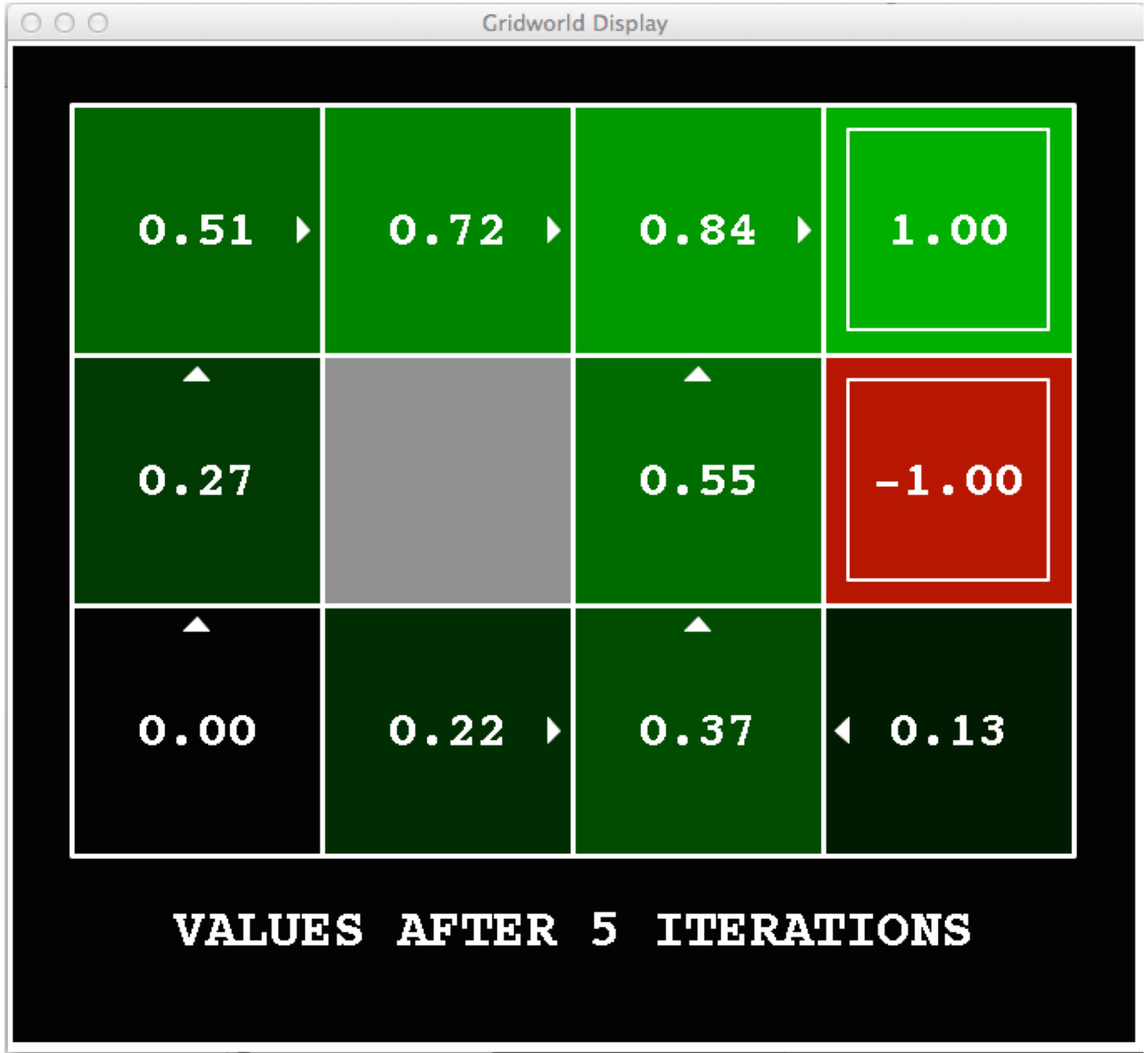

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=4



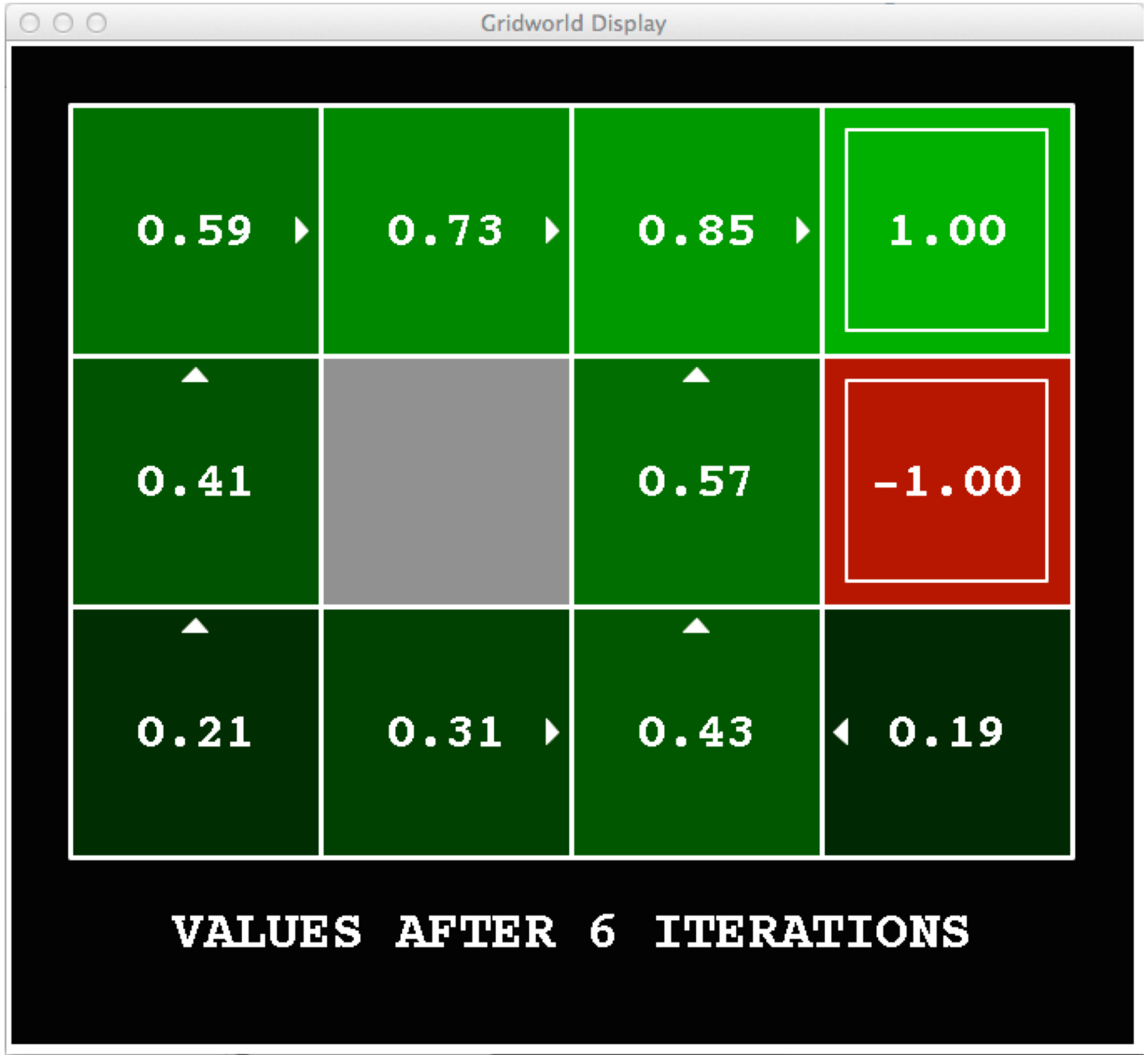

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=5



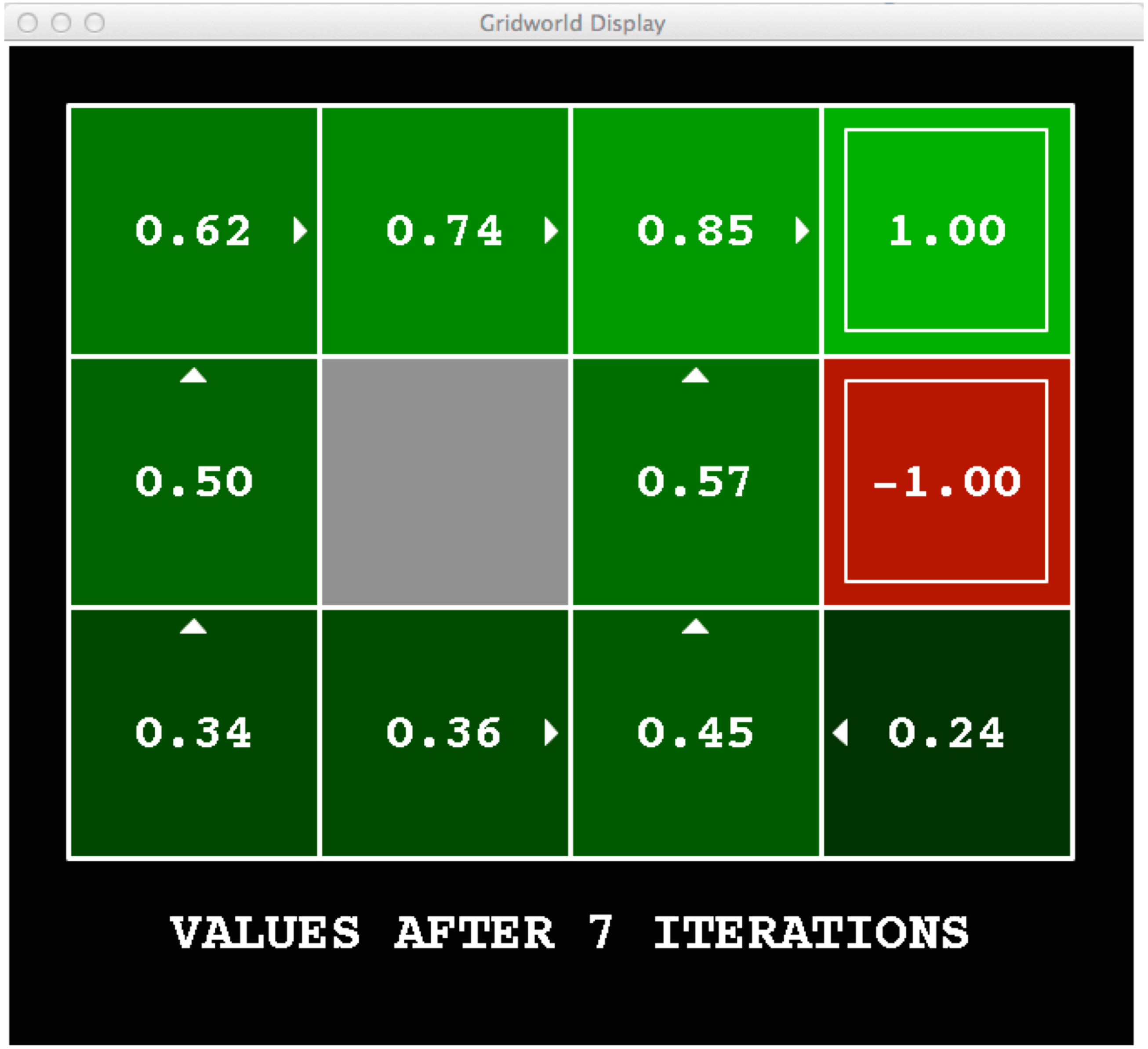

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=6



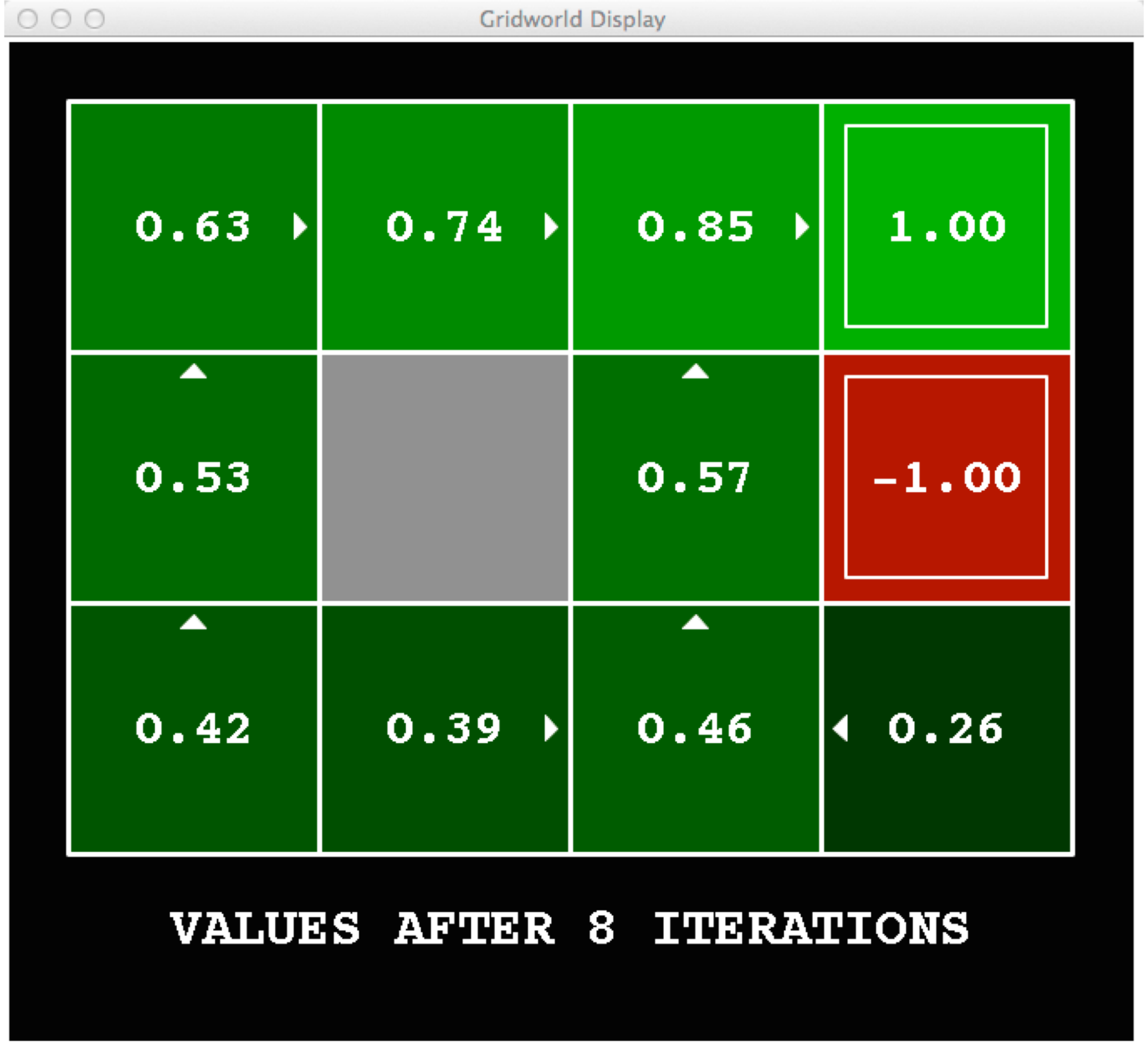

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=7



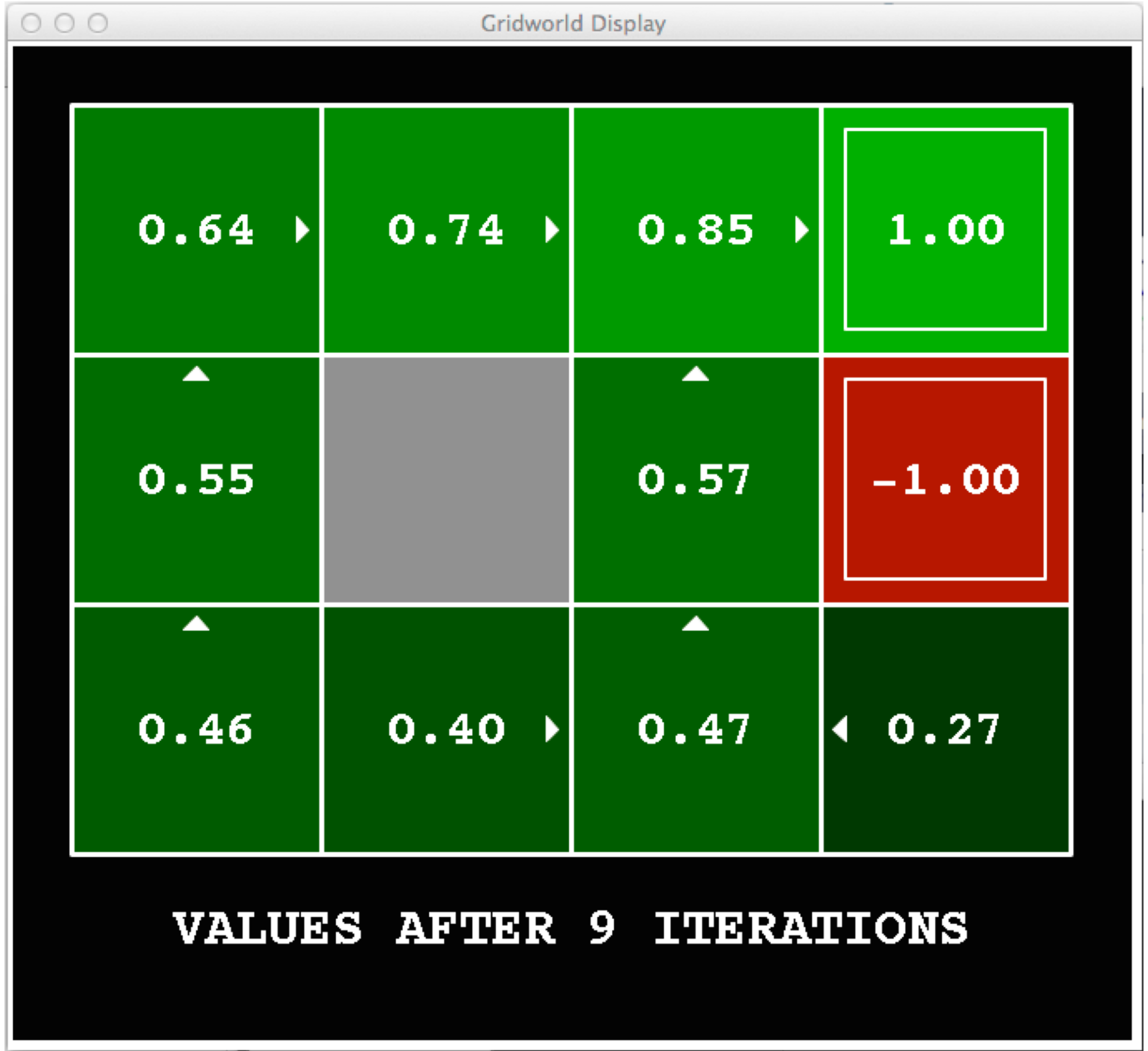

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=8



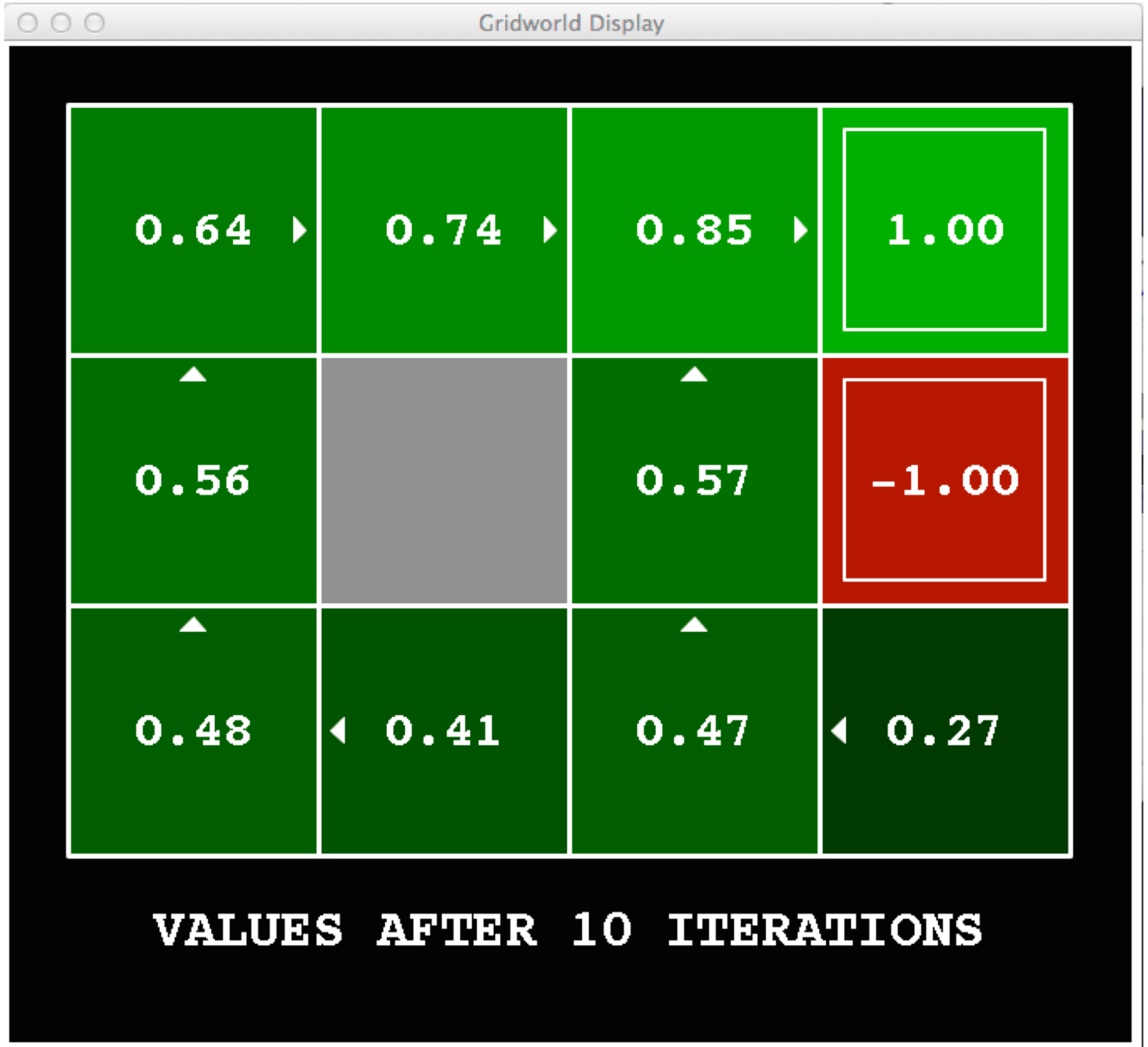

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=9



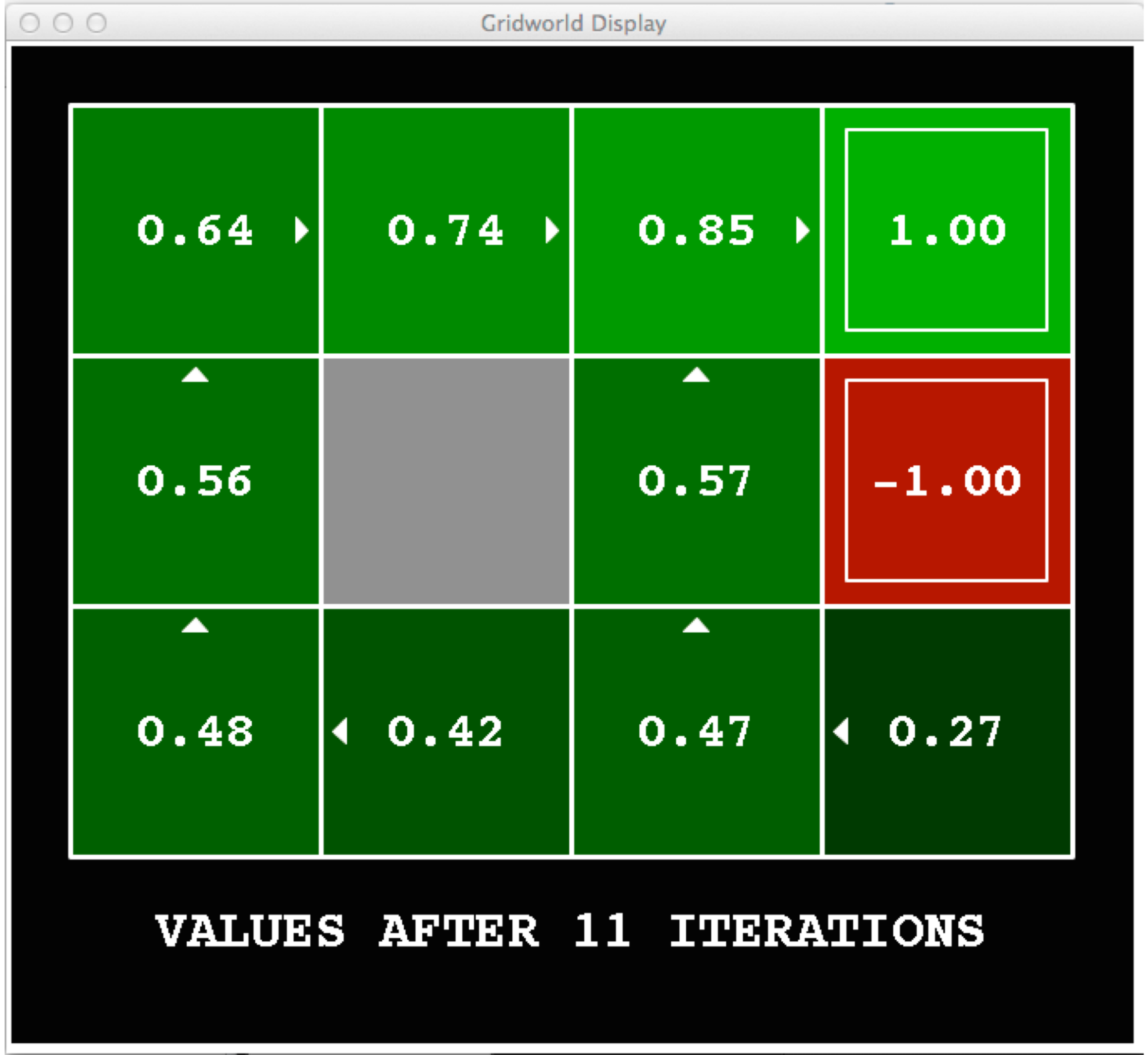

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=10



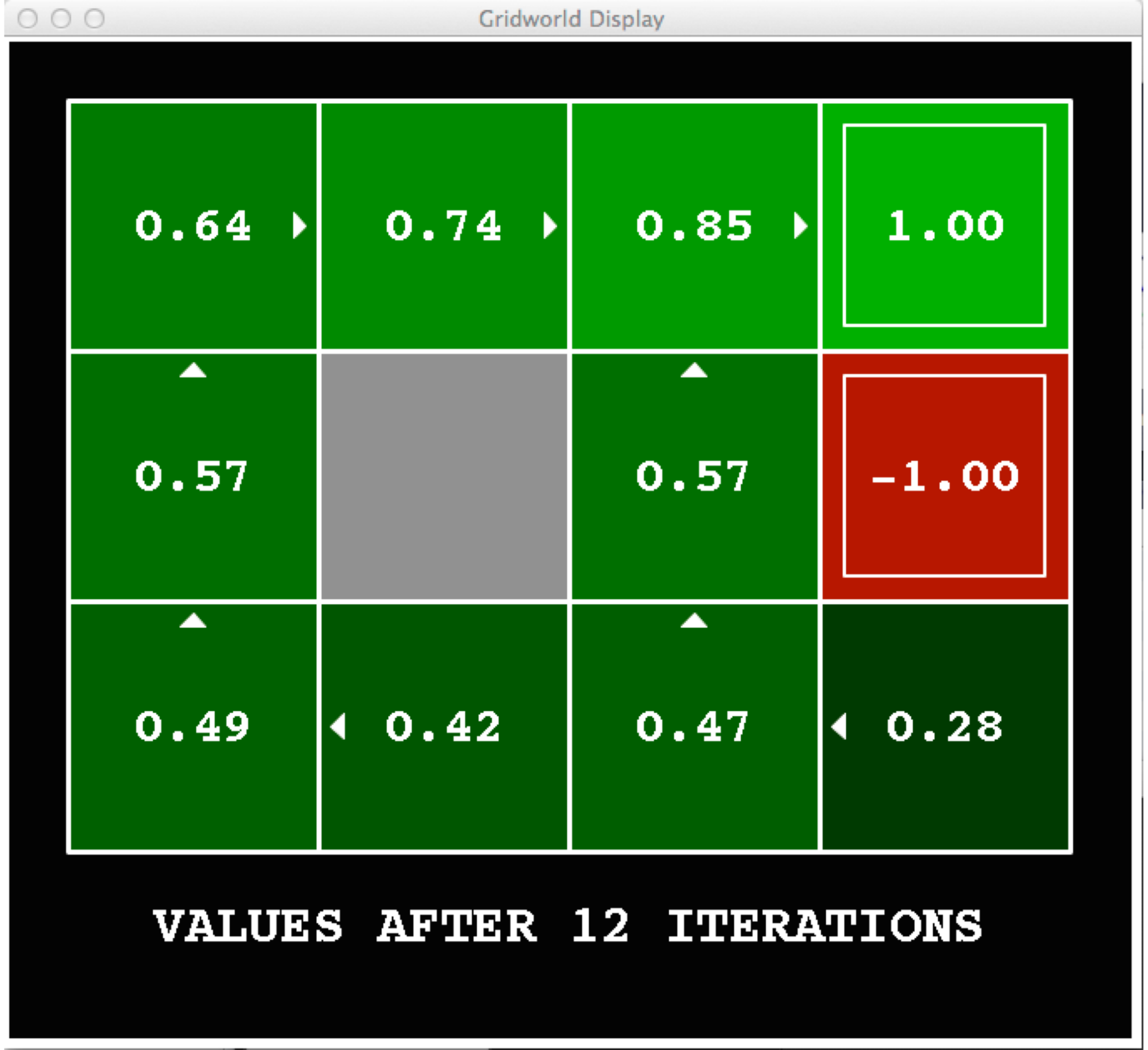

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=11



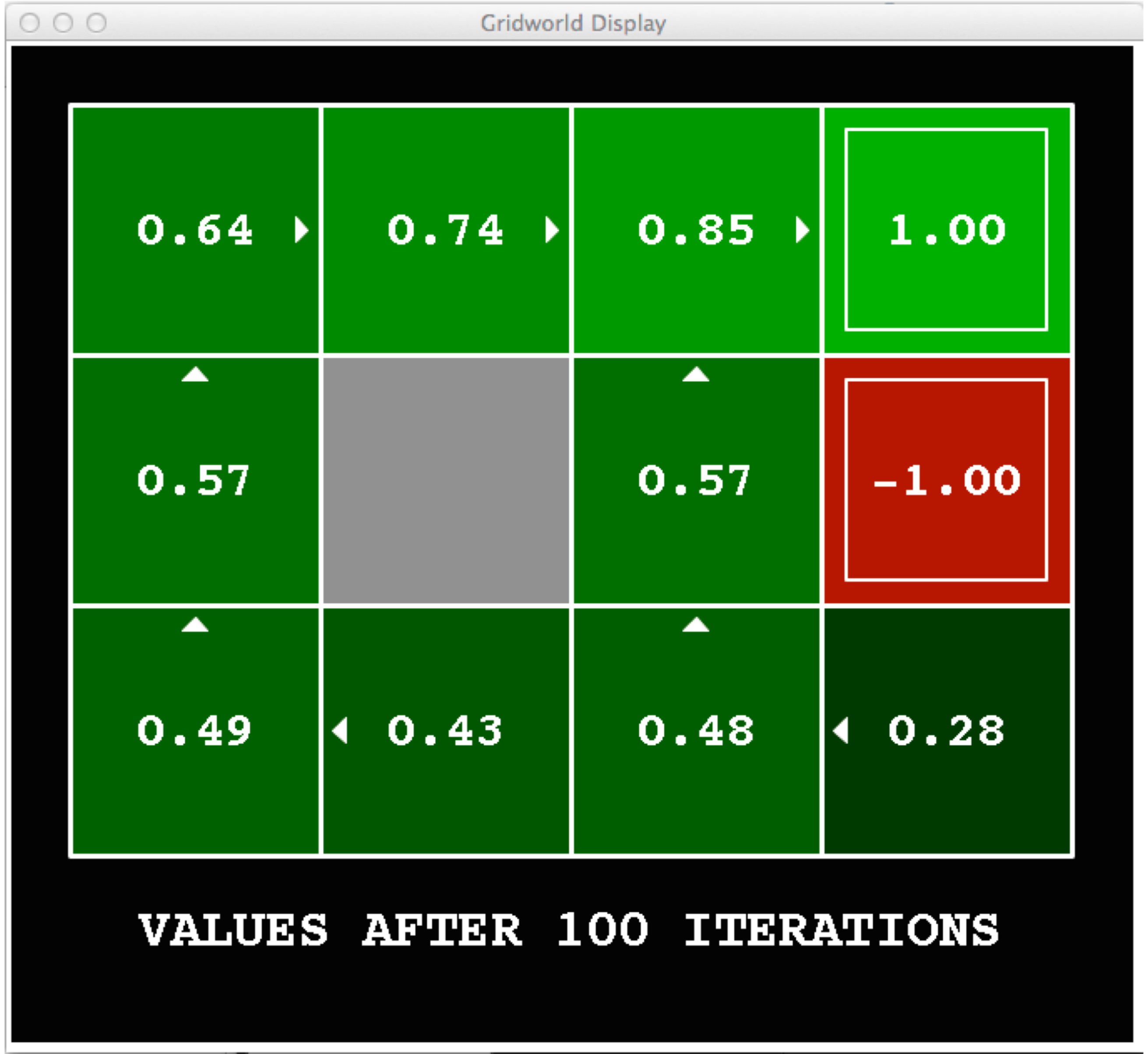

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=12



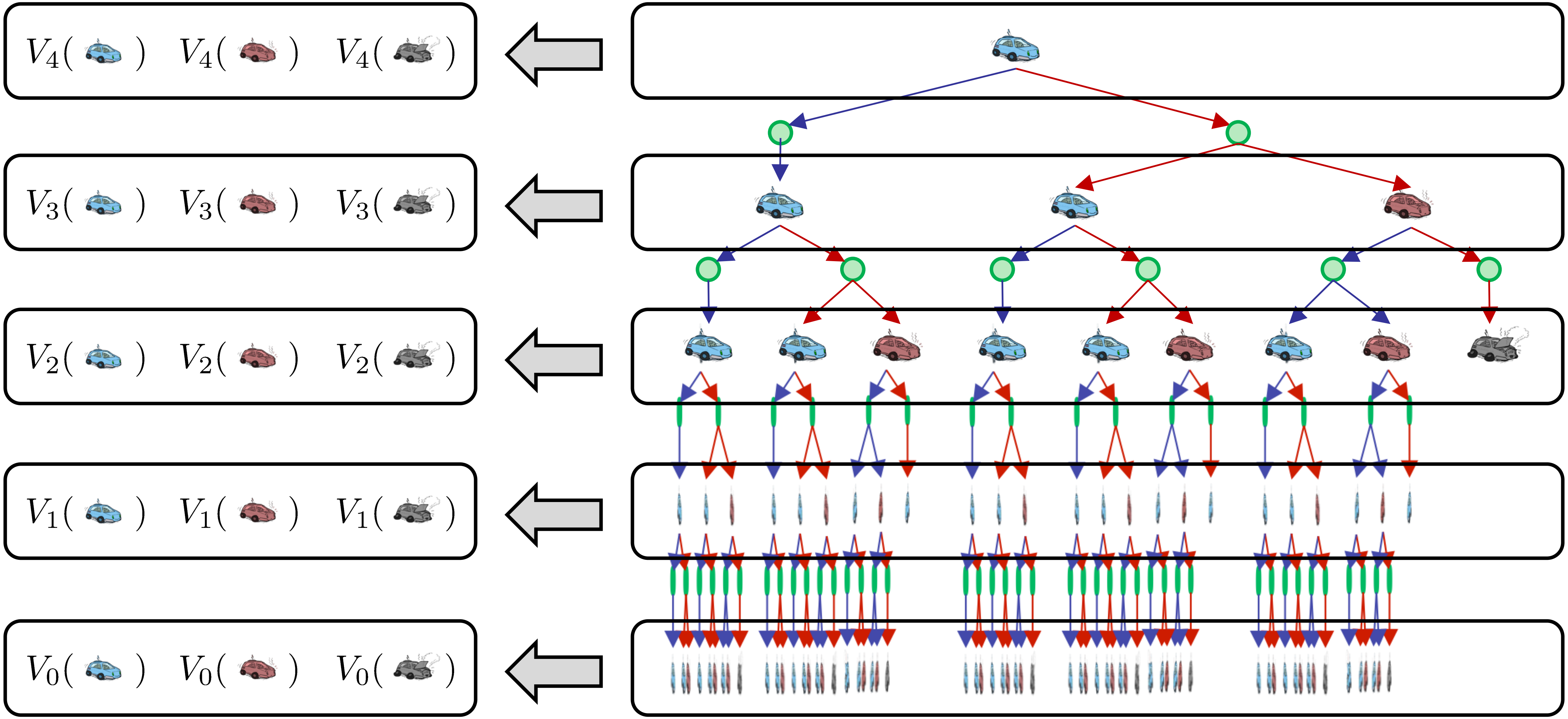

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=100

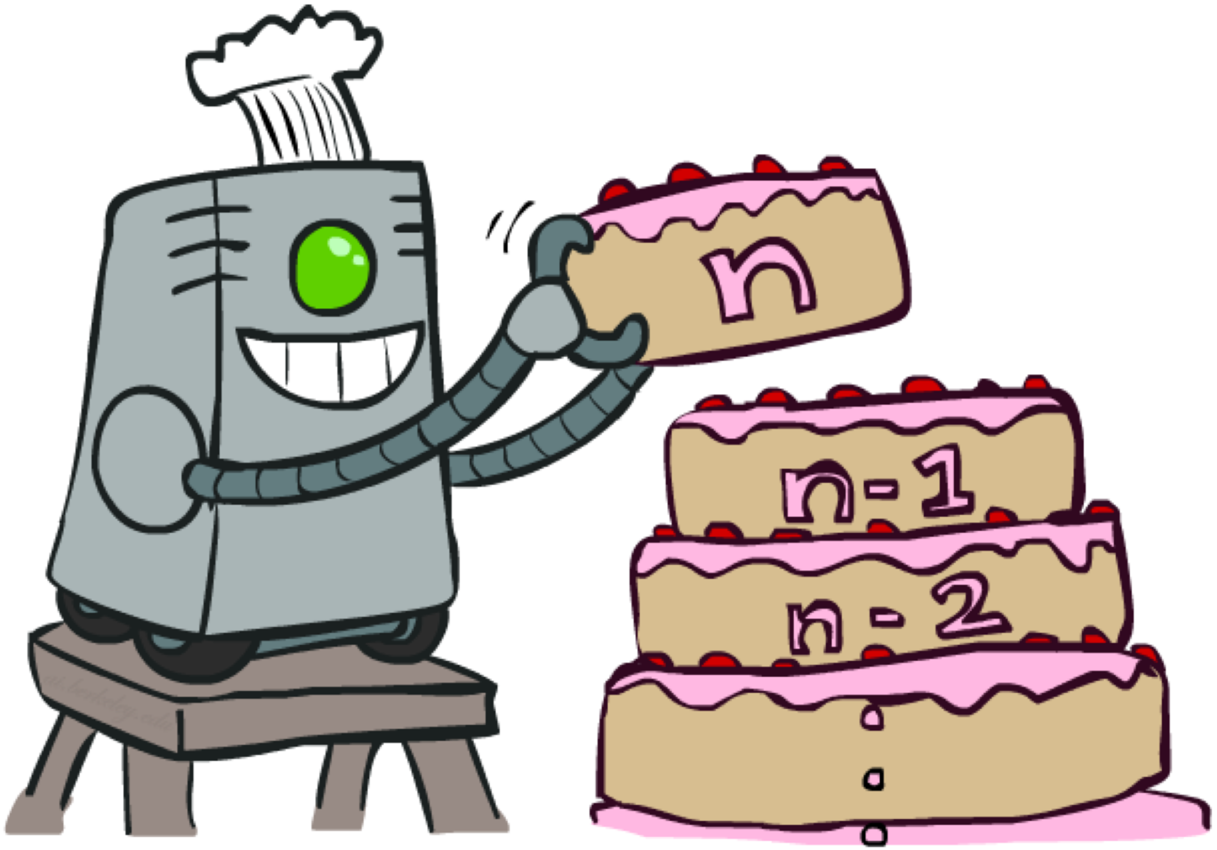



Noise = 0.2
Discount = 0.9
Living reward = 0

# Computing Time-Limited Values



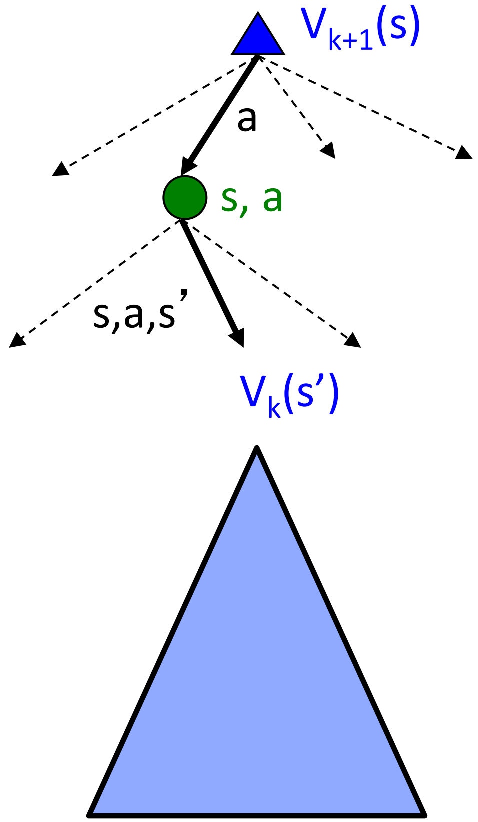
# Value Iteration



# Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

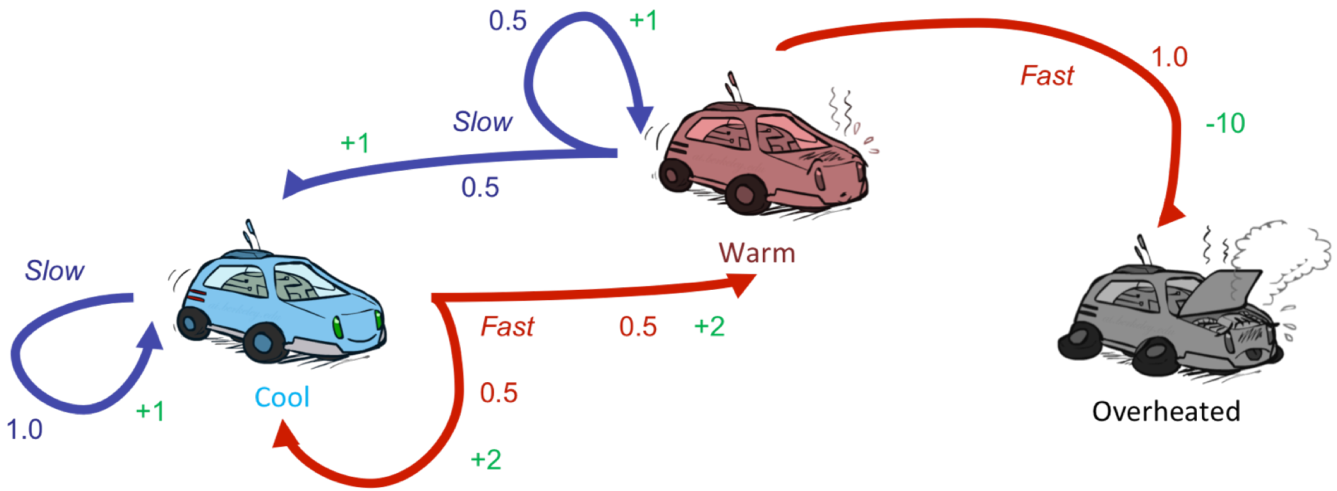
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$




- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

# Example: Value Iteration

| | | | |
|---|---|---|---|
| $V_2$ | | | |
| $V_1$ | S: 1<br>F: .5*2+.5*2=2 | | |
| $V_0$ | 0 | 0 | 0 |

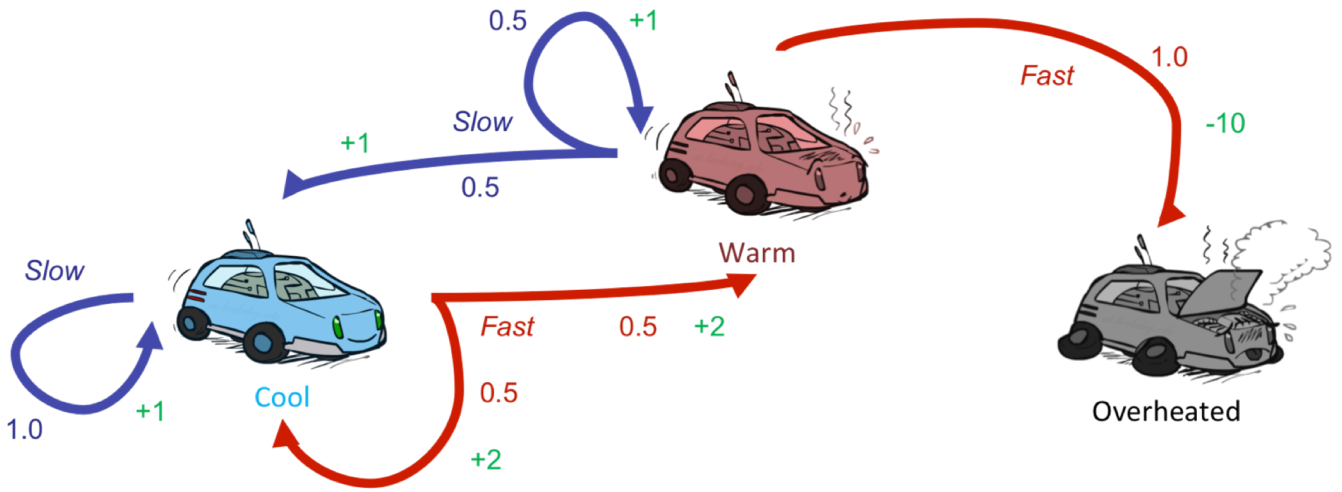



*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

# Example: Value Iteration

| | | | |
|---|---|---|---|
| $V_2$ | | | |
| $V_1$ | 2 | S: .5*1+.5*1=1<br>F: -10 | |
| $V_0$ | 0 | 0 | 0 |

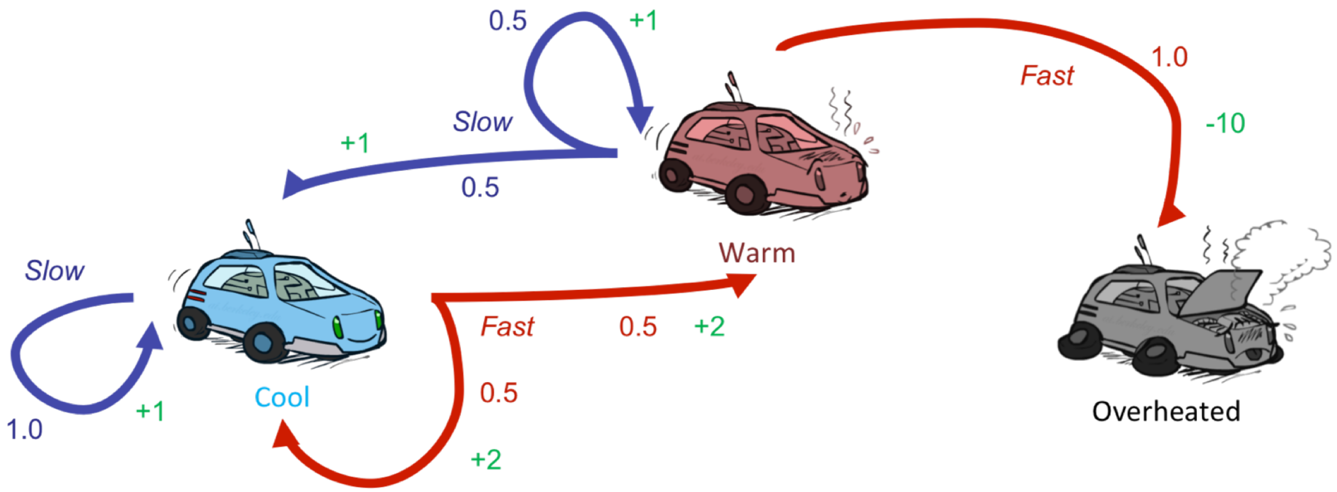



*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V_k(s')\right]$$

# Example: Value Iteration

| | | | |
|---|---|---|---|
| $V_2$ | | | |
| $V_1$ | 2 | 1 | 0 |
| $V_0$ | 0 | 0 | 0 |




*Assume no discount!*

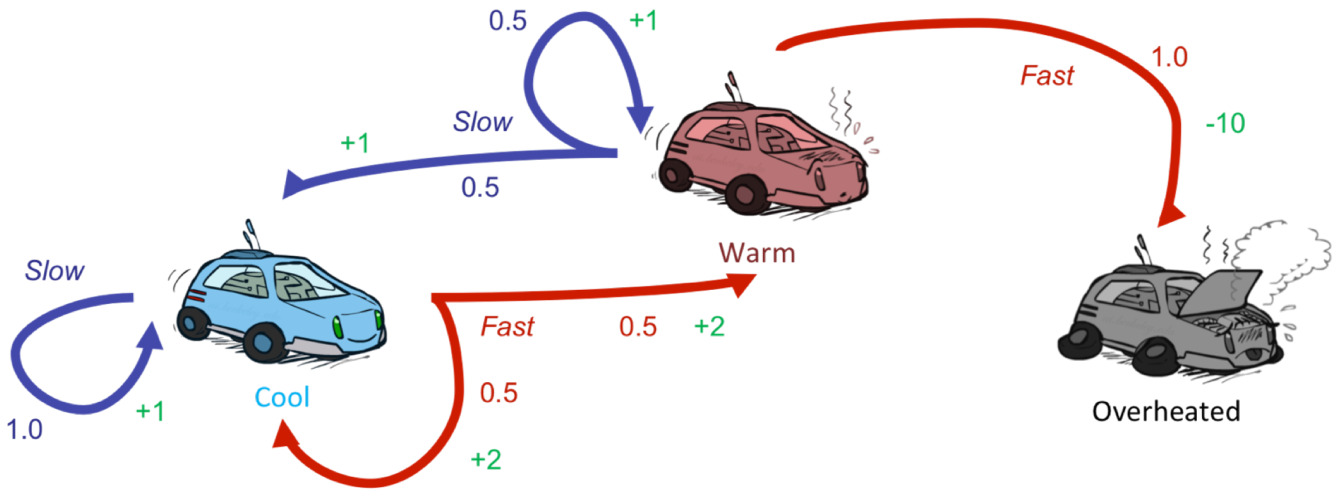
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

# Example: Value Iteration

$V_2$

S: 1+2=3
F:
.5*(2+2)+.5*(2+1)=3.5

| | | | |
|---|---|---|---|
| $V_1$ | 2 | 1 | 0 |
| $V_0$ | 0 | 0 | 0 |

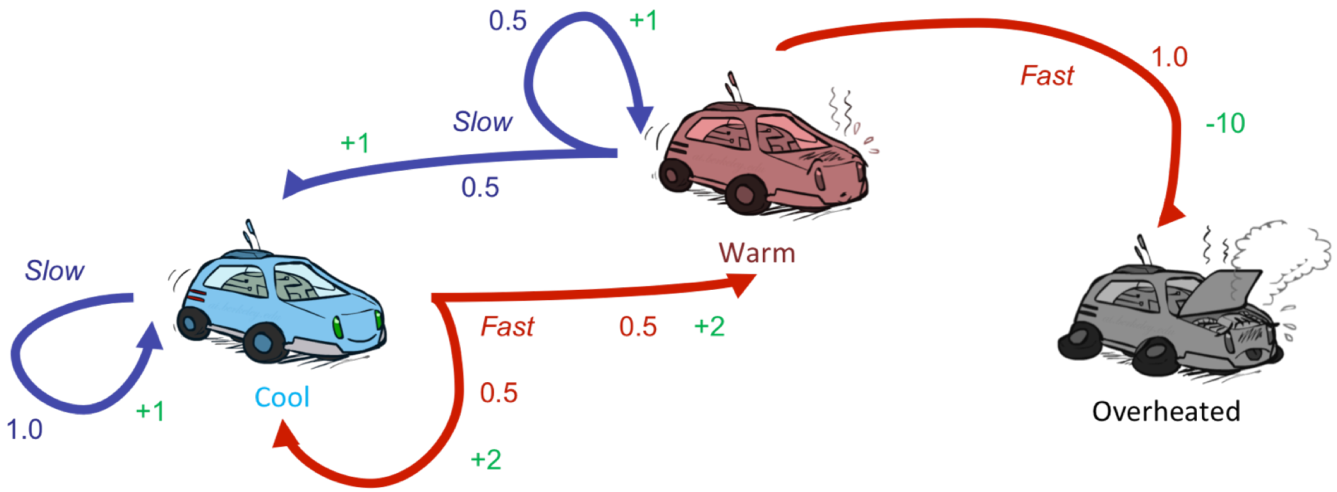



*Assume no discount!*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

# Example: Value Iteration

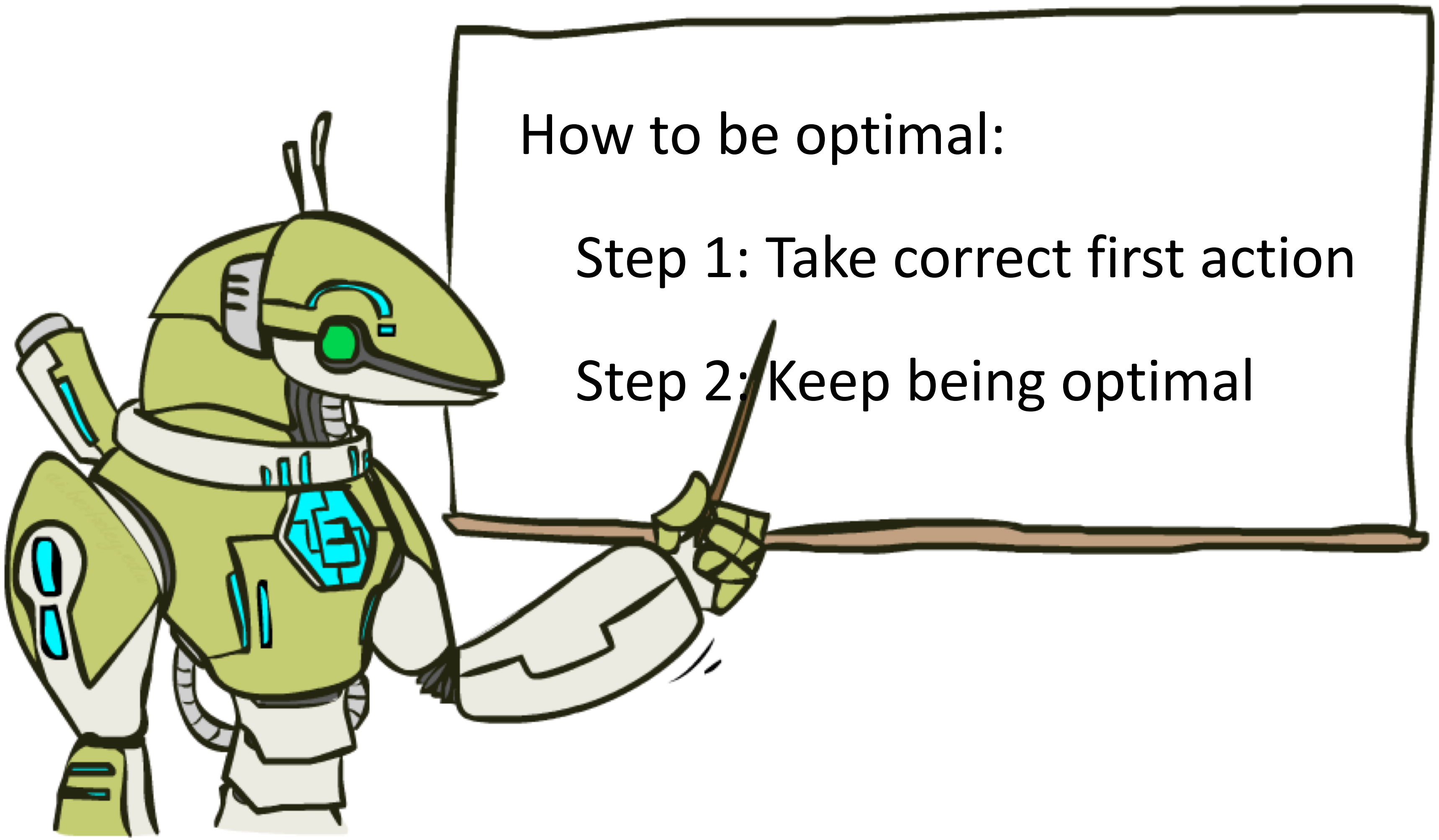
| | | | |
|---|---|---|---|
| $V_2$ | 3.5 | 2.5 | 0 |
| $V_1$ | 2 | 1 | 0 |
| $V_0$ | 0 | 0 | 0 |




*Assume no discount!*

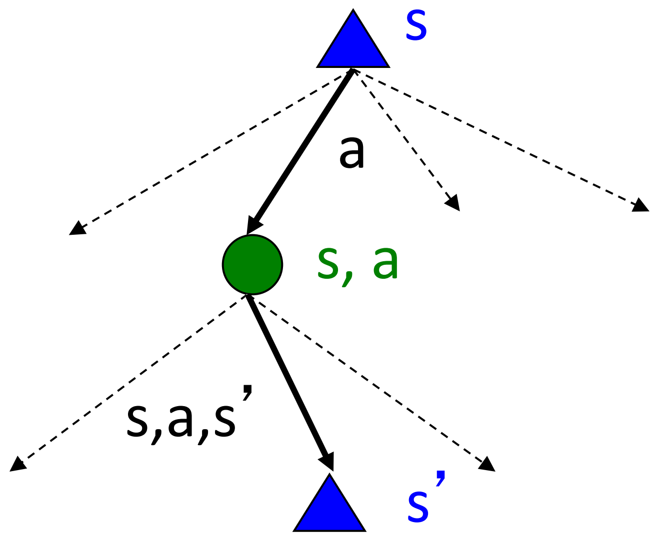
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

# The Bellman Equations



# The Bellman Equations

- Definition of "optimal utility" via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values




$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

- These are the Bellman equations, and they characterize optimal values in a way we'll use over and over

# Value Iteration

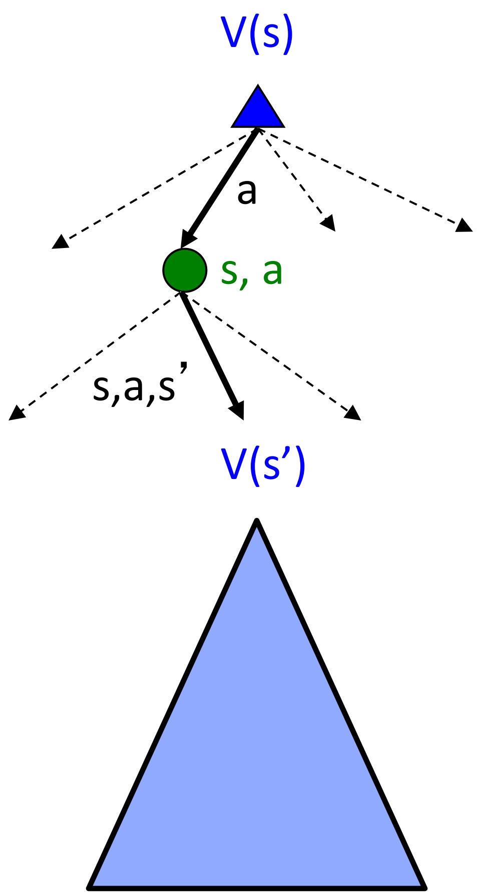
- Bellman equations characterize the optimal values:

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

- Value iteration computes them:

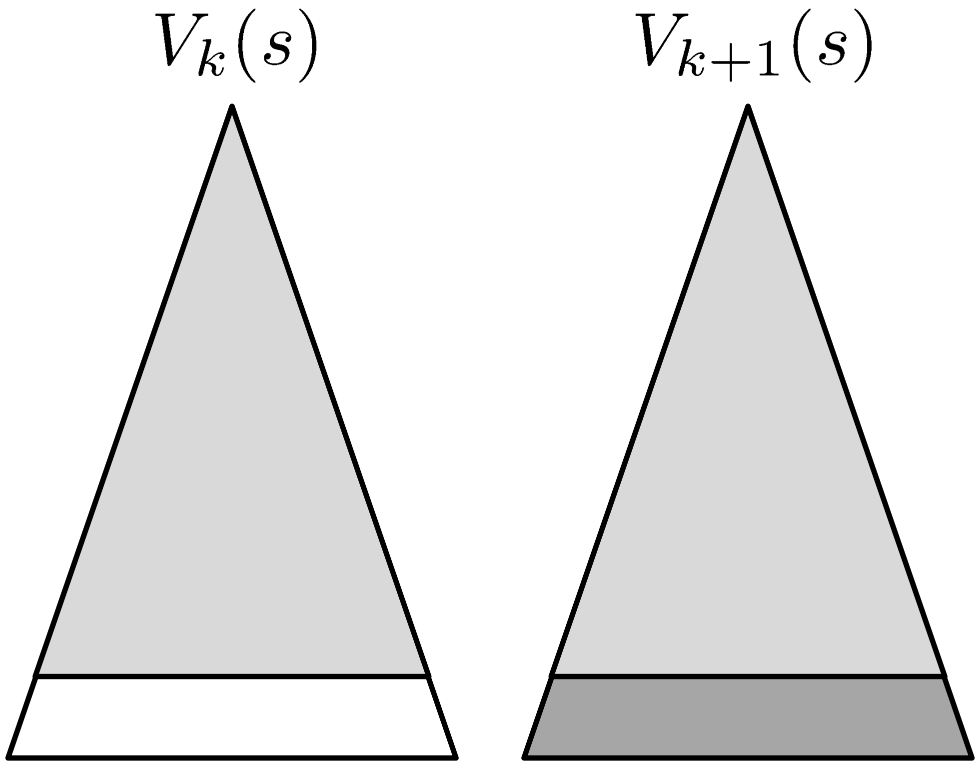
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

- Value iteration is just a fixed point solution method
  - … though the $V_k$ vectors are also interpretable as time-limited values

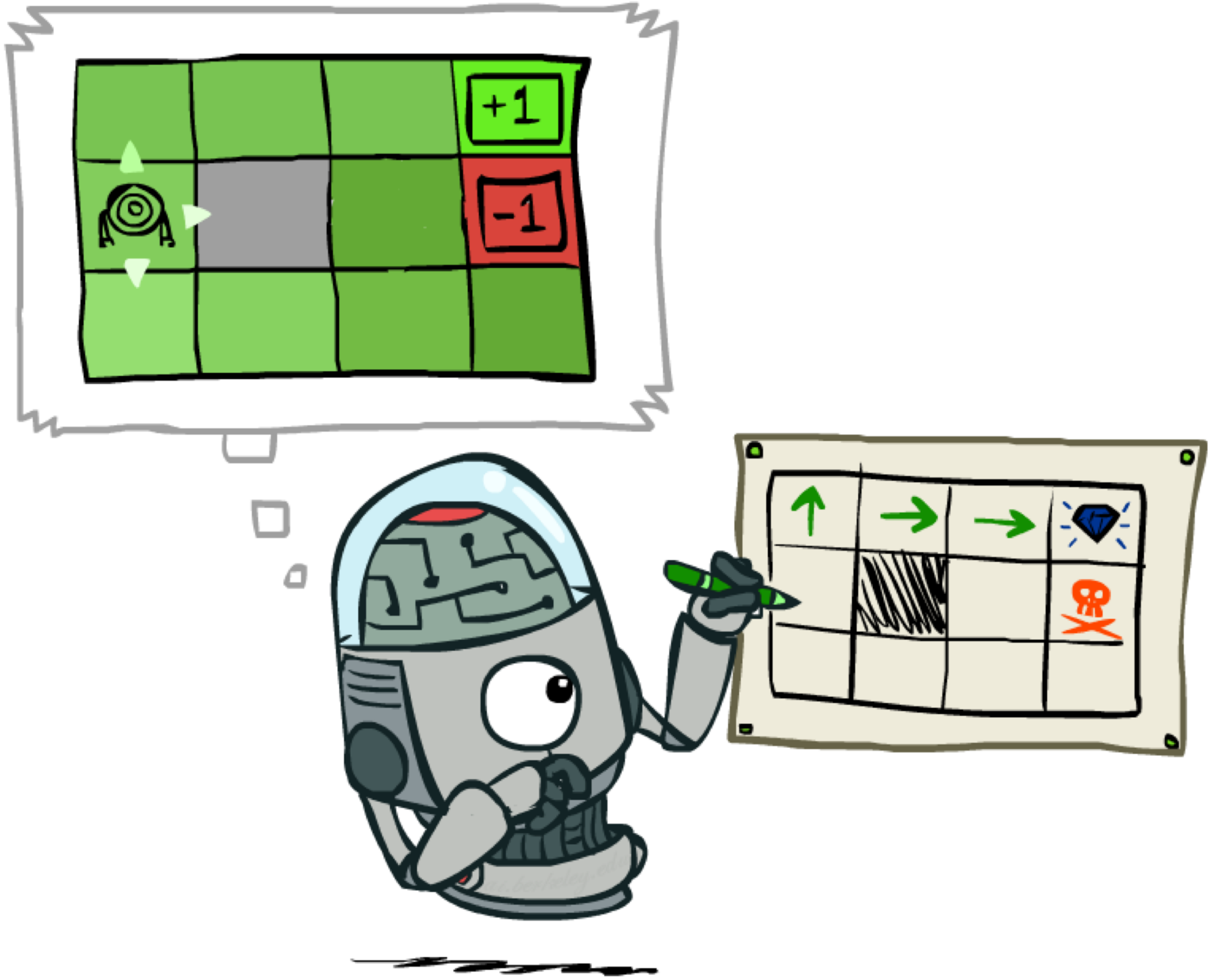


# Convergence*

- How do we know the $V_k$ vectors are going to converge?
- Case 1: If the tree has maximum depth M, then $V_M$ holds the actual untruncated values
- Case 2: If the discount is less than 1
  - Sketch: For any state $V_k$ and $V_{k+1}$ can be viewed as depth k+1 expectimax results in nearly identical search trees
  - The difference is that on the bottom layer, $V_{k+1}$ has actual rewards while $V_k$ has zeros
  - That last layer is at best all $R_{MAX}$
  - It is at worst $R_{MIN}$
  - But everything is discounted by $\gamma^k$ that far out
  - So $V_k$ and $V_{k+1}$ are at most $\gamma^k \max|R|$ different
  - So as k increases, the values converge



# Policy Extraction

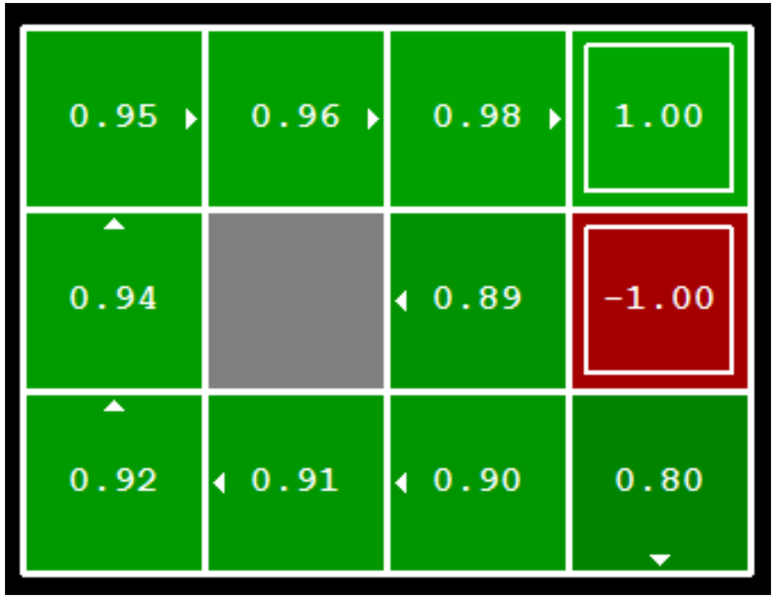


# Computing Actions from Values

- Let's imagine we have the optimal values V*(s)

- How should we act?
  - It's not obvious!

- We need to do a mini-expectimax (one step)

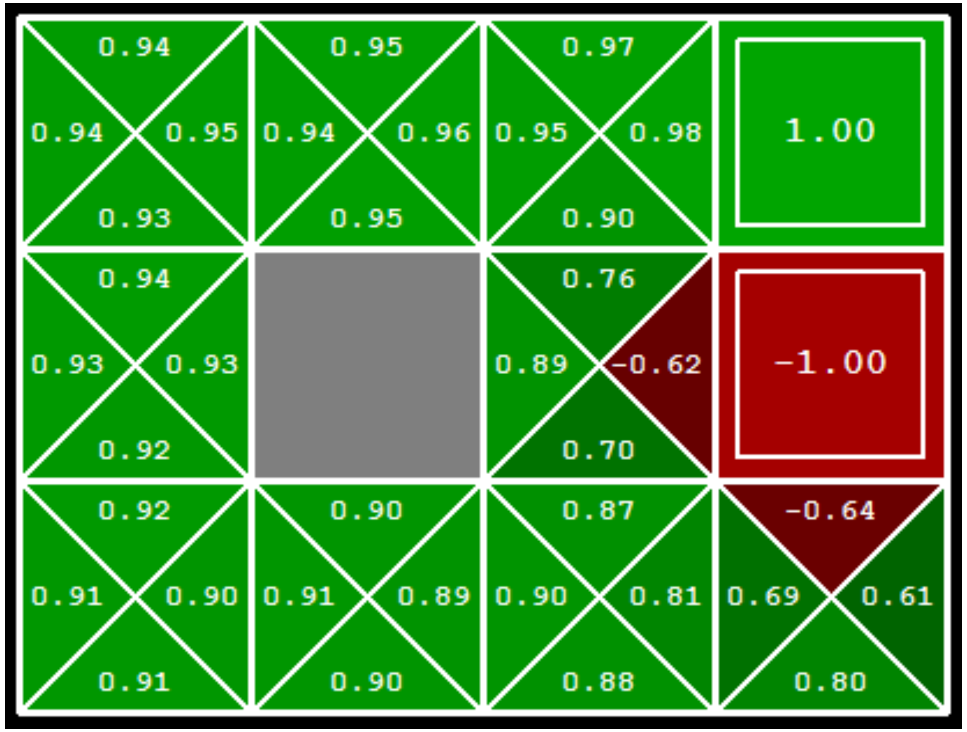



$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- This is called policy extraction, since it gets the policy implied by the values

# Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:

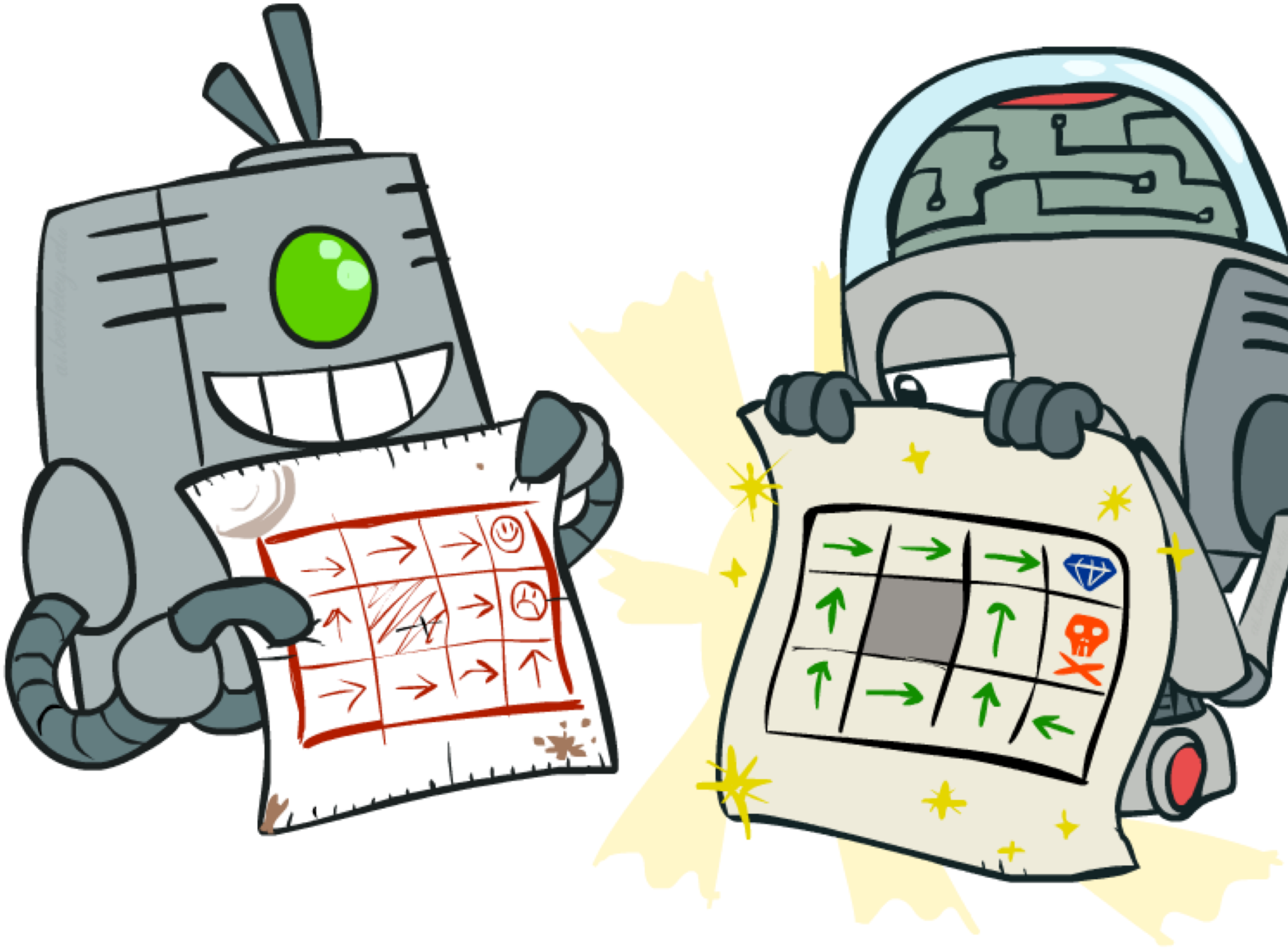



- How should we act?
  - Completely trivial to decide!

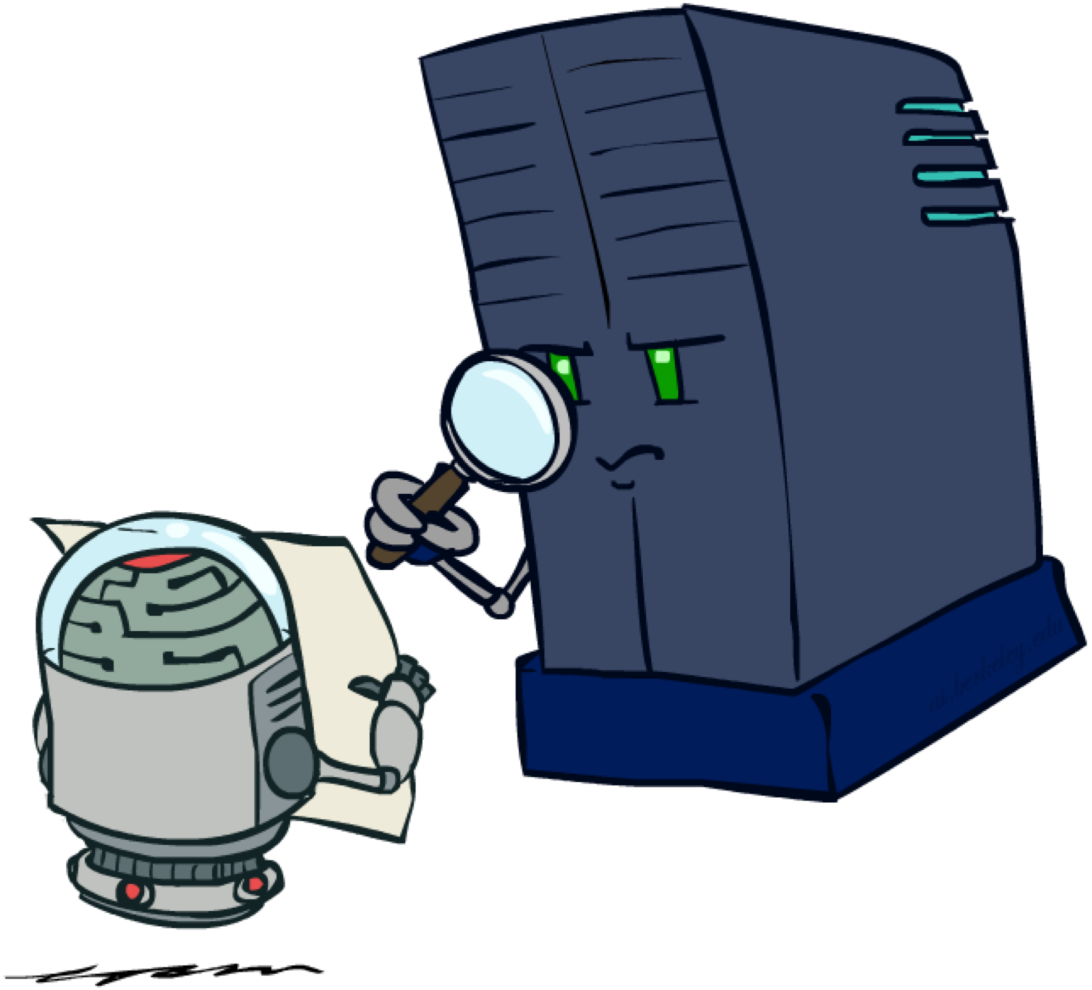
$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

- Important lesson: actions are easier to select from q-values than values!

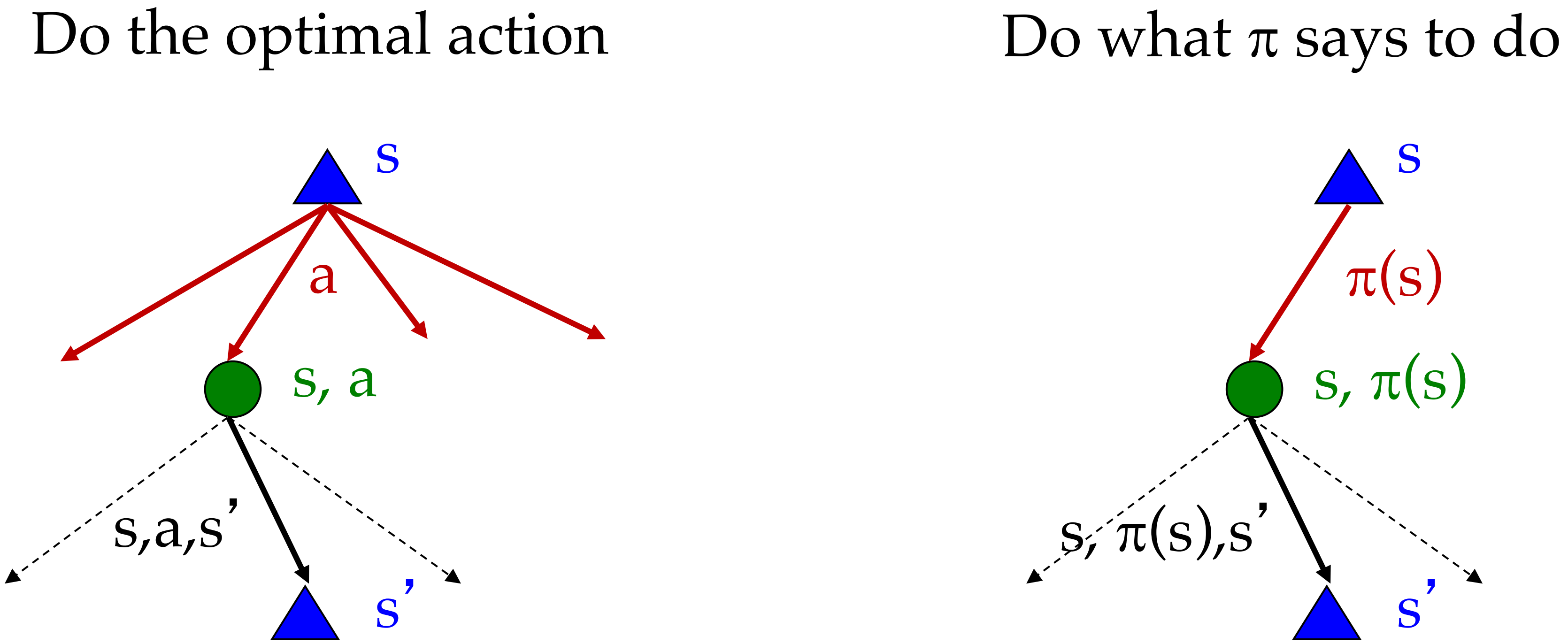
# Policy Methods



# Policy Evaluation



# Fixed Policies




- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
  - … though the tree's value would depend on which policy we fixed

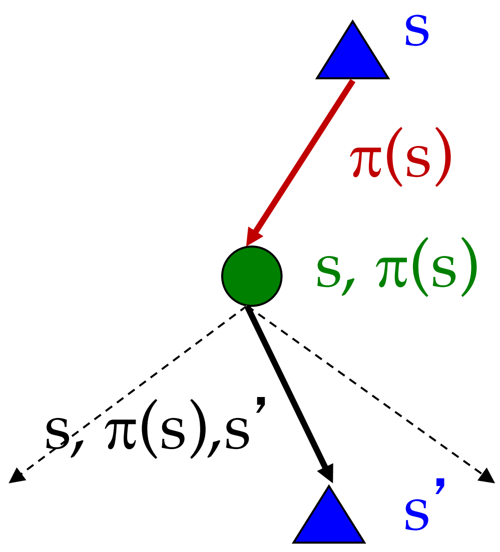
# Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy

- Define the utility of a state s, under a fixed policy π:
  $V^{\pi}(s)$ = expected total discounted rewards starting in s and following π

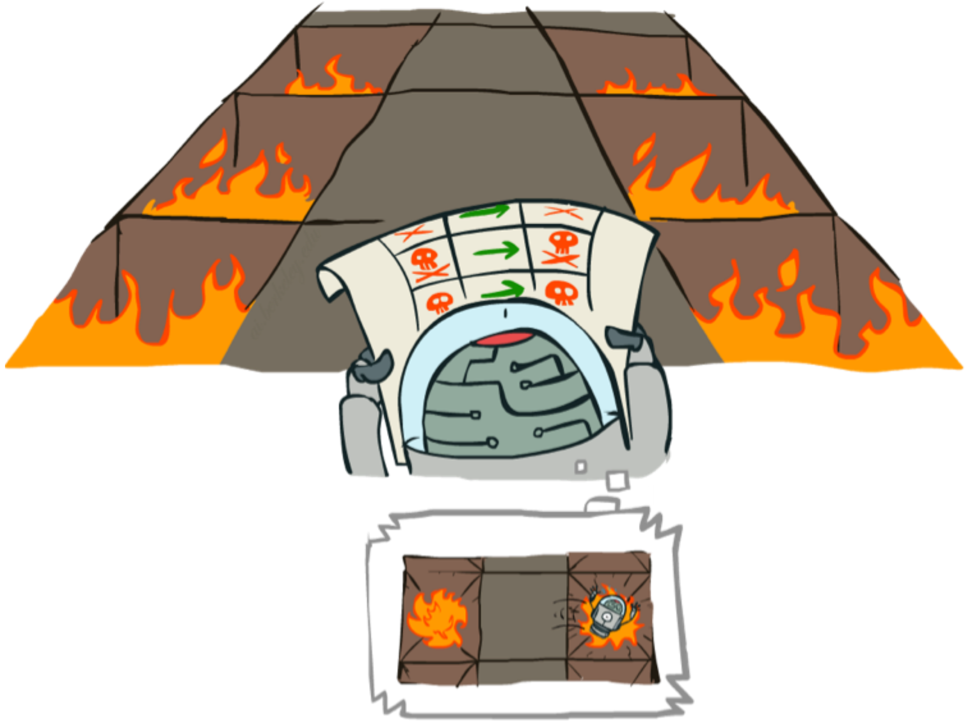
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

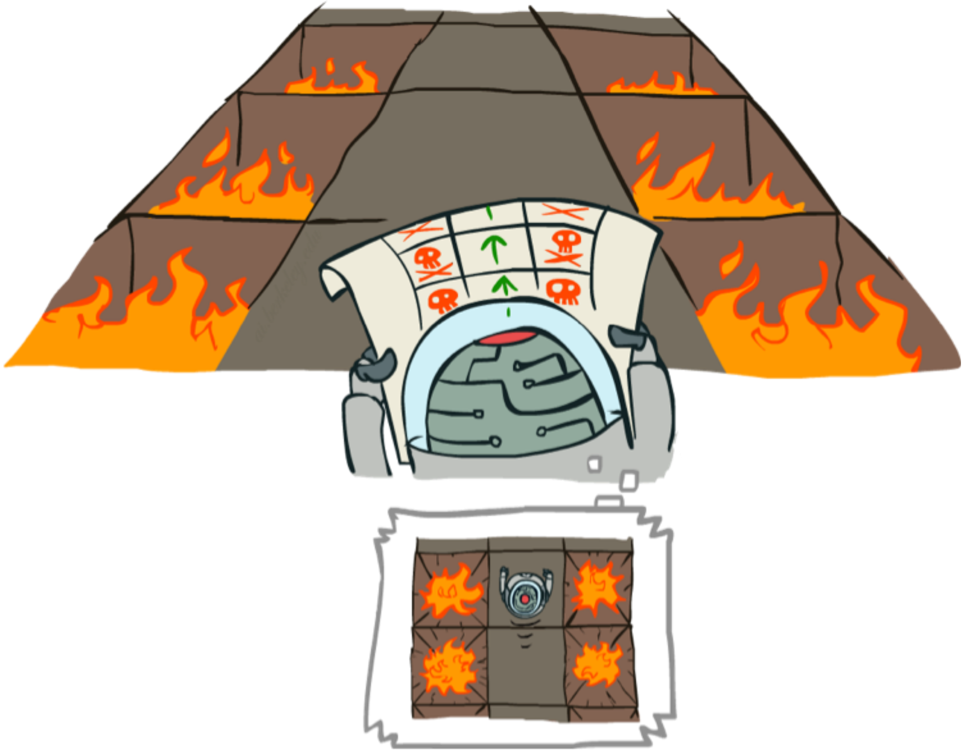


# Example: Policy Evaluation

Always Go Right



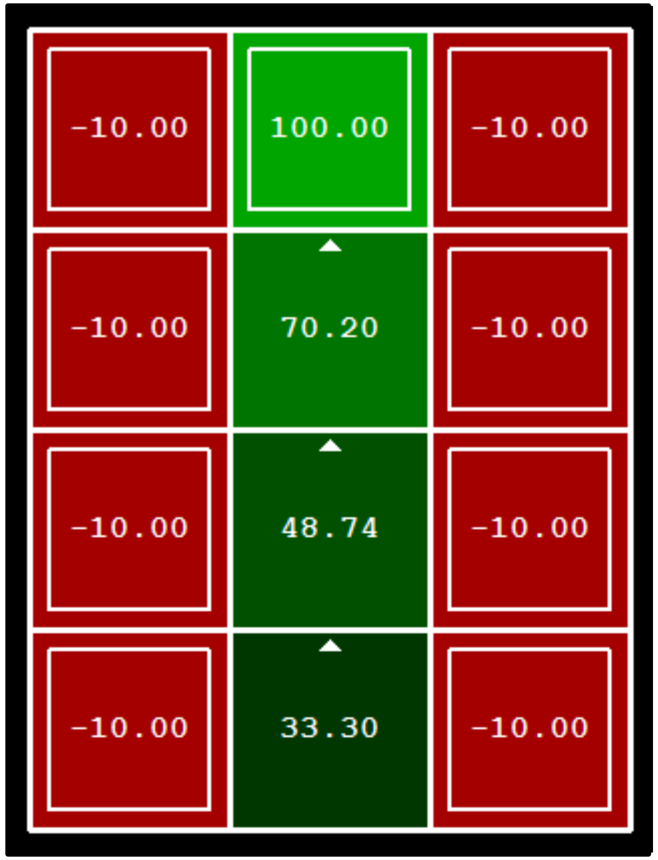
Always Go Forward



# Example: Policy Evaluation

Always Go Right


Always Go Forward



# Policy Evaluation

- How do we calculate the V's for a fixed policy π?

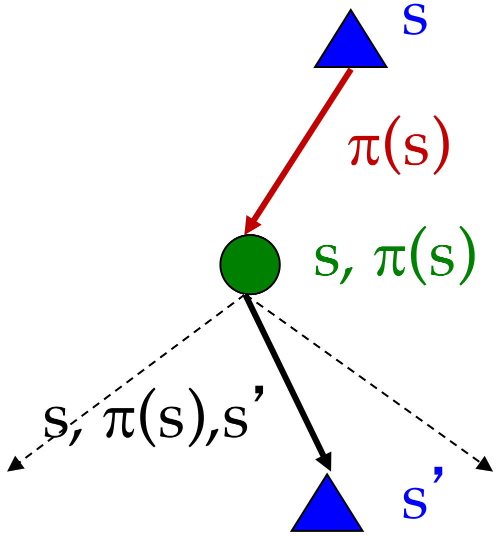
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

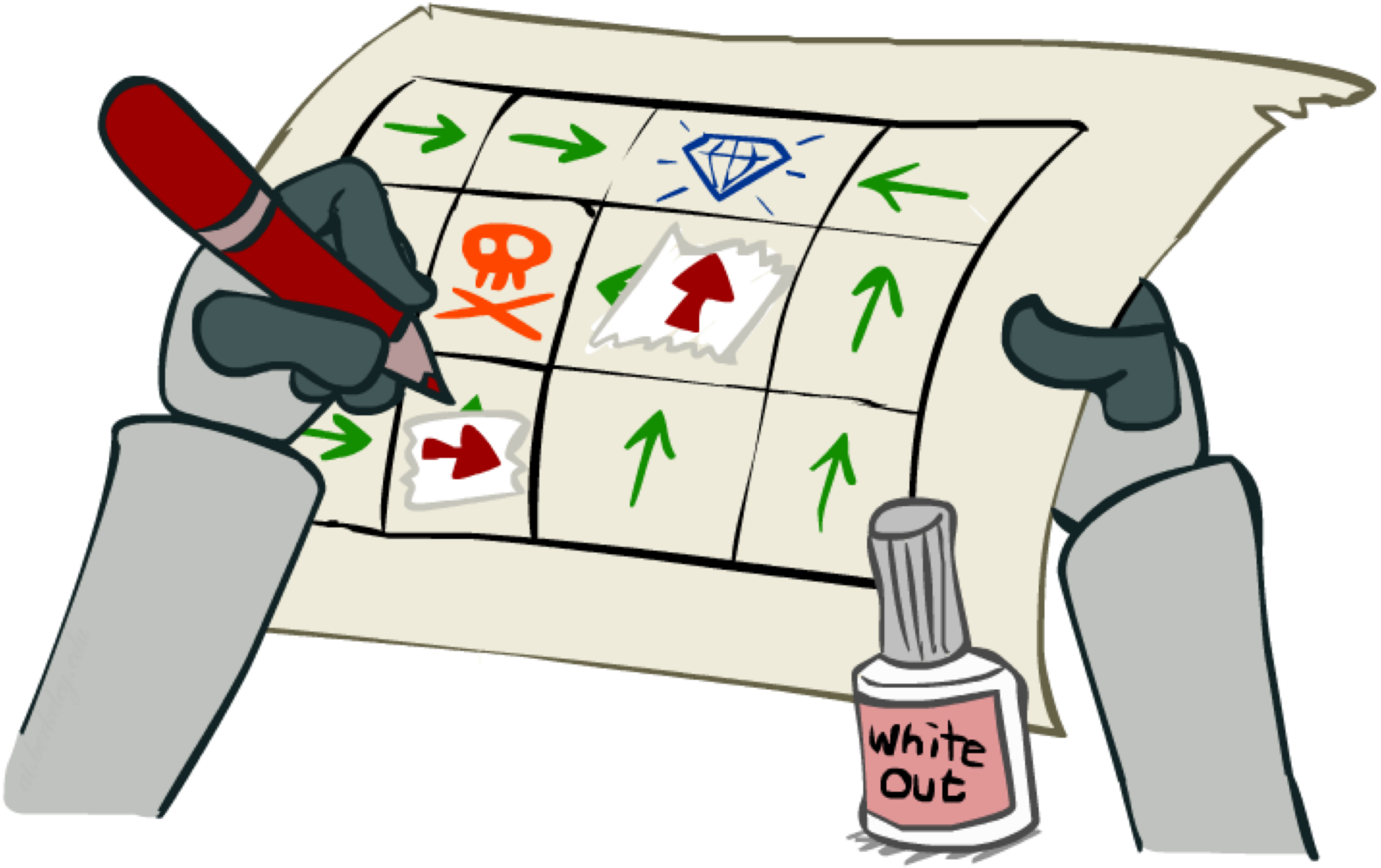
$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V_k^\pi(s')]$$

- Efficiency: $O(S^2)$ per iteration

- Idea 2: Without the maxes, the Bellman equations are just a linear system
  - Solve with Matlab (or your favorite linear system solver)



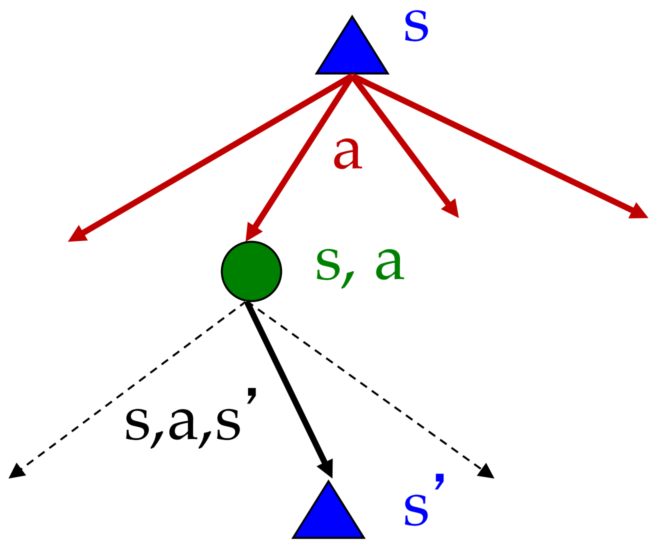
# Policy Iteration



# Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

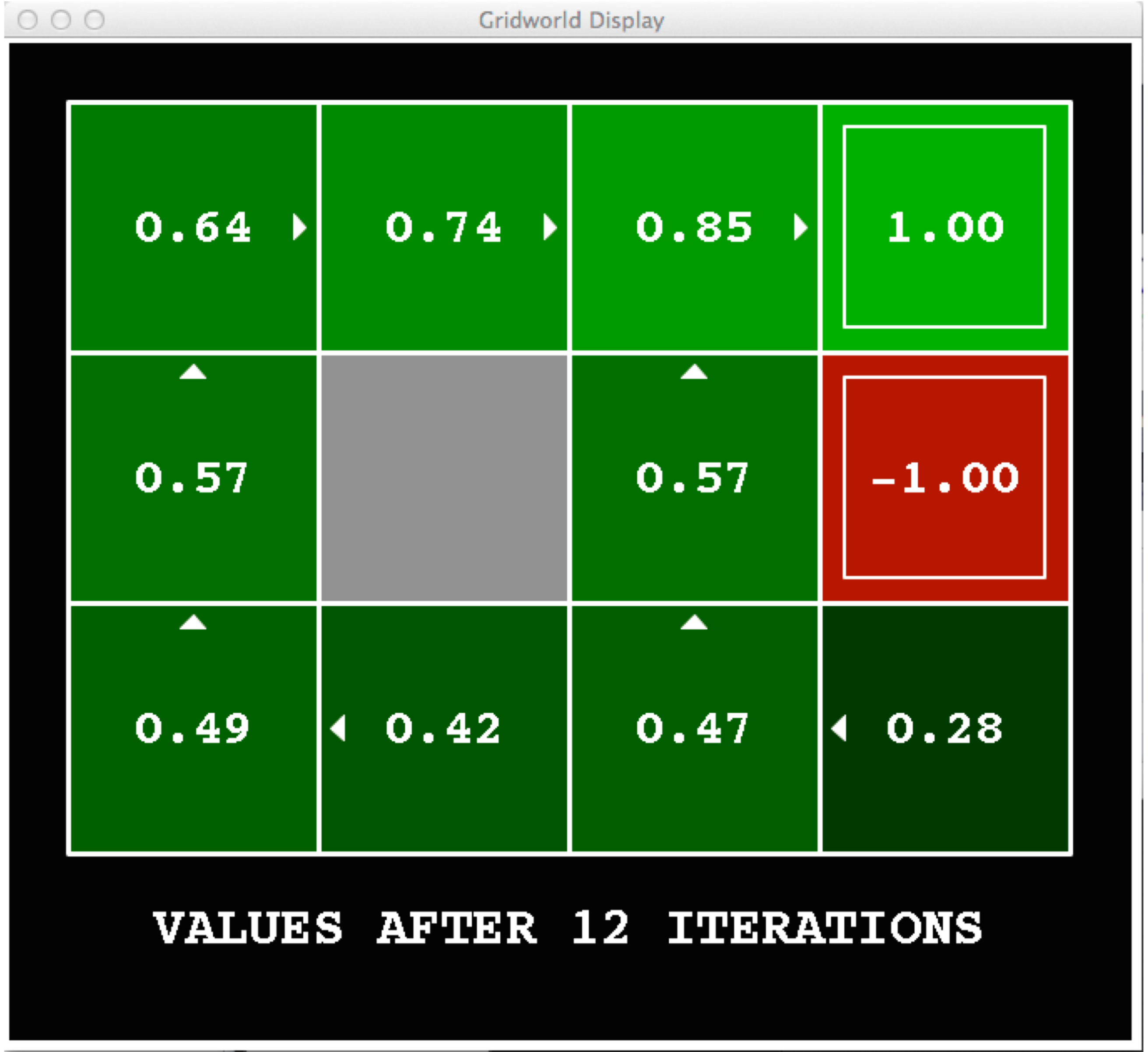



- Problem 1: It's slow – $O(S^2A)$ per iteration

- Problem 2: The "max" at each state rarely changes

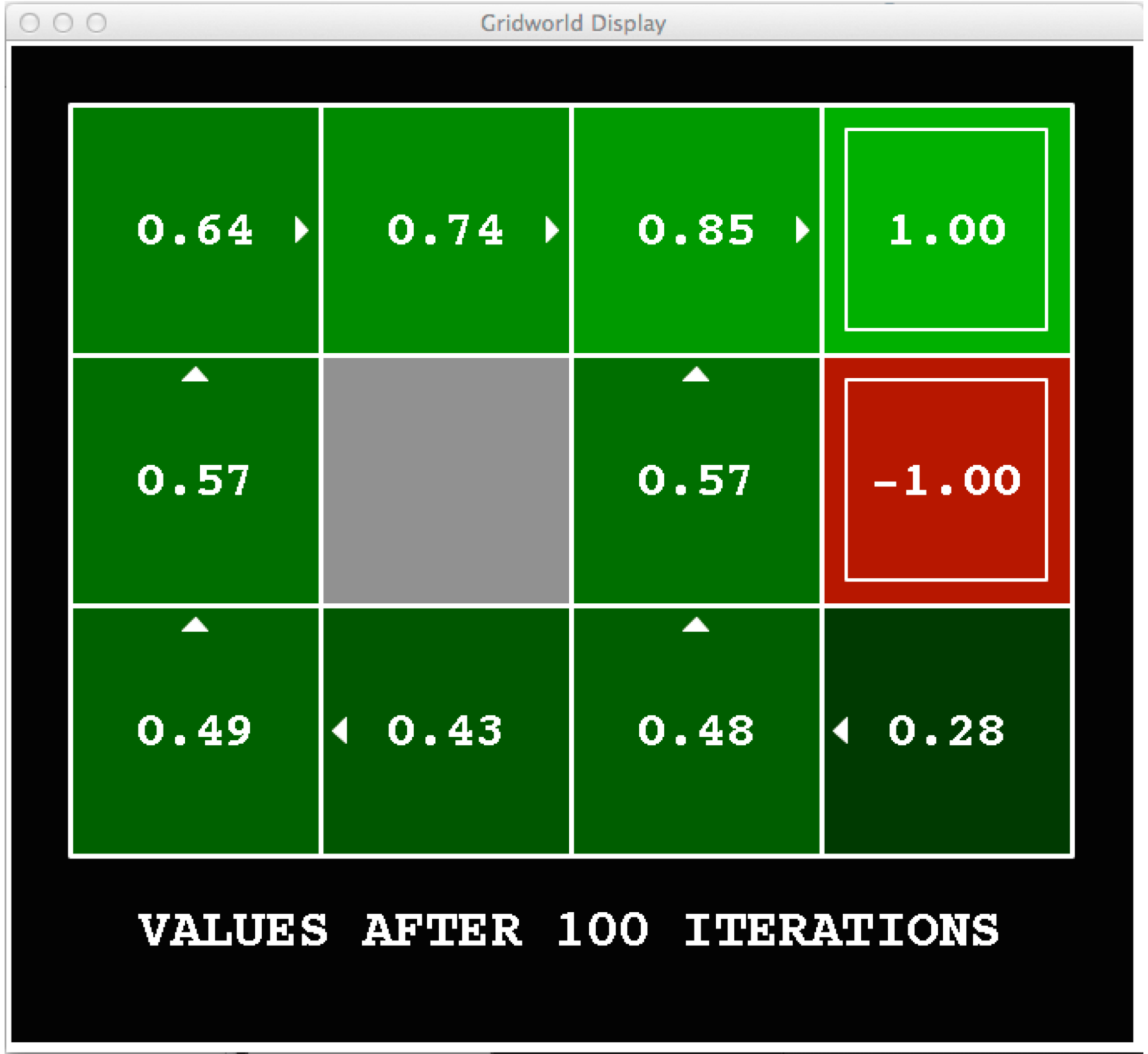
- Problem 3: The policy often converges long before the values

# k=12




Noise = 0.2
Discount = 0.9
Living reward = 0

# k=100




Noise = 0.2
Discount = 0.9
Living reward = 0

# Policy Iteration

- Alternative approach for optimal values:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

- This is policy iteration
  - It's still optimal!
  - Can converge (much) faster under some conditions

# Policy Iteration

- Evaluation: For fixed current policy π, find values with policy evaluation:
  - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')\right]$$

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

# Comparison

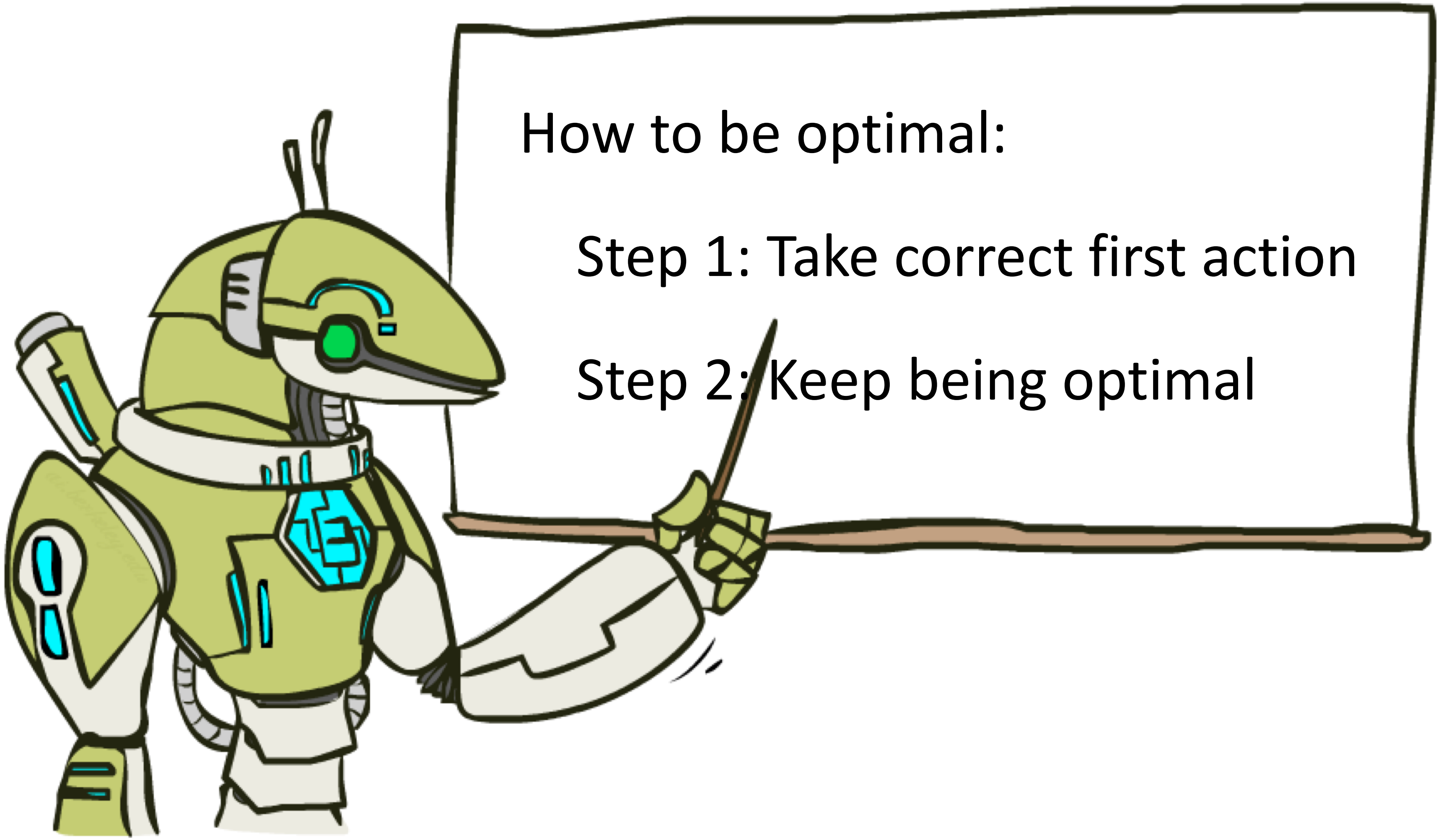
- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
  - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
  - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

# Summary: MDP Algorithms

- So you want to….
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)

- These all look the same!
  - They basically are – they are all variations of Bellman updates
  - They all use one-step lookahead expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions

# The Bellman Equations



# Next Time: Reinforcement Learning!