Heuristic Search and A* Algorithms

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With slides from :
Dieter Fox, Dan Weld, Dan Klein, Stuart Russell, Andrew Moore, Luke Zettlemoyer
Today

- A* Search
- Heuristic Design
- Graph search
Recap: Search

- **Search problem:**
  - States (configurations of the world)
  - Successor function: a function from states to lists of (state, action, cost) triples; drawn as a graph
  - Start state and goal test

- **Search tree:**
  - Nodes: represent plans for reaching states
  - Plans have costs (sum of action costs)

- **Search Algorithm:**
  - Systematically builds a search tree
  - Chooses an ordering of the fringe (unexplored nodes)
Example: Pancake Problem

Action: Flip over the top $n$ pancakes

Cost: Number of pancakes flipped
Example: Pancake Problem

BOUNDS FOR SORTING BY PREFIX REVERSAL

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Revised 28 August 1978

For a permutation $\sigma$ of the integers from 1 to $n$, let $f(\sigma)$ be the smallest number of prefix reversals that will transform $\sigma$ to the identity permutation, and let $f(n)$ be the largest such $f(\sigma)$ for all $\sigma$ in (the symmetric group) $S_n$. We show that $f(n) \leq (5n + 5)/3$, and that $f(n) \geq 17n/16$ for $n$ a multiple of 16. If, furthermore, each integer is required to participate in an even number of reversed prefixes, the corresponding function $g(n)$ is shown to obey $3n/2 - 1 \leq g(n) \leq 2n + 3$. 
Example: Pancake Problem

State space graph with costs as weights
General Tree Search

function TREE-SEARCH(problem, strategy) returns a solution, or failure

initialize the search tree using the initial state of problem

loop do
    if there are no candidates for expansion then return failure
    choose a leaf node for expansion according to strategy
    if the node contains a goal state then return the corresponding solution
    else expand the node and add the resulting nodes to the search tree
end

Action: flip top two
Cost: 2

Action: flip all four
Cost: 4

Path to reach goal: Flip four, flip three
Total cost: 7
Example: Heuristic Function

Heuristic: the largest pancake that is still out of place
What is a **Heuristic**?

- An *estimate* of how close a state is to a goal
- Designed for a particular search problem

Examples:
- Manhattan distance: $10 + 5 = 15$
- Euclidean distance: $11.2$
Example: Heuristic Function

$h(x)$
Greedy Search
Best First (Greedy)

- **Strategy:** expand a node that you think is closest to a goal state
  - Heuristic: estimate of distance to nearest goal for each state

- **A common case:**
  - Best-first takes you straight to the (wrong) goal

- **Worst-case:** like a badly-guided DFS
Greedy Search

- Expand the node that seems closest...

- What can go wrong?
A* Search
Combining UCS and Greedy

- **Uniform-cost** orders by path cost, or *backward cost* $g(n)$
- **Greedy** orders by goal proximity, or *forward cost* $h(n)$
- **A* Search** orders by the sum: $f(n) = g(n) + h(n)$

Example: Teg Grenager
When should A* terminate?

- Should we stop when we enqueue a goal?
  - No: only stop when we dequeue a goal
What went wrong?
- Actual bad goal cost < estimated good path cost
- We need estimates to be less than or equal to actual costs!
Admissible Heuristics

- A heuristic $h$ is **admissible** (optimistic) if:

  \[ 0 \leq h(n) \leq h^*(n) \]

  where $h^*(n)$ is the true cost to a nearest goal

- Examples:

  - Coming up with admissible heuristics is most of what’s involved in using A* in practice.
Optimality of A* Tree Search

Assume:
- A is an optimal goal node
- B is a suboptimal goal node
- h is admissible

Claim:
- A will exit the fringe before B
Optimality of A* Tree Search

Proof:

- Imagine B is on the fringe
- Some ancestor n of A is on the fringe, too (maybe A!)
- Claim: n will be expanded before B
  1. \( f(n) \) is less or equal to \( f(A) \)

\[
\begin{align*}
  f(n) &= g(n) + h(n) & \text{Definition of f-cost} \\
  f(n) &\leq g(A) & \text{Admissibility of h} \\
  g(A) &= f(A) & h = 0 \text{ at a goal}
\end{align*}
\]
Optimality of A* Tree Search

Proof:

- Imagine B is on the fringe
- Some ancestor $n$ of A is on the fringe, too (maybe A!)
- Claim: $n$ will be expanded before B
  1. $f(n)$ is less or equal to $f(A)$
  2. $f(A)$ is less than $f(B)$

$$g(A) < g(B) \quad \text{B is suboptimal}$$
$$f(A) < f(B) \quad \text{h = 0 at a goal}$$
Optimality of A* Tree Search

Proof:

- Imagine B is on the fringe
- Some ancestor $n$ of A is on the fringe, too (maybe A!)
- Claim: $n$ will be expanded before B
  1. $f(n)$ is less or equal to $f(A)$
  2. $f(A)$ is less than $f(B)$
  3. $n$ expands before B
- All ancestors of A expand before B
- A expands before B
- A* search is optimal

$f(n) \leq f(A) < f(B)$
UCS vs A* Contours

- Uniform-cost expanded in all directions
- A* expands mainly toward the goal, but hedges its bets to ensure optimality
Which Algorithm?

- Uniform cost search (UCS):
Which Algorithm?

- A*, Manhattan Heuristic:
Which Algorithm?

- Best First / Greedy, Manhattan Heuristic:
Creating Admissible Heuristics

- Most of the work in solving hard search problems optimally is in coming up with admissible heuristics.

- Often, admissible heuristics are solutions to *relaxed problems*, where new actions are available.

- Inadmissible heuristics are often useful too.
Creating Heuristics

8-puzzle:

- What are the states?
- How many states?
- What are the actions?
- What states can I reach from the start state?
- What should the costs be?
8 Puzzle I

- Heuristic: Number of tiles misplaced
  - $h(\text{start}) = 8$
- Is it admissible?

<table>
<thead>
<tr>
<th>Average nodes expanded when optimal path has length…</th>
<th>UCS</th>
<th>TILES</th>
</tr>
</thead>
<tbody>
<tr>
<td>…4 steps</td>
<td>112</td>
<td>13</td>
</tr>
<tr>
<td>…8 steps</td>
<td>6,300</td>
<td>39</td>
</tr>
<tr>
<td>…12 steps</td>
<td>$3.6 \times 10^6$</td>
<td>227</td>
</tr>
</tbody>
</table>
8 Puzzle II

- What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?
- Total *Manhattan* distance
  - \( h(\text{start}) = 3 + 1 + 2 + \ldots \)
  - \( = 18 \)

- Admissible?

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<table>
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<tr>
<th>TILES</th>
<th>13</th>
<th>39</th>
<th>227</th>
</tr>
</thead>
<tbody>
<tr>
<td>MANHATTAN</td>
<td>12</td>
<td>25</td>
<td>73</td>
</tr>
</tbody>
</table>

**Average nodes expanded when optimal path has length…**

- …4 steps
- …8 steps
- …12 steps

Start State

```
7 2 4
5 6
8 3 1
```

Goal State

```
1 2
3 4 5
6 7 8
```
8 Puzzle III

- How about using the *actual cost* as a heuristic?
  - Would it be admissible?
  - Would we save on nodes expanded?

- What’s wrong with it?

- With A*: a trade-off between quality of estimate and work per node!
Trivial Heuristics, Dominance

- Dominance: \( h_a(n) \geq h_c(n) \) if
  \[
  \forall n : h_a(n) \geq h_c(n)
  \]

- Heuristics form a semi-lattice:
  - Max of admissible heuristics is admissible
    \[
    h(n) = \max(h_a(n), h_b(n))
    \]

- Trivial heuristics
  - Bottom of lattice is the zero heuristic (what does this give us?)
  - Top of lattice is the exact heuristic
A* Applications

- Pathing / routing problems
- Resource planning problems
- Robot motion planning
- Language analysis
- Machine translation
- Speech recognition
- …
Tree Search: Extra Work!

- Failure to detect repeated states can cause exponentially more work. Why?
Graph Search

- In BFS, for example, we shouldn’t bother expanding some nodes (which, and why?)
Graph Search

- Idea: never expand a state twice

- How to implement:
  - Tree search + set of expanded states ("closed set")
  - Expand the search tree node-by-node, but...
  - Before expanding a node, check to make sure its state has never been expanded before
  - If not new, skip it, if new add to closed set

- Hint: in python, store the closed set as a set, not a list

- Can graph search wreck completeness? Why/why not?

- How about optimality?
A* Graph Search Gone Wrong

State space graph

- S (h=2)
  - A (h=4)
    - B (h=1)
      - G (h=0)
- C (h=1)

Search tree

- S (0+2) → A (1+4) → C (2+1) → G (5+0)
  - B (1+1) → C (3+1) → G (6+0)
Consistency of Heuristics

- **Main idea:** estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal
    \[ h(A) \leq \text{actual cost from A to G} \]
  - Consistency: heuristic “arc” cost ≤ actual cost for each arc
    \[ h(A) - h(C) \leq \text{cost(A to C)} \]

- **Consequences of consistency:**
  - The f value along a path never decreases
    \[ h(A) \leq \text{cost(A to C)} + h(C) \]
    \[ f(A) = g(A) + h(A) \leq g(A) + \text{cost(A to C)} + h(C) = f(C) \]
  - A* graph search is optimal
Optimality of A* Graph Search

- Sketch: consider what A* does with a consistent heuristic:
  - Nodes are popped with non-decreasing f-scores: for all \( n, n' \) with \( n' \) popped after \( n \) : \( f(n') \geq f(n) \)
    - Proof by induction: (1) always pop the lowest f-score from the fringe, (2) all new nodes have larger (or equal) scores, (3) add them to the fringe, (4) repeat!
  - For every state \( s \), nodes that reach \( s \) optimally are expanded before nodes that reach \( s \) sub-optimally
  - Result: A* graph search is optimal
Optimality

- **Tree search:**
  - A* optimal if heuristic is admissible (and non-negative)
  - UCS is a special case \((h = 0)\)

- **Graph search:**
  - A* optimal if heuristic is consistent
  - UCS optimal \((h = 0\) is consistent)

- Consistency implies admissibility

- In general, natural admissible heuristics tend to be consistent, especially if from relaxed problems
Summary: A*

- A* uses both backward costs and (estimates of) forward costs
- A* is optimal with admissible / consistent heuristics
- Heuristic design is key: often use relaxed problems