# CSE 473: Artificial Intelligence

## Autumn 2015

Constraint Satisfaction

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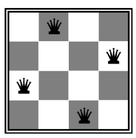
## What is Search For?

- Models of the world: single agent, deterministic actions, fully observed state, discrete state space
- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics to guide, fringe to keep backups
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems

## Constraint Satisfaction Problems

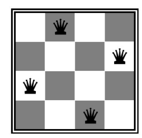


- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test: any function over states
  - Successor function can be anything
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables $X_i$ with values from a domain $D$ (sometimes $D$ depends on $i$)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Simple example of a *formal representation language*
- Allows useful general-purpose algorithms with more power than standard search algorithms

## Example: N-Queens

- Formulation 1:
  - Variables: $X_{ij}$
  - Domains: $\{0, 1\}$
  - Constraints

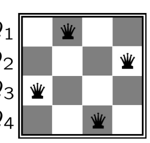


$$\forall i, j, k \ \ (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \ \ (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\sum_{i,j} X_{ij} = N$$

  - Note: need to make sure that constraints refer to different squares

## Example: N-Queens

- Formulation 2:
  - Variables: $Q_k$
  - Domains: $\{1, 2, 3, \dots N\}$
  - Constraints:




Implicit: $\forall i, j$ non-threatening$(Q_i, Q_j)$

-or-

Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \dots\}$

$\cdots$

## Example: Map-Coloring

- Variables: $WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T$
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors


$$WA \neq NT$$

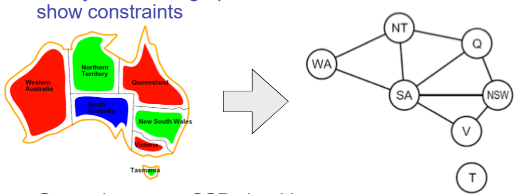
$$(WA, NT) \in \{(red, green), (red, blue), (green, red), \dots\}$$

- Solutions are assignments satisfying all constraints, e.g.:

$$\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$$

# Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints




- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!

# Example: Cryptarithmetic

- Variables (circles):

  $F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$
- Domains:

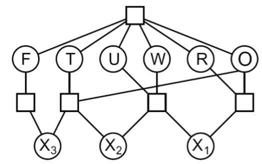
  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Constraints (boxes):

  $\text{alldiff}(F, T, U, W, R, O)$

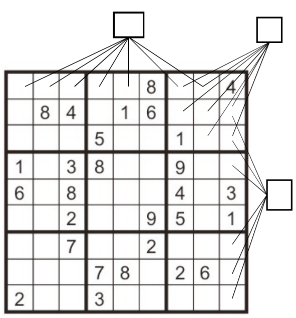
  $O + O = R + 10 \cdot X_1$

  $\cdots$

```
  T W O
+ T W O
-------
F O U R
```




# Example: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

9-way alldiff for each row

9-way alldiff for each region

# Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size $d$ means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable
- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by LP methods

# Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equiv. to shrinking domains):

    $SA \neq green$
  - Binary constraints involve pairs of variables:

    $SA \neq WA$
  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints
- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes' nets)

# Real-World CSPs

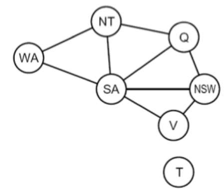
- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Floorplanning
- Fault diagnosis
- … lots more!
- Many real-world problems involve real-valued variables…

## Standard Search Formulation

- Standard search formulation of CSPs (incremental)
- Let's start with a straightforward, dumb approach, then fix it
- States are defined by the values assigned so far
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

## Search Methods

- What does BFS do?
- What does DFS do?




## Backtracking Search

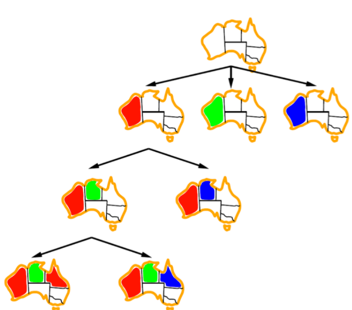
- Idea 1: Only consider a single variable at each point
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
  - How many leaves are there?
- Idea 2: Only allow legal assignments at each point
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to figure out whether a value is ok
  - "Incremental goal test"
- Depth-first search for CSPs with these two improvements is called *backtracking search*
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for n ≈ 25

## Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

- What are the choice points?

## Backtracking Example

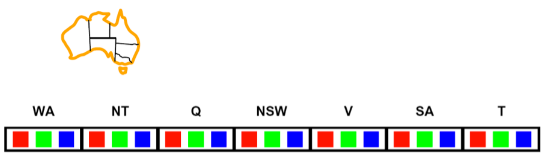


## Improving Backtracking

- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?

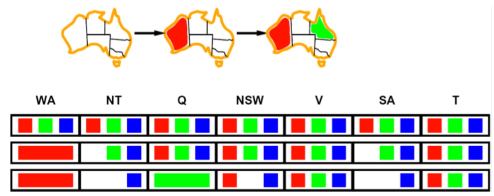
# Forward Checking

- Idea: Keep track of remaining legal values for unassigned variables (using immediate constraints)
- Idea: Terminate when any variable has no legal values




# Constraint Propagation

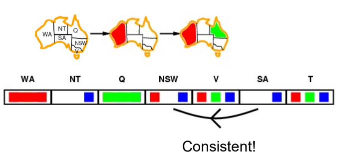
- Forward checking propagates information from assigned to adjacent unassigned variables, but doesn't detect more distant failures:




- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation* repeatedly enforces constraints (locally)

# Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*

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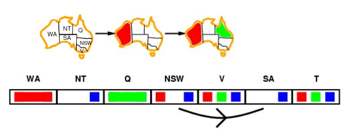
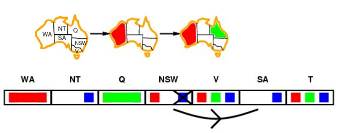



# Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*
  - When checking *X* → *Y*, throw out any values of X for which there isn't an allowed value of Y




- If *X* loses a value, all pairs *Z* → *X* need to be rechecked

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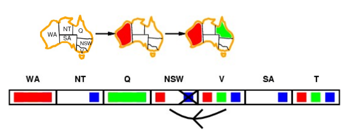


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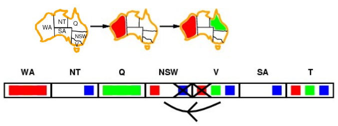
## Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - $X \rightarrow Y$ is consistent iff for every value of $X$ there is some allowed value of $Y$
  - When checking $X \rightarrow Y$, throw out any values of X for which there isn't an allowed value of Y




- If $X$ loses a value, all pairs $Z \rightarrow X$ need to be rechecked

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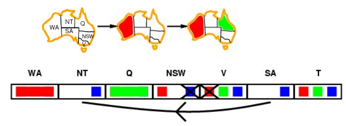


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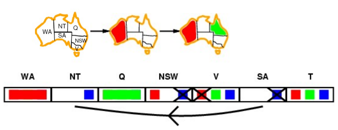



- Arc consistency detects failure earlier than forward checking
- Can be run before or after each assignment

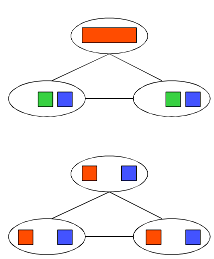
## Arc Consistency

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue

function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

- Runtime: $O(nd^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

## Limitations of Arc Consistency

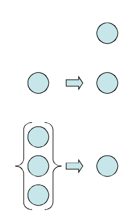
- After running arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)



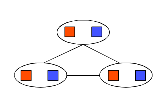
*What went wrong here?*

## K-Consistency*

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the $k^{th}$ node.



- Higher k more expensive to compute
- (You need to know the k=2 algorithm)



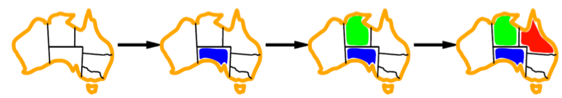
## Ordering: Minimum Remaining Values

- Minimum remaining values (MRV):
  - Choose the variable with the fewest legal values

- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering

## Ordering: Degree Heuristic

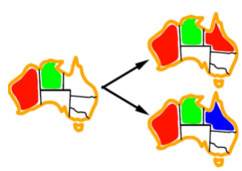
- Tie-breaker among MRV variables
- Degree heuristic:
  - Choose the variable participating in the most constraints on remaining variables



- Why most rather than fewest constraints?

## Ordering: Least Constraining Value

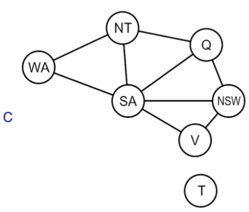
- Given a choice of variable:
  - Choose the *least constraining value*
  - The one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!



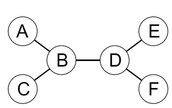
- Why least rather than most?
- Combining these heuristics makes 1000 queens feasible

## Problem Structure

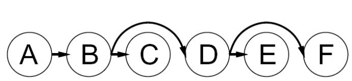
- Tasmania and mainland are independent subproblems
- Identifiable as connected components of constraint graph
- Suppose each subproblem has c variables out of n total
- Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - $2^{80}$ = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec




## Tree-Structured CSPs

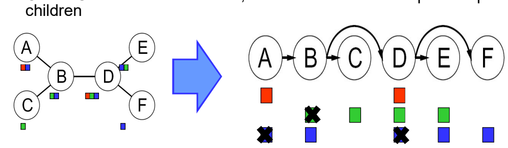



- Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



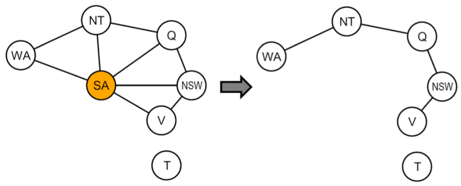

## Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children



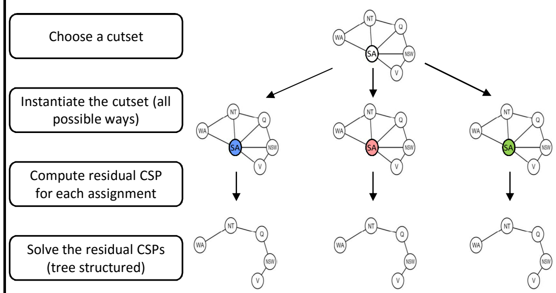

  - Remove backward:
    For i = n : 2, apply RemoveInconsistent(Parent($X_i$),$X_i$)
  - Assign forward:
    For i = 1 : n, assign $X_i$ consistently with Parent($X_i$)
- Runtime: $O(n\ d^2)$ (why?)

## Nearly Tree-Structured CSPs




- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime O( ($d^c$) (n-c) $d^2$ ), very fast for small c

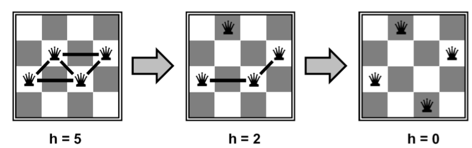
## Cutset Conditioning




## Iterative Algorithms for CSPs

- Greedy and local methods typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - Allow states with unsatisfied constraints
  - Operators *reassign* variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - Choose value that violates the fewest constraints
  - I.e., hill climb with h(n) = total number of violated constraints

## Example: 4-Queens

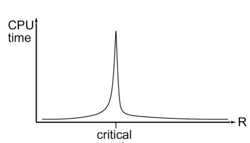



- States: 4 queens in 4 columns ($4^4$ = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks

## Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



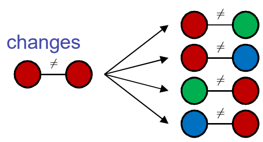

## Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- Backtracking = depth-first search with one legal variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The constraint graph representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice

## Local Search

## Local Search

- Tree search keeps unexplored alternatives on the fringe (ensures completeness)
- Local search: improve a single option until you can't make it better (no fringe!)
- New successor function: local changes

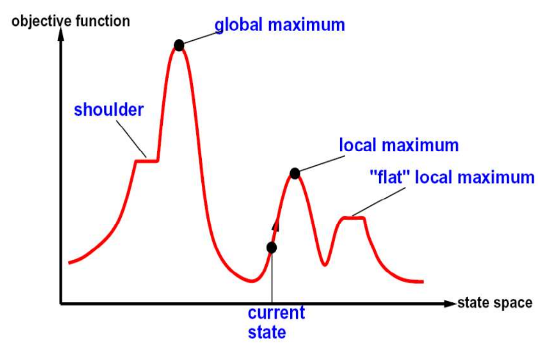


- Generally much faster and more memory efficient (but incomplete and suboptimal)

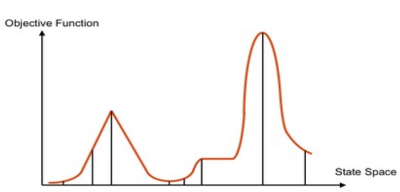
## Hill Climbing

- Simple, general idea:
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit
- What's bad about this approach?
  - Complete?
  - Optimal?
- What's good about it?

## Hill Climbing Diagram




## Hill Climbing




Starting from X, where do you end up ?

Starting from Y, where do you end up ?

Starting from Z, where do you end up ?

## Simulated Annealing

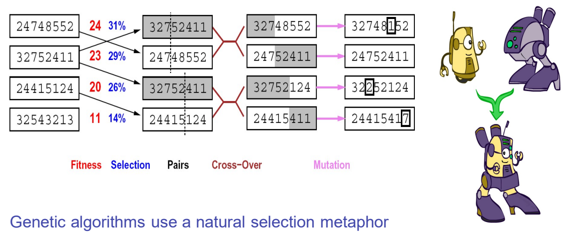
- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on

```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
  inputs: problem, a problem
          schedule, a mapping from time to "temperature"
  local variables: current, a node
                   next, a node
                   T, a "temperature" controlling prob. of downward steps

  current ← MAKE-NODE(INITIAL-STATE[problem])
  for t ← 1 to ∞ do
     T ← schedule[t]
     if T = 0 then return current
     next ← a randomly selected successor of current
     ΔE ← VALUE[next] – VALUE[current]
     if ΔE > 0 then current ← next
     else current ← next only with probability e^(ΔE/T)
```

## Simulated Annealing

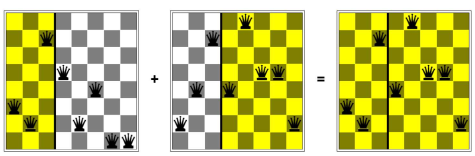
- Theoretical guarantee:
  - Stationary distribution: $p(x) \propto e^{\frac{E(x)}{kT}}$
  - If T decreased slowly enough, will converge to optimal state!
- Is this an interesting guarantee?
- Sounds like magic, but reality is reality:
  - The more downhill steps you need to escape a local optimum, the less likely you are to ever make them all in a row
  - People think hard about *ridge operators* which let you jump around the space in better ways

## Genetic Algorithms




- Genetic algorithms use a natural selection metaphor
  - Keep best N hypotheses at each step (selection) based on a fitness function
  - Also have pairwise crossover operators, with optional mutation to give variety
- Possibly the most misunderstood, misapplied (and even maligned) technique around

## Example: N-Queens




- Why does crossover make sense here?
- When wouldn't it make sense?
- What would mutation be?
- What would a good fitness function be?

## GA's for Locomotion

Ever wonder what it would be like to see evolution happening right before your eyes?

Hod Lipson's Creative Machines Lab @ Cornell