CSE 473: Artificial Intelligence

Markov Models

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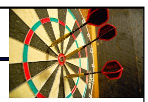
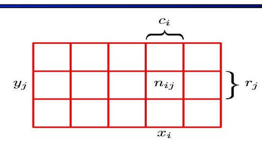
Announcements

- No class on Wed…

Terminology





Marginal Probability

$p(X = x_i) =$

Conditional Probability

$p(Y = y_j | X = x_i) =$

X value is given

Joint Probability

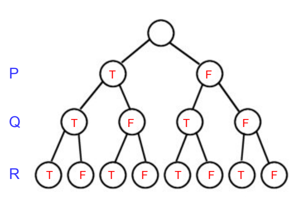
$p(X = x_i, Y = y_j) =$

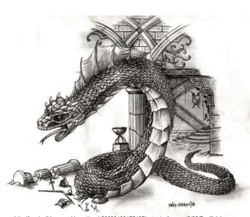
Don't be Fooled

- It may look cute…

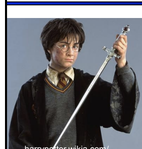
Don't be Fooled

- It gets big…



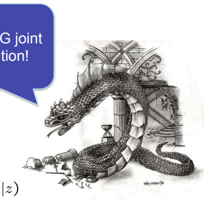


The Sword of Conditional Independence!



Slay the Basilisk!

I am a BIG joint distribution!



$X \perp\!\!\!\perp Y | Z$ Means: $\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$

Or, equivalently: $\forall x, y, z : P(x|z, y) = P(x|z)$

Probability Recap

- Conditional probability $P(x|y) = \frac{P(x,y)}{P(y)}$
- Product rule $P(x,y) = P(x|y)P(y)$
- Chain rule $P(X_1, X_2, \ldots X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots = \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1})$
- Bayes rule $P(x|y) = \frac{P(y|x)}{P(y)}P(x)$
- X, Y independent if and only if: $\forall x, y : P(x,y) = P(x)P(y)$
- X and Y are conditionally independent given Z: $X \perp\!\!\!\perp Y|Z$
  if and only if: $\forall x, y, z : P(x,y|z) = P(x|z)P(y|z)$

Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time (or space) into our models

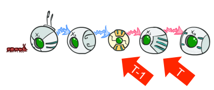
Markov Models Recap

- Explicit assumption for all t : $X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$
$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

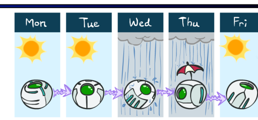
- Additional explicit assumption:

$P(X_t \mid X_{t-1})$ is the same for all t



Example Markov Chain: Weather

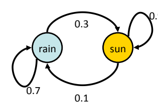
- States: X = {rain, sun}

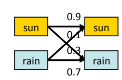


- Initial distribution: 1.0 sun
- CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t\|X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

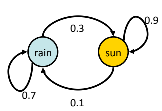
Two new ways of representing the same CPT





Example Markov Chain: Weather

- Initial distribution: 1.0 sun



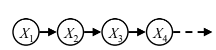
- What is the probability distribution after one step?

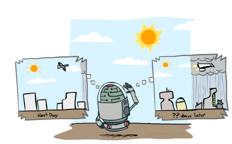
$$P(X_2 = \text{sun}) = P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

Mini-Forward Algorithm

- Question: What's P(X) on some day t?





$P(x_1)$ = known

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$
$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1})P(x_{t-1})$$

Forward simulation

Example Run of Mini-Forward Algorithm

- From initial observation of sun

$\langle 1.0, 0.0 \rangle$ P(X_1) $\quad \langle 0.9, 0.1 \rangle$ P(X_2) $\quad \langle 0.84, 0.16 \rangle$ P(X_3) $\quad \langle 0.804, 0.196 \rangle$ P(X_4) $\Rightarrow \langle 0.75, 0.25 \rangle$ P(X_∞)

- From initial observation of rain

$\langle 0.0, 1.0 \rangle$ P(X_1) $\quad \langle 0.3, 0.7 \rangle$ P(X_2) $\quad \langle 0.48, 0.52 \rangle$ P(X_3) $\quad \langle 0.588, 0.412 \rangle$ P(X_4) $\Rightarrow \langle 0.75, 0.25 \rangle$ P(X_∞)

- From yet another initial distribution P(X_1):

$\langle p, 1-p \rangle$ P(X_1) $\quad \dots \quad \Rightarrow \langle 0.75, 0.25 \rangle$ P(X_∞)

[Demo: L13D1,2,3]

Video of Demo Ghostbusters Basic Dynamics

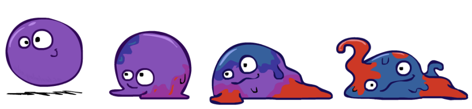
Video of Demo Ghostbusters Circular Dynamics

Video of Demo Ghostbusters Whirlpool Dynamics

Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution
- Stationary distribution:
  - The distribution we end up with is called the stationary distribution P_∞ of the chain
  - It satisfies

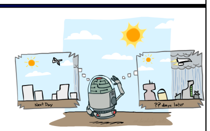
$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$



Example: Stationary Distributions

- Question: What's P(X) at time t = infinity?

$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$



$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

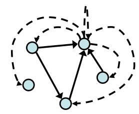
$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

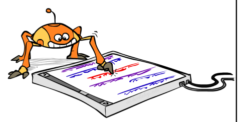
X_{t-1}	X_t	$P(X_t\|X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Also: $P_\infty(sun) + P_\infty(rain) = 1$ $\Rightarrow$ $P_\infty(sun) = 3/4$, $P_\infty(rain) = 1/4$

Application of Stationary Distribution: Web Link Analysis

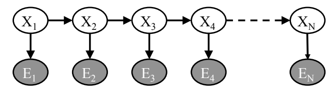
- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page (dotted lines, not all shown)
    - With prob. 1-c, follow a random outlink (solid lines)
- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



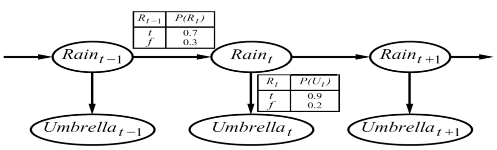


Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don't know anything anymore
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states S
  - You observe outputs (effects) at each time step
  - As a Bayes' net:

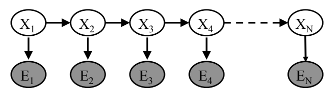


Example



- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t|X_{t-1})$
  - Emissions: $P(E|X)$

Hidden Markov Models



- Defines a joint probability distribution:

$$P(X_1, \ldots, X_n, E_1, \ldots, E_n) =$$
$$P(X_{1:n}, E_{1:n}) =$$
$$P(X_1)P(E_1|X_1)\prod_{t=2}^{N} P(X_t|X_{t-1})P(E_t|X_t)$$

Ghostbusters HMM

- $P(X_1)$ = uniform
- $P(X'|X)$ = ghosts usually move clockwise, but sometimes move in a random direction or stay put
- $P(E|X)$ = same sensor model as before: red means close, green means far away.

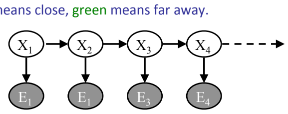
1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$

1/6	1/6	1/2
0	1/6	0
0	0	0

Etc…

P(X'|X=<1,2>)



P(E|X)

P(red | 3)	P(orange | 3)	P(yellow | 3)	P(green | 3)
0.05	0.15	0.5	0.3

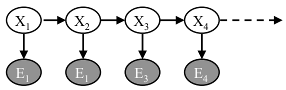
Etc… (must specify for other distances)

HMM Computations

- Given
  - parameters
  - evidence $E_{1:n} = e_{1:n}$
- Inference problems include:
  - Filtering, find $P(X_t|e_{1:t})$ for all t
  - Smoothing, find $P(X_t|e_{1:n})$ for all t
  - Most probable explanation, find $x^*_{1:n} = \text{argmax}_{x_{1:n}} P(x_{1:n}|e_{1:n})$

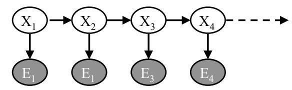
Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)



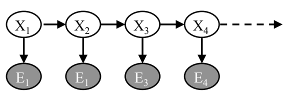
Real HMM Examples

- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options



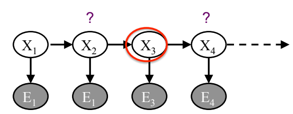
Real HMM Examples

- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)



Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process, future depends on past via the present



Conditional Independence

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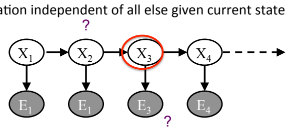

Conditional Independence

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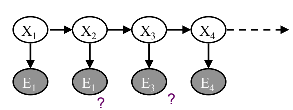


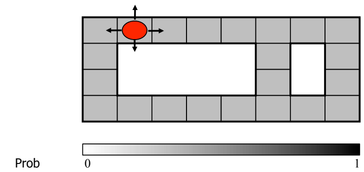
- Quiz: does this mean that observations are independent given no evidence?
  - [No, correlated by the hidden state]

Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution B(X) (the belief state) over time
- We start with B(X) in an initial setting, usually uniform
- As time passes, or we get observations, we update B(X)
- The Kalman filter (one method – Real valued values)
  - invented in the 60's as a method of trajectory estimation for the Apollo program

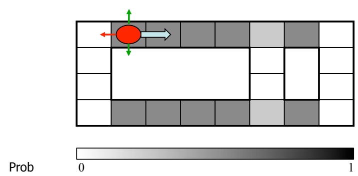
Example: Robot Localization

Example from Michael Pfeiffer



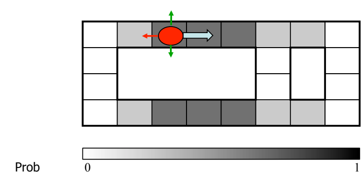
t=0
Sensor model: never more than 1 mistake
Motion model: may not execute action with small prob.

Example: Robot Localization



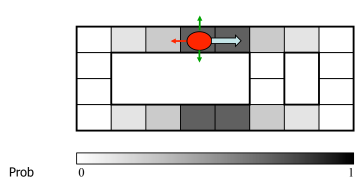
t=1

Example: Robot Localization



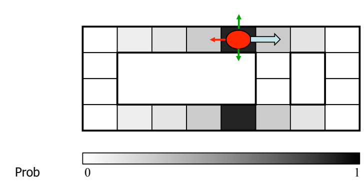
t=2

Example: Robot Localization



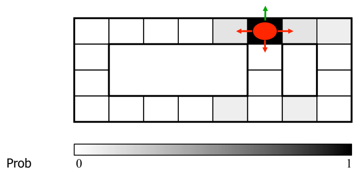
t=3

Example: Robot Localization



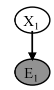
t=4

Example: Robot Localization



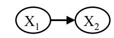
t=5

Inference Recap: Simple Cases



$P(X_1|e_1)$

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$
$$\propto_{X_1} P(x_1, e_1)$$
$$= P(x_1)P(e_1|x_1)$$

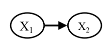


$P(X_2)$

$$P(x_2) = \sum_{x_1} P(x_1, x_2)$$
$$= \sum_{x_1} P(x_1)P(x_2|x_1)$$

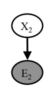
Online Belief Updates

- Every time step, we start with current P(X | evidence)
- We update for time:



$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

- We update for evidence:



$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

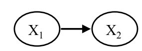
- The forward algorithm does both at once (and doesn't normalize)
- Problem: space is |X| and time is $|X|^2$ per time step

Passage of Time

- Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$

- Then, after one time step passes:



$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

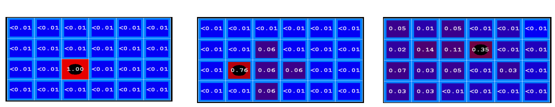
- Or, compactly:

$$B'(X') = \sum_{x} P(X'|x)B(x)$$

- Basic idea: beliefs get "pushed" through the transitions
  - With the "B" notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Example: Passage of Time

- As time passes, uncertainty "accumulates"



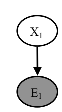
T = 1　　T = 2　　T = 5

$$B'(X') = \sum_{x} P(X'|x)B(x)$$

Transition model: ghosts usually go clockwise

Observation

- Assume we have current belief P(X | previous evidence):



$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

- Then:

$$P(X_{t+1}|e_{1:t+1}) \propto P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

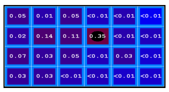
- Or:

$$B(X_{t+1}) \propto P(e|X)B'(X_{t+1})$$

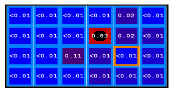
- Basic idea: beliefs reweighted by likelihood of evidence
- Unlike passage of time, we have to renormalize

Example: Observation

- As we get observations, beliefs get reweighted, uncertainty "decreases"



Before observation



After observation

$$B(X) \propto P(e|X)B'(X)$$

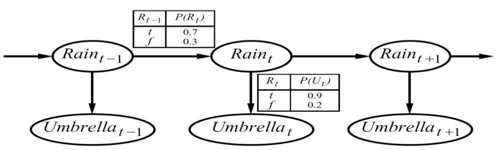
The Forward Algorithm

- We want to know: $B_t(X) = P(X_t|e_{1:t})$
- We can derive the following updates

$$P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t})$$
$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$
$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1})P(x_t|x_{t-1})P(e_t|x_t)$$
$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1}, e_{1:t-1})$$

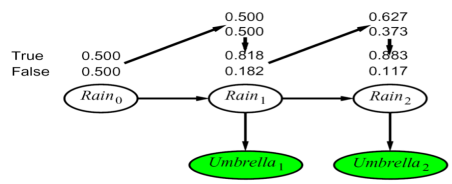
- To get $B_t(X)$, compute each entry and normalize

Example: Run the Filter

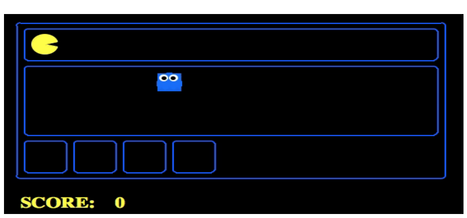


- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t|X_{t-1})$
  - Emissions: $P(E|X)$

Example HMM



Example Pac-man



Summary: Filtering

- Filtering is the inference process of finding a distribution over X_T given e_1 through e_T : $P(X_T | e_{1:t})$
- We first compute $P(X_1 | e_1)$: $P(x_1|e_1) \propto P(x_1) \cdot P(e_1|x_1)$
- For each t from 2 to T, we have $P(X_{t-1} | e_{1:t-1})$
- Elapse time: compute $P(X_t | e_{1:t-1})$

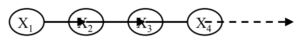
$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

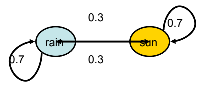
- Observe: compute $P(X_t | e_{1:t-1}, e_t) = P(X_t | e_{1:t})$

$$P(x_t|e_{1:t}) \propto P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

Recap: Reasoning Over Time

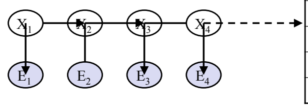
- Stationary Markov models





$P(X_1)$ $\quad$ $P(X|X_{-1})$

- Hidden Markov models



$P(E|X)$

X	E	P
rain	umbrella	0.9
rain	no umbrella	0.1
sun	umbrella	0.2
sun	no umbrella	0.8

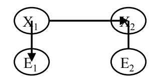
Recap: Filtering

Elapse time: compute P(X_t | $e_{1:t-1}$)

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

Observe: compute P(X_t | $e_{1:t}$)

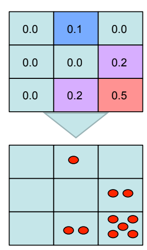
$$P(x_t|e_{1:t}) \propto P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$



	Belief: <P(rain), P(sun)>	
$P(X_1)$	<0.5, 0.5>	Prior on X_1
$P(X_1 \mid E_1 = umbrella)$	<0.82, 0.18>	Observe
$P(X_2 \mid E_1 = umbrella)$	<0.63, 0.37>	Elapse time
$P(X_2 \mid E_1 = umb, E_2 = umb)$	<0.88, 0.12>	Observe

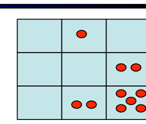
Particle Filtering

- Sometimes |X| is too big to use exact inference
  - |X| may be too big to even store B(X)
  - E.g. X is continuous
  - $|X|^2$ may be too big to do updates
- Solution: approximate inference
  - Track samples of X, not all values
  - Samples are called particles
  - Time per step is linear in the number of samples
  - But: number needed may be large
  - In memory: list of particles, not states
- This is how robot localization works in practice



Representation: Particles

- Our representation of P(X) is now a list of N particles (samples)
  - Generally, N << |X|
  - Storing map from X to counts would defeat the point
- P(x) approximated by number of particles with value x
  - So, many x will have P(x) = 0!
  - More particles, more accuracy
- For now, all particles have a weight of 1



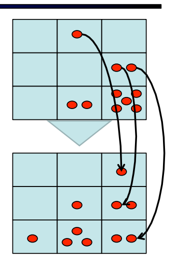
Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(2,1)
(3,3)
(3,3)
(2,1)

Particle Filtering: Elapse Time

- Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

  - This is like prior sampling – samples' frequencies reflect the transition probs
  - Here, most samples move clockwise, but some move in another direction or stay in place
- This captures the passage of time
  - If we have enough samples, close to the exact values before and after (consistent)



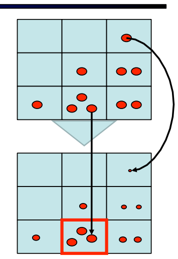
Particle Filtering: Observe

- Slightly trickier:
  - Don't do rejection sampling (why not?)
  - We don't sample the observation, we fix it
  - This is similar to likelihood weighting, so we downweight our samples based on the evidence

$$w(x) = P(e|x)$$

$$B(X) \propto P(e|X)B'(X)$$

  - Note that, as before, the probabilities don't sum to one, since most have been downweighted (in fact they sum to an approximation of P(e))

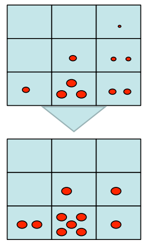


Particle Filtering: Resample

- Rather than tracking weighted samples, we resample
- N times, we choose from our weighted sample distribution (i.e. draw with replacement)
- This is equivalent to renormalizing the distribution
- Now the update is complete for this time step, continue with the next one

Old Particles:
(3,3) w=0.1
(2,1) w=0.9
(2,1) w=0.9
(3,1) w=0.4
(3,2) w=0.3
(2,2) w=0.4
(1,1) w=0.4
(3,1) w=0.4
(2,1) w=0.9
(3,2) w=0.3

New Particles:
(2,1) w=1
(2,1) w=1
(2,1) w=1
(3,2) w=1
(2,2) w=1
(2,1) w=1
(1,1) w=1
(3,1) w=1
(2,1) w=1
(1,1) w=1



Recap: Particle Filtering

At each time step t, we have a set of N particles / samples
- Initialization: Sample from prior, reweight and resample
- Three step procedure, to move to time t+1:
  1. Sample transitions: for each each particle x, sample next state

$$x' = \text{sample}(P(X'|x))$$

  2. Reweight: for each particle, compute its weight given the actual observation e

  - Resample $w(x) = P(e|x)$ nts, and sample N new particles from the resulting distribution over states

Particle Filtering Summary

- Represent current belief P(X | evidence to date) as set of n samples (actual assignments X=x)
- For each new observation e:
  1. Sample transition, once for each current particle x

$$x' = \text{sample}(P(X'|x))$$

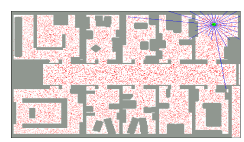
  2. For each new sample x', compute importance weights for the new evidence e:

$$w(x') = P(e|x')$$

  3. Finally, normalize the importance weights and resample N new particles

Robot Localization

- In robot localization:
  - We know the map, but not the robot's position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store B(X)
  - Particle filtering is a main technique

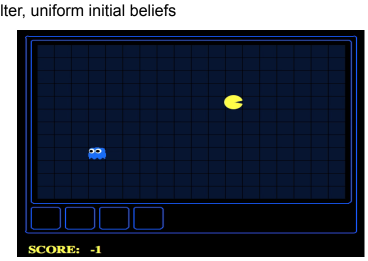


Robot Localization

QuickTime™ and a
GIF decompressor
are needed to see this picture.

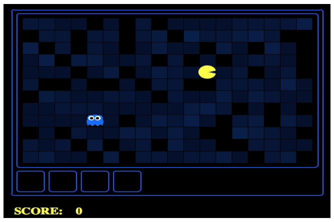
Which Algorithm?

Exact filter, uniform initial beliefs



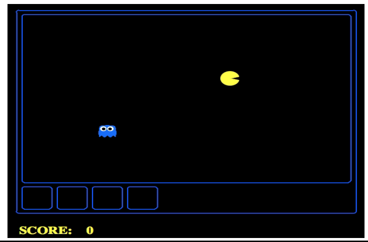
Which Algorithm?

Particle filter, uniform initial beliefs, 300 particles



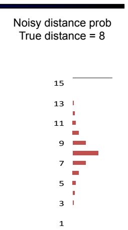
Which Algorithm?

Particle filter, uniform initial beliefs, 25 particles



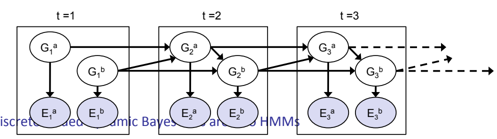
P4: Ghostbusters

- Plot: Pacman's grandfather, Grandpac, learned to hunt ghosts for sport.
- He was blinded by his power, but could hear the ghosts' banging and clanging.
- Transition Model: All ghosts move randomly, but are sometimes biased
- Emission Model: Pacman knows a "noisy" distance to each ghost



Dynamic Bayes Nets (DBNs)

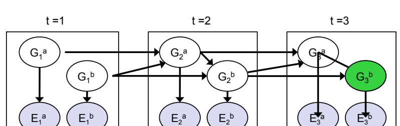
- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net structure at each time
- Variables from time t can condition on those from t-1



- Discrete valued dynamic Bayes nets are also HMMs

Exact Inference in DBNs

- Variable elimination applies to dynamic Bayes nets
- Procedure: "unroll" the network for T time steps, then eliminate variables until $P(X_T | e_{1:T})$ is computed



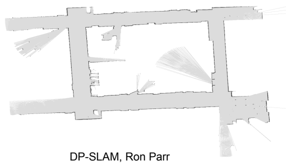
- Online belief updates: Eliminate all variables from the previous time step; store factors for current time only

DBN Particle Filters

- A particle is a complete sample for a time step
- Initialize: Generate prior samples for the t=1 Bayes net
  - Example particle: $G_1^a = (3,3)$ $G_1^b = (5,3)$
- Elapse time: Sample a successor for each particle
  - Example successor: $G_2^a = (2,3)$ $G_2^b = (6,3)$
- Observe: Weight each entire sample by the likelihood of the evidence conditioned on the sample
  - Likelihood: $P(E_1^a | G_1^a) * P(E_1^b | G_1^b)$
- Resample: Select prior samples (tuples of values) in proportion to their likelihood

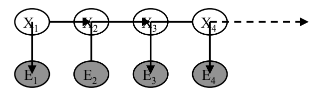
SLAM

- SLAM = Simultaneous Localization And Mapping
  - We do not know the map or our location
  - Our belief state is over maps and positions!
  - Main techniques: Kalman filtering (Gaussian HMMs) and particle methods
- [DEMOS]



DP-SLAM, Ron Parr

Best Explanation Queries

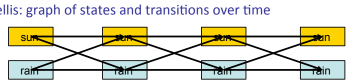


- Query: most likely seq:

$$\arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$$

State Path Trellis

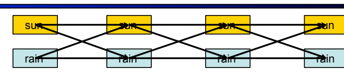
- State trellis: graph of states and transitions over time



$X_1 \quad X_2 \quad \cdots \quad X_N$

- Each arc represents some transition $x_{t-1} \rightarrow x_t$
- Each arc has weight $P(x_t|x_{t-1})P(e_t|x_t)$
- Each path is a sequence of states
- The product of weights on a path is the seq's probability
- Can think of the Forward (and now Viterbi) algorithms as computing sums of all paths (best paths) in this graph

Viterbi Algorithm



$$x^*_{1:T} = \arg\max_{x_{1:T}} P(x_{1:T}|e_{1:T}) = \arg\max_{x_{1:T}} P(x_{1:T}, e_{1:T})$$

$$m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t})$$

$$= \max_{x_{1:t-1}} P(x_{1:t-1}, e_{1:t-1})P(x_t|x_{t-1})P(e_t|x_t)$$

$$= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1}, e_{1:t-1})$$

$$= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}]$$

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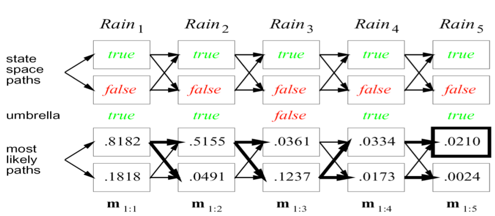
Example



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