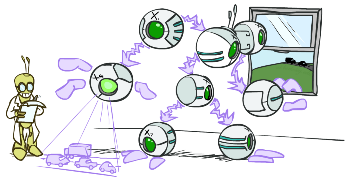
CSE 473: Artificial Intelligence

Bayes' Nets: Inference



Dieter Fox

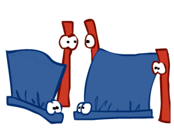
[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

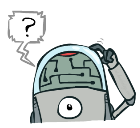
Bayes' Nets

- ✔ Representation
- ✔ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Probabilistic inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data

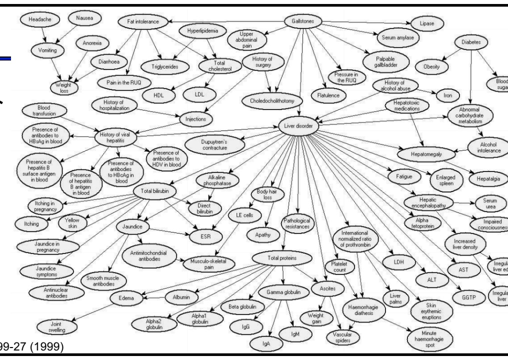
Inference

- Inference: calculating some useful quantity from a joint probability distribution
- Examples:
  - Posterior probability
    $P(Q|E_1 = e_1, \ldots E_k = e_k)$
  - Most likely explanation:
    $\text{argmax}_q \; P(Q = q|E_1 = e_1 \ldots)$





Liver Disorder



Onisko *et al* CBMI-99-27 (1999)

Inference by Enumeration

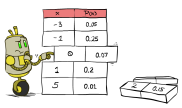
- General case:
  - Evidence variables: $E_1 \ldots E_k = e_1 \ldots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \ldots H_r$

  $\left.\begin{matrix} E_1 \ldots E_k = e_1 \ldots e_k \\ Q \\ H_1 \ldots H_r \end{matrix}\right\} X_1, X_2, \ldots X_n$ *All variables*

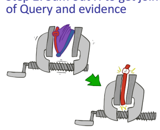
- We want: $P(Q|e_1 \ldots e_k)$

** Works fine with multiple query variables, too*

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \ldots e_k) = \sum_{h_1 \ldots h_r} \underbrace{P(Q, h_1 \ldots h_r, e_1 \ldots e_k)}_{X_1, X_2, \ldots X_n}$$

- Step 3: Normalize

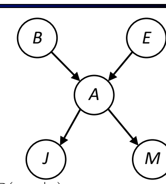
$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference by Enumeration in Bayes' Net

- Given unlimited time, inference in BNs is easy
- Reminder of inference by enumeration by example:



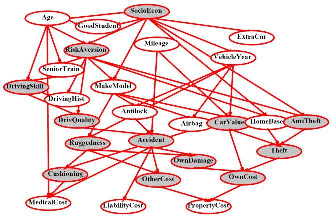
$$P(B \mid +j, +m) \propto_B P(B, +j, +m)$$

$$= \sum_{e,a} P(B, e, a, +j, +m)$$

$$= \sum_{e,a} P(B)P(e)P(a|B,e)P(+j|a)P(+m|a)$$

$$= P(B)P(+e)P(+a|B,+e)P(+j|+a)P(+m|+a) + P(B)P(+e)P(-a|B,+e)P(+j|-a)P(+m|-a$$
$$P(B)P(-e)P(+a|B,-e)P(+j|+a)P(+m|+a) + P(B)P(-e)P(-a|B,-e)P(+j|-a)P(+m|-a$$

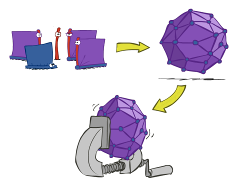
Inference by Enumeration?



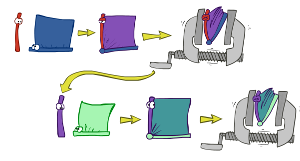
$P(Antilock|observed\ variables) = ?$

Inference by Enumeration vs. Variable Elimination

- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables

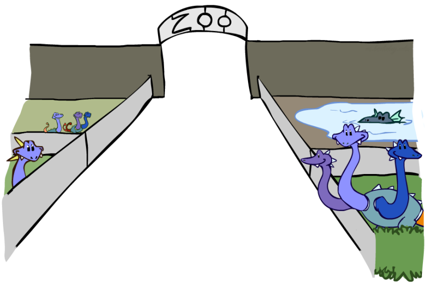


- Idea: interleave joining and marginalizing!
  - Called "Variable Elimination"
  - Still NP-hard, but usually much faster than inference by enumeration



- First we'll need some new notation: factors

Factor Zoo



Factor Zoo I

- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - Sums to 1

$P(T,W)$

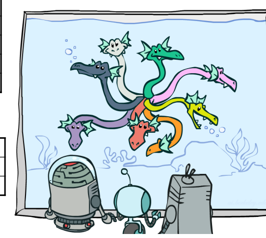
T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Selected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for fixed x, all y
  - Sums to P(x)

$P(cold,W)$

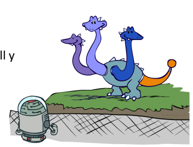
T	W	P
cold	sun	0.2
cold	rain	0.3

- Number of capitals = dimensionality of the table



Factor Zoo II

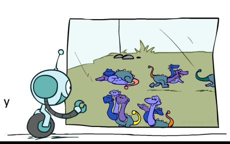
- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, all y
  - Sums to 1



$P(W|cold)$

T	W	P
cold	sun	0.4
cold	rain	0.6

- Family of conditionals: P(X |Y)
  - Multiple conditionals
  - Entries P(x | y) for all x, y
  - Sums to |Y|



$P(W|T)$

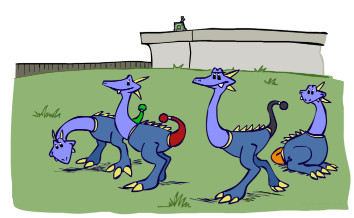
T	W	P	
hot	sun	0.8	$P(W|hot)$
hot	rain	0.2	
cold	sun	0.4	$P(W|cold)$
cold	rain	0.6	

Factor Zoo III

- Specified family: P(y | X)
  - Entries P(y | x) for fixed y, but for all x
  - Sums to … who knows!

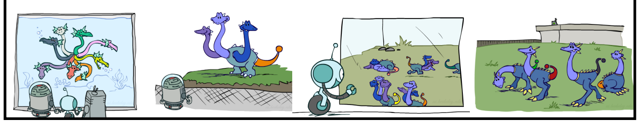
$P(rain|T)$

T	W	P	
hot	rain	0.2	$P(rain|hot)$
cold	rain	0.6	$P(rain|cold)$



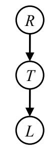
Factor Zoo Summary

- In general, when we write $P(Y_1 \dots Y_N \mid X_1 \dots X_M)$
  - It is a "factor," a multi-dimensional array
  - Its values are $P(y_1 \dots y_N \mid x_1 \dots x_M)$
  - Any assigned (=lower-case) X or Y is a dimension missing (selected) from the array



Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!



$$P(L) = ?$$
$$= \sum_{r,t} P(r,t,L)$$
$$= \sum_{r,t} P(r)P(t|r)P(L|t)$$

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Inference by Enumeration: Procedural Outline

- Track objects called factors
- Initial factors are local CPTs (one per node)

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

- Any known values are selected
  - E.g. if we know $L = +\ell$ the initial factors are

$P(R)$

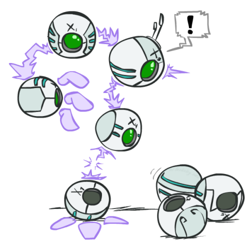
+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(+\ell|T)$

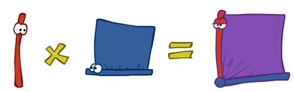
+t	+l	0.3
-t	+l	0.1



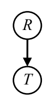
- Procedure: Join all factors, then eliminate all hidden variables

Operation 1: Join Factors

- First basic operation: joining factors
- Combining factors:
  - Just like a database join
  - Get all factors over the joining variable
  - Build a new factor over the union of the variables involved



- Example: Join on R



$$P(R) \times P(T|R) \longrightarrow P(R,T)$$

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

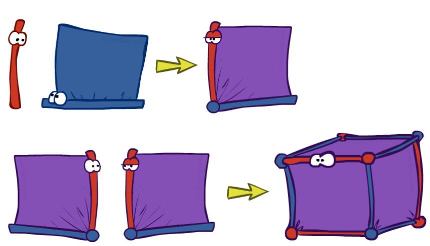
$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

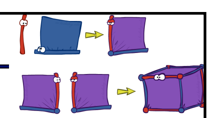
R,T

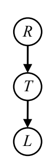
- Computation for each entry: pointwise products $\forall r,t: \quad P(r,t) = P(r) \cdot P(t|r)$

Example: Multiple Joins



Example: Multiple Joins





$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

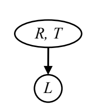
Join R

$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9



Join T

R, T, L

$P(R,T,L)$

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

Operation 2: Eliminate

- Second basic operation: marginalization
- Take a factor and sum out a variable
  - Shrinks a factor to a smaller one
  - A projection operation
- Example:

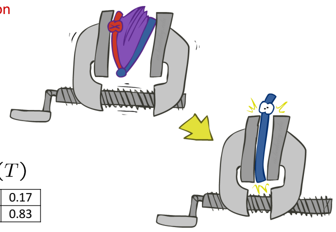
$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

sum R →

$P(T)$

+t	0.17
-t	0.83



Multiple Elimination

$P(R,T,L)$

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729

Sum out R →

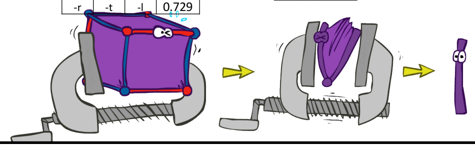
$P(T,L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

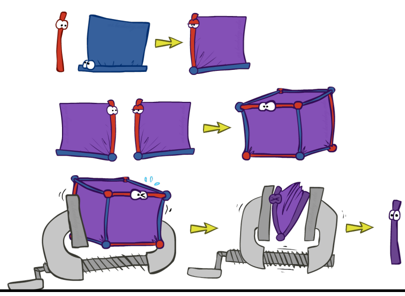
Sum out T →

$P(L)$

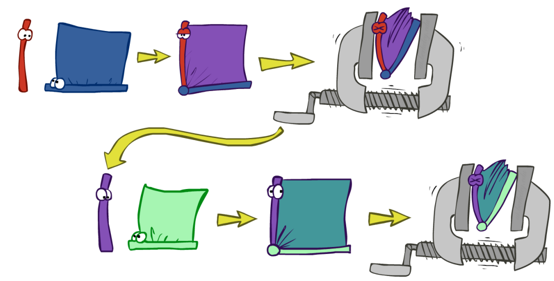
+l	0.134
-l	0.886



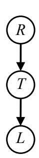
Thus Far: Multiple Join, Multiple Eliminate (= Inference by Enumeration)



Marginalizing Early (= Variable Elimination)



Traffic Domain



$$P(L) = ?$$

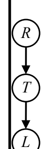
- Inference by Enumeration

$$=\underbrace{\sum_t \underbrace{\sum_r \underbrace{P(L|t)\underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Join on t}}}_{\text{Eliminate r}}}_{\text{Eliminate t}}$$

- Variable Elimination

$$=\underbrace{\sum_t \underbrace{P(L|t)\underbrace{\sum_r \underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Eliminate r}}}_{\text{Join on t}}}_{\text{Eliminate t}}$$

Marginalizing Early! (aka VE)



$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

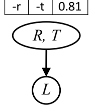
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join R →

$P(R,T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81



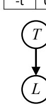
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Sum out R →

$P(T)$

+t	0.17
-t	0.83



$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join T →

$P(T,L)$

+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Sum out T →

$P(L)$

+l	0.134
-l	0.866

Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

  - Computing $P(L|+r)$ the initial factors become:

$P(+r)$

+r	0.1

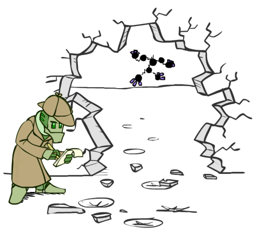
$P(T|+r)$

+r	+t	0.8
+r	-t	0.2

$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

- We eliminate all vars other than query + evidence



Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we would end up with:

$P(+r, L)$

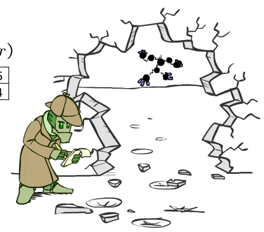
+r	+l	0.026
+r	-l	0.074

Normalize

$P(L|+r)$

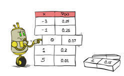
+l	0.26
-l	0.74

- To get our answer, just normalize this!
- That 's it!

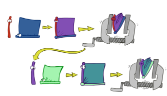


General Variable Elimination

- Query: $P(Q|E_1 = e_1, \dots E_k = e_k)$



- Start with initial factors:
  - Local CPTs (but instantiated by evidence)
- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H



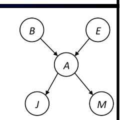
- Join all remaining factors and normalize

 $\times \frac{1}{Z}$

Example

$P(B|j,m) \propto P(B,j,m)$

$\boxed{P(B) \quad P(E) \quad P(A|B,E) \quad P(j|A) \quad P(m|A)}$



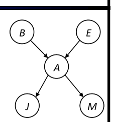
Choose A

$P(A|B,E)$
$P(j|A)$
$P(m|A)$ $\times$ $P(j,m,A|B,E)$ $\sum$ $P(j,m|B,E)$

$\boxed{P(B) \quad P(E) \quad P(j,m|B,E)}$

Example

$\boxed{P(B) \quad P(E) \quad P(j,m|B,E)}$



Choose E

$P(E)$
$P(j,m|B,E)$ $\times$ $P(j,m,E|B)$ $\sum$ $P(j,m|B)$

$\boxed{P(B) \quad P(j,m|B)}$

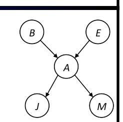
Finish with B

$P(B)$
$P(j,m|B)$ $\times$ $P(j,m,B)$ Normalize $P(B|j,m)$

Same Example in Equations

$P(B|j,m) \propto P(B,j,m)$

$\boxed{P(B) \quad P(E) \quad P(A|B,E) \quad P(j|A) \quad P(m|A)}$



$$
\begin{aligned}
P(B|j,m) &\propto P(B,j,m) \\
&= \sum_{e,a} P(B,j,m,e,a) && \text{marginal can be obtained from joint by summing out} \\
&= \sum_{e,a} P(B)P(e)P(a|B,e)P(j|a)P(m|a) && \text{use Bayes' net joint distribution expression} \\
&= \sum_{e} P(B)P(e)\sum_{a} P(a|B,e)P(j|a)P(m|a) && \text{use x*(y+z) = xy + xz} \\
&= \sum_{e} P(B)P(e)f_1(B,e,j,m) && \text{joining on a, and then summing out gives } f_1 \\
&= P(B)\sum_{e} P(e)f_1(B,e,j,m) && \text{use x*(y+z) = xy + xz} \\
&= P(B)f_2(B,j,m) && \text{joining on e, and then summing out gives } f_2
\end{aligned}
$$

All we are doing is exploiting uwy + uwz + uxy + uxz + vwy + vwz + vxy +vxz = (u+v)(w+x)(y+z) to improve computational efficiency!

Another Variable Elimination Example

Query: $P(X_3|Y_1 = y_1, Y_2 = y_2, Y_3 = y_3)$

Start by inserting evidence, which gives the following initial factors:

$$p(Z)p(X_1|Z)p(X_2|Z)p(X_3|Z)p(y_1|X_1)p(y_2|X_2)p(y_3|X_3)$$

Eliminate X_1, this introduces the factor $f_1(Z, y_1) = \sum_{x_1} p(x_1|Z)p(y_1|x_1)$, and we are left with:

$$p(Z)f_1(Z, y_1)p(X_2|Z)p(X_3|Z)p(y_2|X_2)p(y_3|X_3)$$

Eliminate X_2, this introduces the factor $f_2(Z, y_2) = \sum_{x_2} p(x_2|Z)p(y_2|x_2)$, and we are left with:

$$p(Z)f_1(Z, y_1)f_2(Z, y_2)p(X_3|Z)p(y_3|X_3)$$

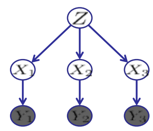
Eliminate Z, this introduces the factor $f_3(y_1, y_2, X_3) = \sum_z p(z)f_1(z, y_1)f_2(z, y_2)p(X_3|z)$, and we are left:

$$p(y_3|X_3), f_3(y_1, y_2, X_3)$$

No hidden variables left. Join the remaining factors to get:

$$f_4(y_1, y_2, y_3, X_3) = P(y_3|X_3)f_3(y_1, y_2, X_3).$$

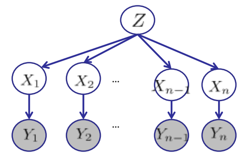
Normalizing over X_3 gives $P(X_3|y_1, y_2, y_3)$.



Computational complexity critically depends on the largest factor being generated in this process. Size of factor = number of entries in table. In example above (assuming binary) all factors generated are of size 2 --- as they all only have one variable (Z, Z, and X_3 respectively).

Variable Elimination Ordering

- For the query $P(X_n|y_1,\ldots,y_n)$ work through the following two different orderings as done in previous slide: $Z, X_1, \ldots, X_{n-1}$ and $X_1, \ldots, X_{n-1}, Z$. What is the size of the maximum factor generated for each of the orderings?



- Answer: 2^n versus 2^1 (assuming binary)
- In general: the ordering can greatly affect efficiency.

VE: Computational and Space Complexity

- The computational and space complexity of variable elimination is determined by the largest factor
- The elimination ordering can greatly affect the size of the largest factor.
  - E.g., previous slide's example 2^n vs. 2
- Does there always exist an ordering that only results in small factors?
  - No!

Worst Case Complexity?

- CSP:

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_2 \vee x_4) \wedge (\neg x_3 \vee \neg x_4 \vee \neg x_5) \wedge (x_2 \vee x_5 \vee x_7) \wedge (x_4 \vee x_5 \vee x_6) \wedge (\neg x_5 \vee x_6 \vee \neg x_7) \wedge (\neg x_5 \vee \neg x_6 \vee x_7)$$

$P(X_i = 0) = P(X_i = 1) = 0.5$

$Y_1 = X_1 \vee X_2 \vee \neg X_3$

...

$Y_8 = \neg X_5 \vee X_6 \vee X_7$

$Y_{1,2} = Y_1 \wedge Y_2$

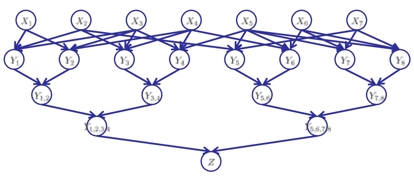
...

$Y_{7,8} = Y_7 \wedge Y_8$

$Y_{1,2,3,4} = Y_{1,2} \wedge Y_{3,4}$

$Y_{5,6,7,8} = Y_{5,6} \wedge Y_{7,8}$

$Z = Y_{1,2,3,4} \wedge Y_{5,6,7,8}$



- If we can answer P(z) equal to zero or not, we answered whether the 3-SAT problem has a solution.
- Hence inference in Bayes' nets is NP-hard. No known efficient probabilistic inference in general.

Polytrees

- A polytree is a directed graph with no undirected cycles
- For poly-trees you can always find an ordering that is efficient
  - Try it!!
- Cut-set conditioning for Bayes' net inference
  - Choose set of variables such that if removed only a polytree remains
  - Exercise: Think about how the specifics would work out!

Bayes' Nets

- ✔ Representation
- ✔ Conditional Independences
- Probabilistic Inference
  - ✔ Enumeration (exact, exponential complexity)
  - ✔ Variable elimination (exact, worst-case exponential complexity, often better)
  - ✔ Inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data