# CSE 473: Artificial Intelligence

## Markov Decision Processes

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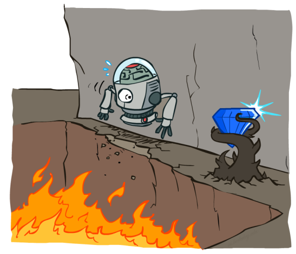
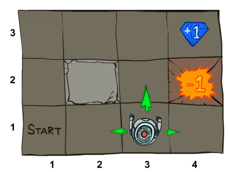
## Non-Deterministic Search



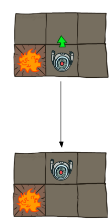
## Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

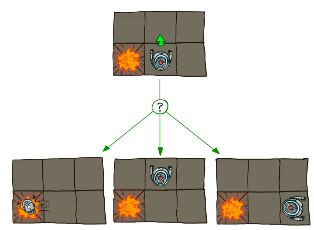



## Grid World Actions

Deterministic Grid World



Stochastic Grid World




## Markov Decision Processes

- An MDP is defined by:
  - A set of states s in S
  - A set of actions a in A
  - A transition function T(s, a, s')
    - Probability that a from s leads to s', i.e., P(s'| s, a)
    - Also called the model or the dynamics

$T(s_{11}, E, \ldots$
…
$T(s_{31}, N, s_{11}) = 0$
…
$T(s_{31}, N, s_{32}) = 0.8$
$T(s_{31}, N, s_{21}) = 0.1$
$T(s_{31}, N, s_{41}) = 0.1$
…

*T is a Big Table!*
*11 X 4 x 11 = 484 entries*

**For now, we give this as input to the agent**


## Markov Decision Processes

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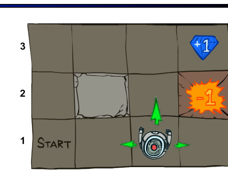
  - A reward function R(s, a, s')

…
$R(s_{32}, N, s_{33}) = -0.01$ ← ***Cost of breathing***
…
$R(s_{32}, N, s_{42}) = -1.01$
…
$R(s_{33}, E, s_{43}) = 0.99$

R is also a Big Table!

For now, we also give this to the agent



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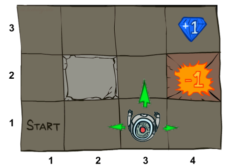
…

$R(s_{33}) = -0.01$

$R(s_{42}) = -1.01$

$R(s_{43}) = 0.99$

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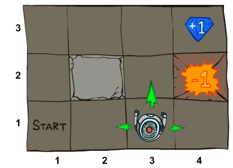
    - Also called the model or the dynamics
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state
  - Maybe a terminal state

- MDPs are non-deterministic search problems
  - One way to solve them is with expectimax search
  - We'll have a new tool soon




## What is Markov about MDPs?

- "Markov" generally means that given the present state, the future and the past are independent

- For Markov decision processes, "Markov" means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots S_0 = s_0)$$
$$=$$
$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

- This is just like search, where the successor function could only depend on the current state (not the history)

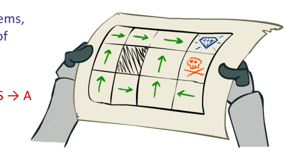
Andrey Markov (1856-1922)

## Policies

- In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal

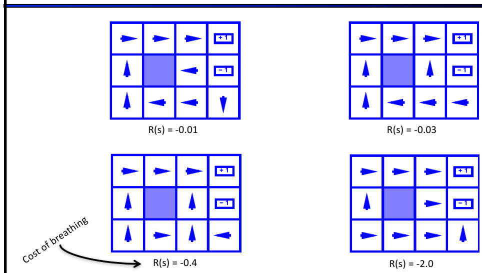
- For MDPs, we want an optimal policy π*: S → A
  - A policy π gives an action for each state
  - An optimal policy is one that maximizes expected utility if followed
  - An explicit policy defines a reflex agent

- Expectimax didn't compute entire policies
  - It computed the action for a single state only

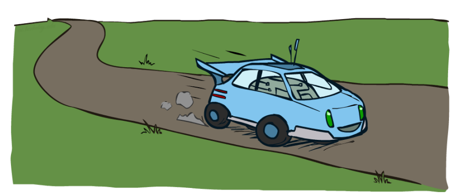


Optimal policy when R(s, a, s') = -0.03 for all non-terminals s

## Optimal Policies

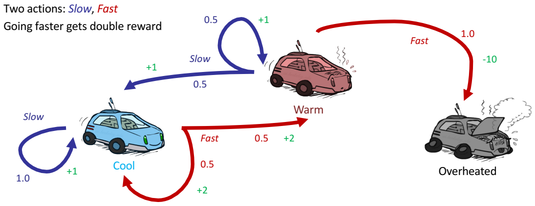



## Example: Racing

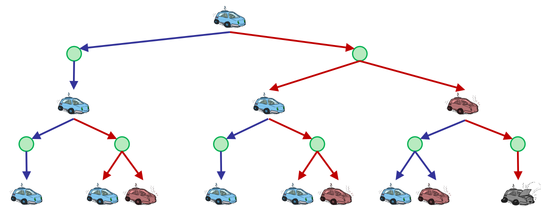


## Example: Racing

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

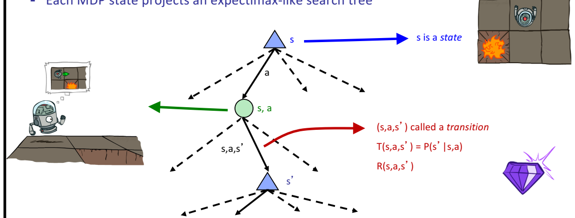



## Racing Search Tree

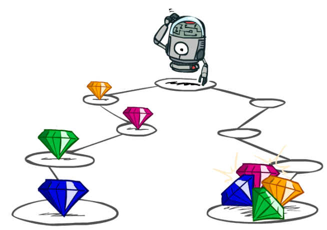


## MDP Search Trees

- Each MDP state projects an expectimax-like search tree

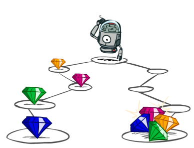



## Utilities of Sequences



## Utilities of Sequences

- What preferences should an agent have over reward sequences?
- More or less? [1, 2, 2] or [2, 3, 4]
- Now or later? [0, 0, 1] or [1, 0, 0]



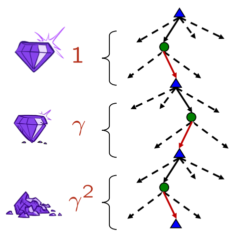
## Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially

| $1$ | $\gamma$ | $\gamma^2$ |
|---|---|---|
| Worth Now | Worth Next Step | Worth In Two Steps |

## Discounting

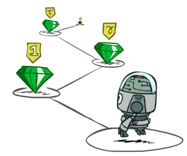
- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Sooner rewards probably do have higher utility than later rewards
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])




## Stationary Preferences

- Theorem: if we assume stationary preferences:

$$[a_1, a_2, \ldots] \succ [b_1, b_2, \ldots]$$
$$\Updownarrow$$
$$[r, a_1, a_2, \ldots] \succ [r, b_1, b_2, \ldots]$$



- Then: there are only two ways to define utilities
  - Additive utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$
  - Discounted utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$

## Quiz: Discounting

- Given:

| a | b | c | d | e |
|---|---|---|---|---|
| 10 | | | | 1 |

$$10 * \gamma^3 = 1 * \gamma$$
$$\gamma^2 = \frac{1}{10}$$

  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic
- Quiz 1: For γ = 1, what is the optimal policy?

| | | | | |
|---|---|---|---|---|
| 10 | | | | 1 |

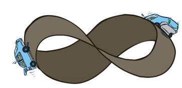
- Quiz 2: For γ = 0.1, what is the optimal policy?

| | | | | |
|---|---|---|---|---|
| 10 | | | | 1 |

- Quiz 3: For which γ are West and East equally good when in state d?

## Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?
- Solutions:
  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (γ depends on time left)



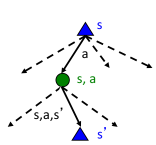
  - Discounting: use 0 < γ < 1

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)

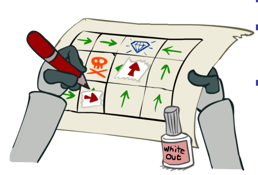
## Recap: Defining MDPs

- Markov decision processes:
  - Set of states S
  - Start state $s_0$
  - Set of actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)




- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility = sum of (discounted) rewards

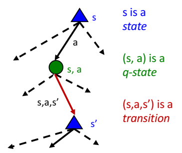
## Solving MDPs



- Value Iteration
- Policy Iteration
- Reinforcement Learning

## Optimal Quantities

- The value (utility) of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
  $\pi^*(s)$ = optimal action from state s



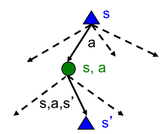

## Values of States

- Fundamental operation: compute the (expectimax) value of a state
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is just what expectimax computed!
- Recursive definition of value:

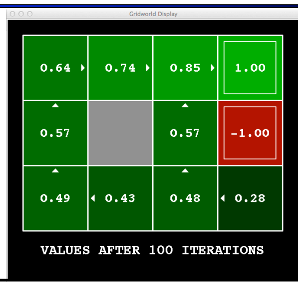
$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

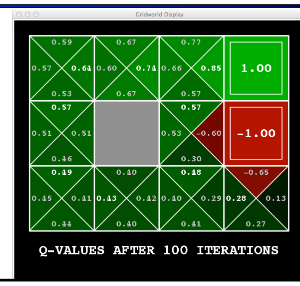



## Snapshot of Demo – Gridworld V Values




Noise = 0.2
Discount = 0.9
Living reward = 0

## Snapshot of Demo – Gridworld Q Values

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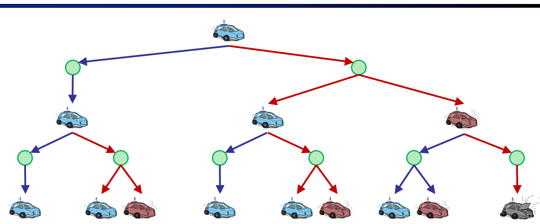
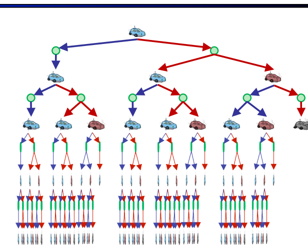
## Racing Search Tree



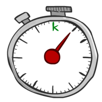
## Racing Search Tree

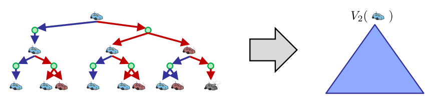
- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$



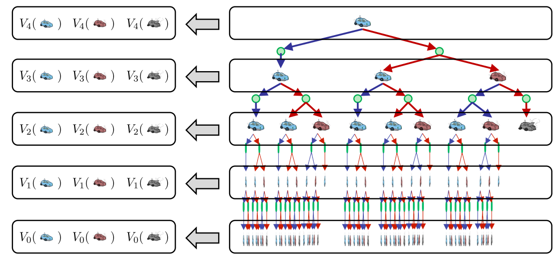
## Time-Limited Values

- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s

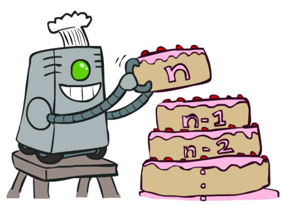


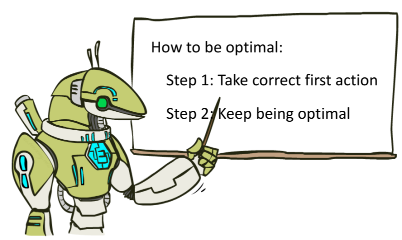

## Computing Time-Limited Values




## Value Iteration




## The Bellman Equations



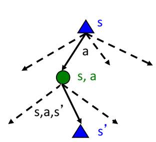

## The Bellman Equations

- Definition of "optimal utility" via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

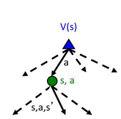



- These are the Bellman equations, and they characterize optimal values in a way we'll use over and over

## Value Iteration

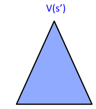
- Bellman equations characterize the optimal values:

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$




- Value iteration computes them:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V_k(s')\right]$$

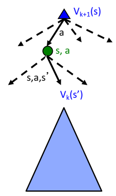


- Value iteration is just a fixed point solution method
  - … though the $V_k$ vectors are also interpretable as time-limited values

## Value Iteration Algorithm

- **Start with $V_0(s) = 0$:**
- **Given vector of $V_k(s)$ values, do one ply of expectimax from each state:**

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

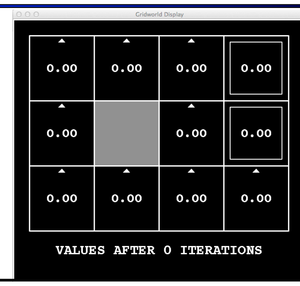



- **Repeat until convergence**

- **Complexity of each iteration: $O(S^2A)$**
- **Number of iterations: poly(|S|, |A|, 1/(1-γ))**

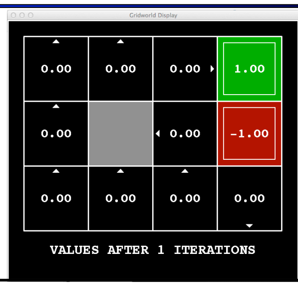
- **Theorem: will converge to unique optimal values**

## k=0



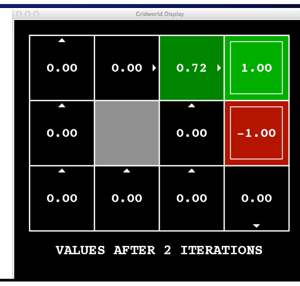

Noise = 0.2
Discount = 0.9
Living reward = 0

## k=1




Noise = 0.2
Discount = 0.9
Living reward = 0

## k=2




Noise = 0.2
Discount = 0.9
Living reward = 0

## k=3


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=4


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=5


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=6


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=7


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=8


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=9


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=10


Noise = 0.2
Discount = 0.9
Living reward = 0

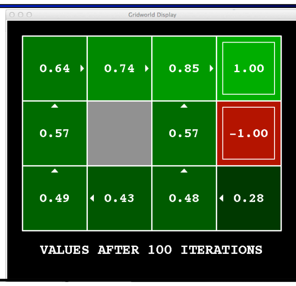
## k=11


Noise = 0.2
Discount = 0.9
Living reward = 0

## k=12


Noise = 0.2
Discount = 0.9
Living reward = 0

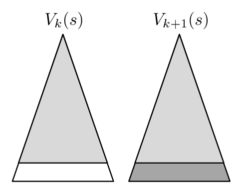
## k=100




Noise = 0.2
Discount = 0.9
Living reward = 0

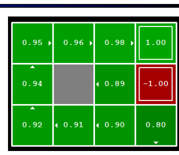
## Convergence*

- How do we know the $V_k$ vectors will converge?
- Case 1: If the tree has maximum depth M, then $V_M$ holds the actual untruncated values
- Case 2: If the discount is less than 1
  - Sketch: For any state $V_k$ and $V_{k+1}$ can be viewed as depth k+1 expectimax results in nearly identical search trees
  - The max difference happens if big reward at k+1 level
  - That last layer is at best all $R_{MAX}$
  - But everything is discounted by $\gamma^k$ that far out
  - So $V_k$ and $V_{k+1}$ are at most $\gamma^k$ max|R| different
  - So as k increases, the values converge




## Computing Actions from Values

- Let's imagine we have the optimal values V*(s)



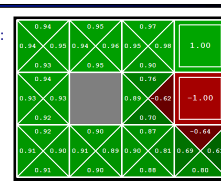

- How should we act?
  - It's not obvious!
- We need to do a mini-expectimax (one step)

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- This is called policy extraction, since it gets the policy implied by the values

## Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:




- How should we act?
  - Completely trivial to decide!

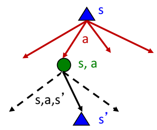
$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

- Important lesson: actions are easier to select from q-values than values!

## Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$




- Problem 1: It's slow – O($S^2A$) per iteration
- Problem 2: The "max" at each state rarely changes
- Problem 3: The policy often converges long before the values

## VI → Asynchronous VI

- Is it essential to back up ***all*** states in each iteration?
  - No!
- States may be backed up
  - many times or not at all
  - in any order
- As long as no state gets starved…
  - convergence properties still hold!!

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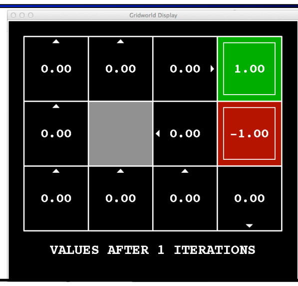
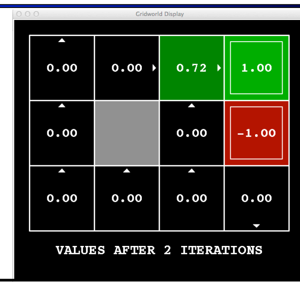
## k=1




Noise = 0.2
Discount = 0.9
Living reward = 0

## k=2




Noise = 0.2
Discount = 0.9
Living reward = 0

## k=3


Noise = 0.2
Discount = 0.9
Living reward = 0

## Asynch VI: Prioritized Sweeping

- Why backup a state if values of successors same?
- Prefer backing a state
  - whose successors had most change
- Priority Queue of (state, expected change in value)
- Backup in the order of priority
- After backing a state update priority queue
  - for all predecessors