CSE 473: Artificial Intelligence
Spring 2012

Reasoning about Uncertainty
&
Hidden Markov Models

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Many slides adapted from Dan Klein, Stuart Russell, Andrew Moore & Luke Zettlemoyer

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Outline

- Probabilistic sequence models (and inference)
  - Bayesian Networks – Preview
  - Markov Chains
  - Hidden Markov Models
  - Exact Inference
  - Particle Filters

Going Hunting

4

Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \dots H_r$

$$\left.\begin{array}{l} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{array}\right\} X_1, X_2, \dots X_n$$

All variables

- We want: $P(Q|e_1 \dots e_k)$
- First, select the entries consistent with the evidence
- Second, sum out H to get joint of Query and evidence:

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

- Finally, normalize the remaining entries to conditionalize
- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

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Bayes' Nets: Big Picture

- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions

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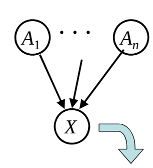
Bayes' Net Semantics

Formally:

- A set of nodes, one per variable X
- A directed, acyclic graph
- A CPT for each node
  - CPT = "Conditional Probability Table"
  - Collection of distributions over X, one for each combination of parents' values

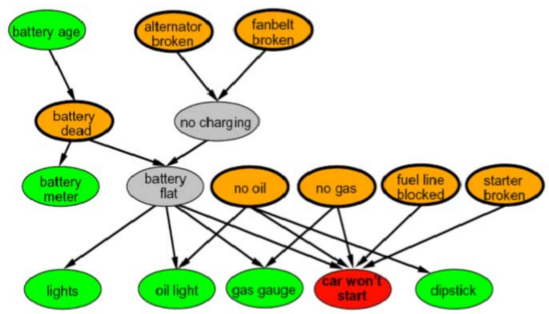
$$P(X|a_1 \dots a_n)$$



$$P(X|A_1 \dots A_n)$$

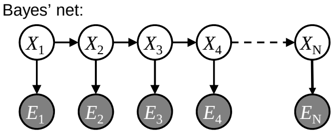
A Bayes net = Topology (graph) + Local Conditional Probabilities

Example Bayes' Net: Car

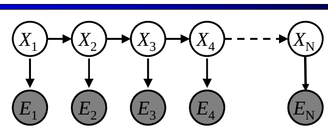


Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don't know anything anymore
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states S
  - You observe outputs (effects) at each time step
  - POMDPs without actions (or rewards).
  - As a Bayes' net:



Hidden Markov Models



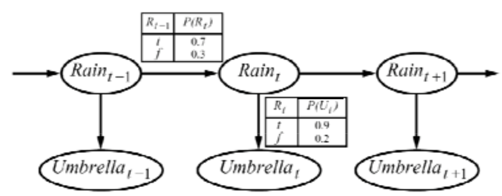
- Defines a joint probability distribution:

$$P(X_1,\ldots,X_n,E_1,\ldots,E_n) =$$

$$P(X_{1:n},E_{1:n}) =$$

$$P(X_1)P(E_1|X_1)\prod_{i=2}^{N} P(X_i|X_{i-1})P(E_i|X_i)$$

Example



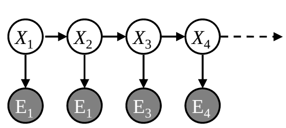
- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t|X_{t-1})$
  - Emissions: $P(E|X)$

Ghostbusters HMM

- $P(X_1)$ = uniform
- $P(X'|X)$ = usually move clockwise, but sometimes move in a random direction or stay in place
- $P(E|X)$ = same sensor model as before: red means close, green means far away.

1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$



1/6	1/6	1/2
0	1/6	0
0	0	0

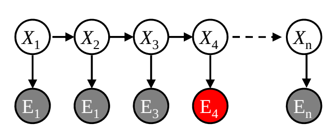
P(X'|X=<1,2>)

P(E|X)

P(red | 3)	P(orange | 3)	P(yellow | 3)	P(green | 3)
0.05	0.15	0.5	0.3

HMM Computations

- Given
  - joint $P(X_{1:n},E_{1:n})$
  - evidence $E_{1:n} = e_{1:n}$



- Inference problems include:
  - Filtering, find $P(X_t|e_{1:t})$ for some t
  - Smoothing, find $P(X_t|e_{1:n})$ for some t

HMM Computations

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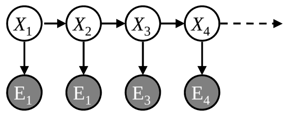
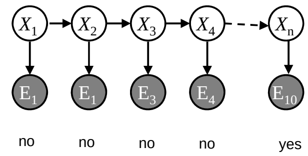


- Inference problems include:
  - Filtering, find $P(X_t|e_{1:t})$ for all t
  - Smoothing, find $P(X_t|e_{1:n})$ for all t
  - Most probable explanation, find

$$x^*_{1:n} = \text{argmax}_{x_{1:n}} P(x_{1:n}|e_{1:n})$$

HMM Computations

- State = number
  - T++ add sum of 2 dice
  - Observation:
    - Yes/no
    - Is sum 75?



- Most probable explanation, find

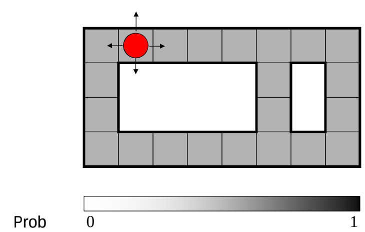
$$x^*_{1:n} = \text{argmax}_{x_{1:n}} P(x_{1:n}|e_{1:n})$$

Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution B(X) (the belief state) over time
- We start with B(X) in an initial setting, usually uniform
- As time passes, or we get observations, we update B(X)
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

Example: Robot Localization

Example from Michael Pfeiffer

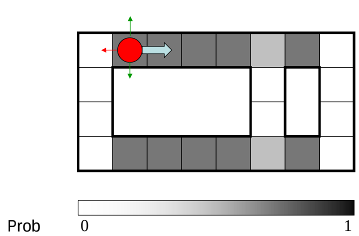


t=0

Sensor model: never more than 1 mistake

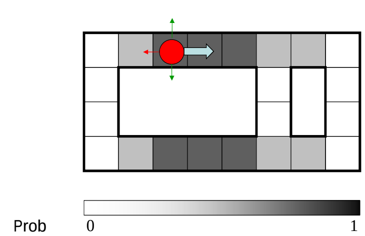
Motion model: may not execute action with small prob.

Example: Robot Localization



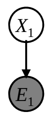
t=1

Example: Robot Localization



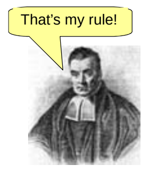
t=2

Inference Recap: Simple Cases



$$P(X_1|e_1)$$

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$
$$\propto_{X_1} P(x_1, e_1)$$
$$= P(x_1)P(e_1|x_1)$$



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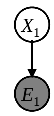

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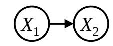
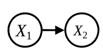


$$P(X_2)$$

$$P(x_2) = \sum_{x_1} P(x_1, x_2)$$
$$= \sum_{x_1} P(x_1)P(x_2|x_1)$$

Online Belief Updates

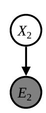
- Every time step, we start with current P(X | evidence)
- We update for time:



$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

- We update for evidence:

$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

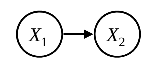


Passage of Time

- Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$

- Then, after one time step passes:



$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

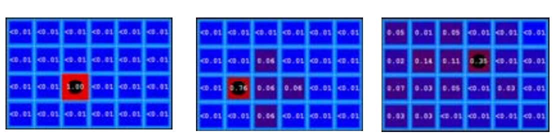
- Or, compactly:

$$B'(X') = \sum_x P(X'|x)B(x)$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Example: Passage of Time

Without observations, uncertainty “accumulates”



T = 1 T = 2 T = 5

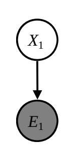
$$B'(X') = \sum_x P(X'|x)B(x)$$

Transition model: ghosts usually go clockwise

Observations

- Assume we have current belief P(X | previous evidence):

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$



- Then:

$$P(X_{t+1}|e_{1:t+1}) \propto P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

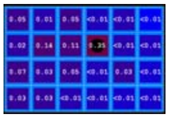
- Or:

$$B(X_{t+1}) \propto P(e|X)B'(X_{t+1})$$

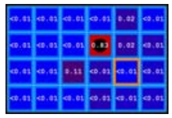
- Basic idea: beliefs reweighted by likelihood of evidence
- Unlike passage of time, we have to renormalize

Example: Observation

- As we get observations, beliefs get reweighted, uncertainty "decreases"



Before observation



After observation

$$B(X) \propto P(e|X)B'(X)$$

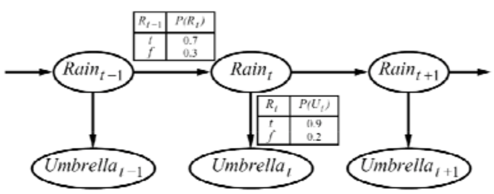
The Forward Algorithm

- We want to know: $B_t(X) = P(X_t|e_{1:t})$
- We can derive the following updates

$$P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t})$$
$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$
$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1})P(x_t|x_{t-1})P(e_t|x_t)$$
$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1}, e_{1:t-1})$$

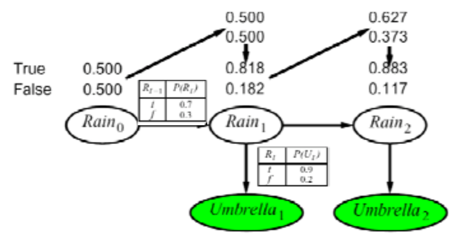
- To get $B_t(X)$ compute each entry and normalize

Example

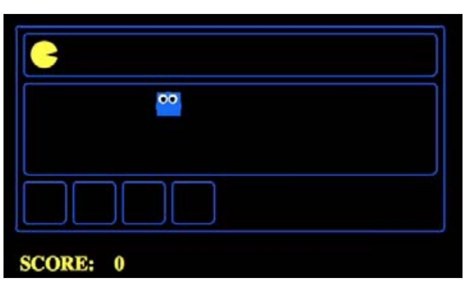


- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t|X_{t-1})$
  - Emissions: $P(E|X)$

Forward Algorithm



Example Pac-man



Summary: Filtering

- Filtering is the inference process of finding a distribution over X_T given e_1 through e_T : $P(X_T | e_{1:t})$
- We first compute $P(X_1 | e_1)$: $P(x_1|e_1) \propto P(x_1) \cdot P(e_1|x_1)$
- For each t from 2 to T, we have $P(X_{t-1} | e_{1:t-1})$
- **Elapse time:** compute $P(X_t | e_{1:t-1})$

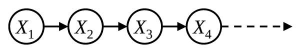
$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

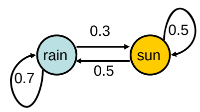
- **Observe:** compute $P(X_t | e_{1:t-1}, e_t) = P(X_t | e_{1:t})$

$$P(x_t|e_{1:t}) \propto P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

Recap: Reasoning Over Time

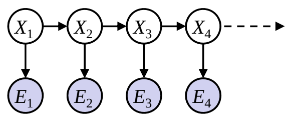
- Stationary Markov models





$P(X_1)$ $\quad$ $P(X|X_{-1})$

- Hidden Markov models

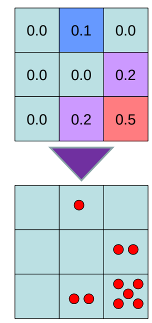


$P(E|X)$

X	E	P
rain	umbrella	0.9
rain	no umbrella	0.1
sun	umbrella	0.2
sun	no umbrella	0.8

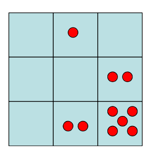
Particle Filtering

- Sometimes |X| is too big for exact inference
  - |X| may be too big to even store B(X)
  - E.g. when X is continuous
  - $|X|^2$ may be too big to do updates
- Solution: approximate inference
  - Track samples of X, not all values
  - Samples are called ***particles***
  - Time per step is linear in the number of samples
  - But: number needed may be large
  - In memory: list of particles, not states
- How robot localization works in practice



Representation: Particles

- Our representation of P(X) is now a list of N particles (samples)
  - ***Generally, N << |X|***
  - Storing map from X to counts would defeat the point
- P(x) approximated by number of particles with value x
  - So, many x will have P(x) = 0!
  - More particles, more accuracy
- For now, all particles have a weight of 1



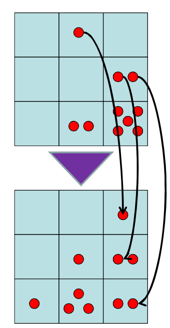
Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(2,1)
(3,3)
(3,3)
(2,1)

Particle Filtering: Elapse Time

- Each particle is moved by sampling its next position from the transition model

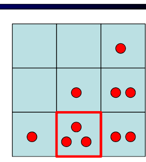
$$x' = \text{sample}(P(X'|x))$$

  - This is like prior sampling – samples' frequencies reflect the transition probs
  - Here, most samples move clockwise, but some move in another direction or stay in place
- This captures the passage of time
  - If we have enough samples, close to the exact values before and after (consistent)



Particle Filtering: Observe

- How handle noisy observations?
- Suppose sensor gives red reading?



Particle Filtering: Observe

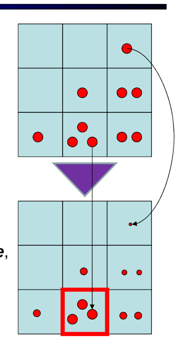
Slightly trickier:

- Don't do ***rejection sampling*** (why not?)
- We don't sample the observation, we fix it
- Instead: ***downweight samples*** based on the evidence (form of likelihood weighting)

$$w(x) = P(e|x)$$

$$B(X) \propto P(e|X)B'(X)$$

- Note: as before, probabilities ***don't sum to one***, since most have been downweighted (in fact they sum to an approximation of P(e))

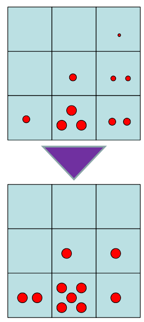


Particle Filtering: Resample

- Rather than tracking weighted samples, we resample
- N times, we choose from our weighted sample distribution (i.e. draw with replacement)
- This is equivalent to renormalizing the distribution
- Now the update is complete for this time step, continue with the next one

Old Particles:
(3,3) w=0.1
(2,1) w=0.9
(2,1) w=0.9
(3,1) w=0.4
(3,2) w=0.3
(2,2) w=0.4
(1,1) w=0.4
(3,1) w=0.4
(2,1) w=0.9
(3,2) w=0.3

New Particles:
(2,1) w=1
(2,1) w=1
(2,1) w=1
(3,2) w=1
(2,2) w=1
(2,1) w=1
(1,1) w=1
(3,1) w=1
(2,1) w=1
(1,1) w=1



Particle Filtering Summary

- Represent current belief P(X | evidence to date) as set of *n* samples (actual assignments X=x)
- For each new observation e:
  1. Sample transition, once for each current particle x
     $$x' = \text{sample}(P(X'|x))$$
  2. For each new sample x', compute importance weights for the new evidence e:
     $$w(x') = P(e|x')$$
  3. Finally, normalize by resampling the importance weights to create N new particles

Robot Localization

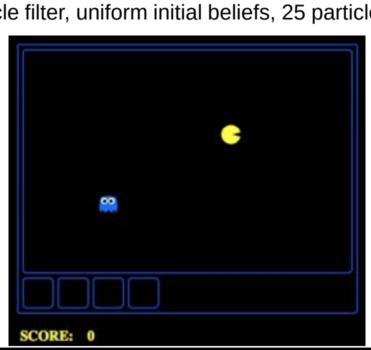
- In robot localization:
  - We know the map, but not the robot's position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store B(X)
  - Particle filtering is a main technique

Robot Localization

QuickTime™ and a
GIF decompressor
are needed to see this picture.

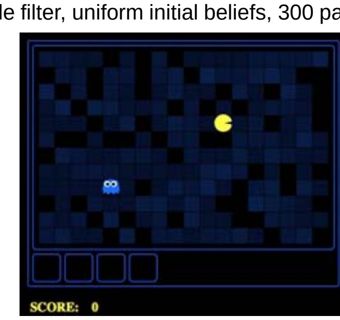
Which Algorithm?

Particle filter, uniform initial beliefs, 25 particles



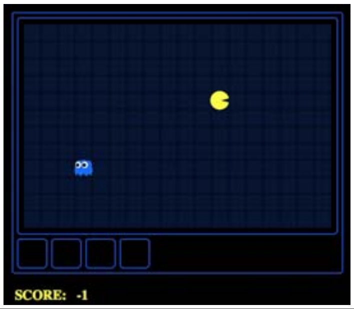
Which Algorithm?

Particle filter, uniform initial beliefs, 300 particles



Which Algorithm?

Exact filter, uniform initial beliefs



P4: Ghostbusters

- **Plot:** Pacman's grandfather, Grandpac, learned to hunt ghosts for sport.
- He was blinded by his power, but could hear the ghosts' banging and clanging.
- **Transition Model:** All ghosts move randomly, but are sometimes biased
- **Emission Model:** Pacman knows a "noisy" distance to each ghost

Noisy distance prob
True distance = 8