CSE 473

Lecture 20
(Chapters 13 & 14)

Probabilistic Inference

Today’s Outline

- Probability Theory
- Probabilistic Inference
- Conditional Independence
- Bayesian Networks

Types of Random Variables

Propositional or Boolean random variables

e.g., $Cavity$ (do I have a cavity?)

Discrete random variables (*finite* or *infinite*)

e.g., $Weather$ is one of $\langle sunny, rain, cloudy, snow \rangle$

$Weather = rain$ is a proposition

Values must be exhaustive and mutually exclusive

Continuous random variables (*bounded* or *unbounded*)

e.g., $Temp = 21.6$; also allow, e.g., $Temp < 22.0$.

Arbitrary Boolean combinations of basic propositions

Axioms of Probability Theory

Just 3 are enough to build entire theory!

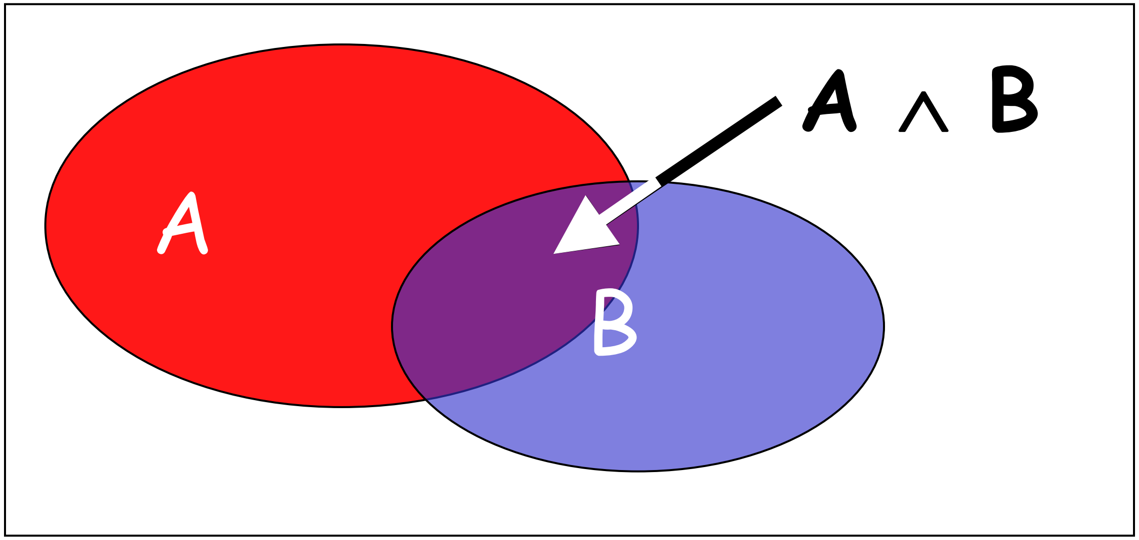
1. All probabilities between 0 and 1

   $0 \leq P(A) \leq 1$

2. P(true) = 1   and P(false) = 0

3. Probability of disjunction of events is:

$$P(A \vee B) = P(A) + P(B)$$



Prior Probability and Probability Distribution

Prior or unconditional probabilities of propositions

e.g., $P(Cavity = true) = 0.2$ and $P(Weather = sunny) = 0.72$

correspond to belief prior to arrival of any (new) evidence

Probability distribution gives values for all possible assignments:

$\mathbf{P}(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$ (*normalized*, i.e., sums to 1)

sunny, rain, cloudy, snow

Joint probability distribution for a set of r.v.s gives the probability of every atomic event on those r.v.s

$\mathbf{P}(Weather, Cavity) =$ a 4×2 matrix of values:

$Weather =$	$sunny$	$rain$	$cloudy$	$snow$
$Cavity = true$	0.144	0.02	0.016	0.02
$Cavity = false$	0.576	0.08	0.064	0.08

We will see later how any question can be answered by the joint distribution

Joint Probability

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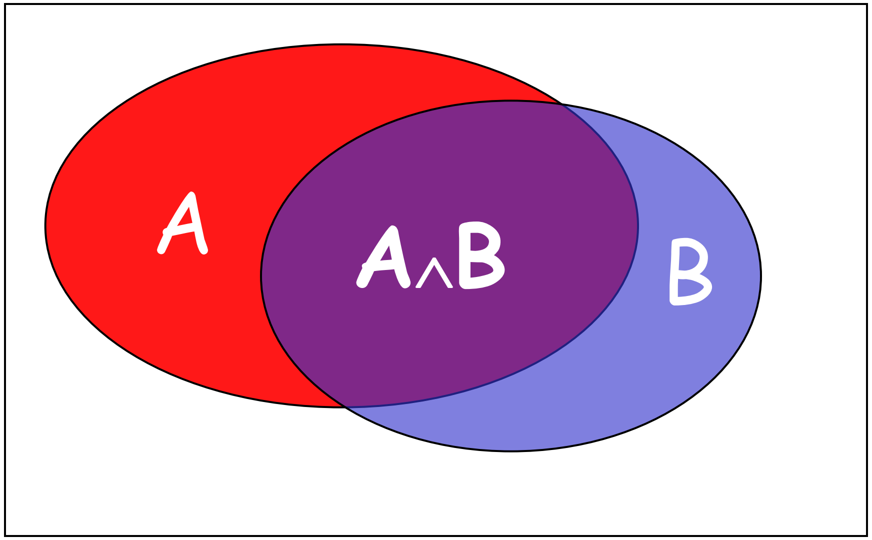
Conditional Probability

- P(*A* | *B*) is the probability of *A* given *B*
- Assumes that *B* is the only info known.
- Defined as:

$$P(A \mid B) = \frac{P(A, B)}{P(B)} = \frac{P(A \wedge B)}{P(B)}$$



Conditional Probability Examples

- P(*Cavity* = *true* | *Toothache* = *true*) = probability of *cavity* given *toothache* (lowercase denotes event is true)
- Notation for conditional distribution:

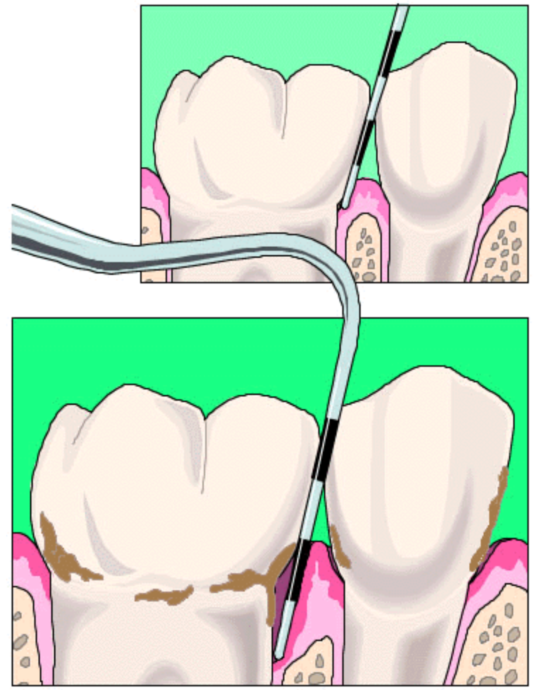
  P(*Cavity* | *Toothache*) = 2-element vector of 2-element vectors (2 Pr values given Toothache is true and 2 Pr values given Toothache is false)

- If we know more, e.g., *Cavity* = *true*, then we have

  P(*Cavity=T* | *toothache, Cavity=T*) = 1

- New evidence may be irrelevant, allowing simplification:
- P(*Cavity=T* | *toothache, Weather* = *sunny*) = P(*Cavity=T* | *toothache*)

Dilemma at the Dentist's



What is the probability of a cavity given a toothache?
What is the probability of a cavity given the probe catches?

Three R.V.'s, each either T or F:
Cavity, Toothache, Catch

Probabilistic Inference by Enumeration

Start with the joint distribution: P(Cavity, Toothache, Catch)

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

For any proposition ϕ, sum the atomic events where it is true:

$$P(\phi) = \Sigma_{\omega:\omega \models \phi} P(\omega)$$

P(toothache)= .108+.012+.016+.064

= .20 or 20%

(Note convention: All lowercase is used to denote that event is true)

Inference by Enumeration

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

P(toothache∨cavity) = ?

.20 + .108 + .012 + .072 + .008 - (.108+.012)

= .28

Inference by Enumeration

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬ *cavity*	**.016**	**.064**	**.144**	**.576**

Can also compute conditional probabilities:

$$P(\neg cavity | toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

Problems with Enumeration

- Worst case time: $O(d^n)$

  where d = max arity of random variables

  e.g., d = 2 for Boolean (T/F)

  and n = number of random variables
- Space complexity also $O(d^n)$
  - Size of joint distribution
- Problem: Hard/impossible to estimate all $O(d^n)$ entries of joint for large problems

Do we need to compute all $O(d^n)$ possible entries of joint distribution?

Independence

- Variables *A* and *B* are *independent* iff:

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = P(B)$$

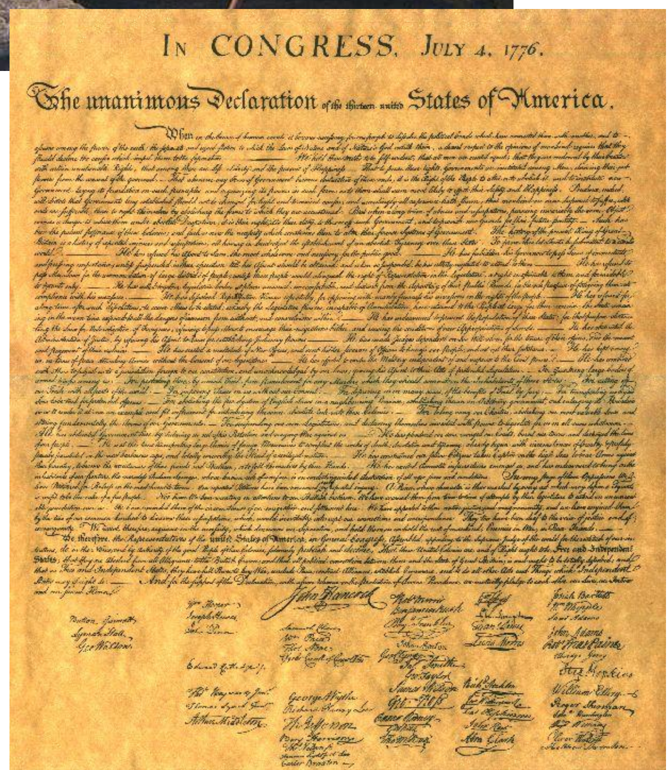
These two constraints are logically equivalent

Therefore, if *A* and *B* are independent:

$$P(A \mid B) = P(A) \text{ and } P(A \mid B) = \frac{P(A \wedge B)}{P(B)}$$

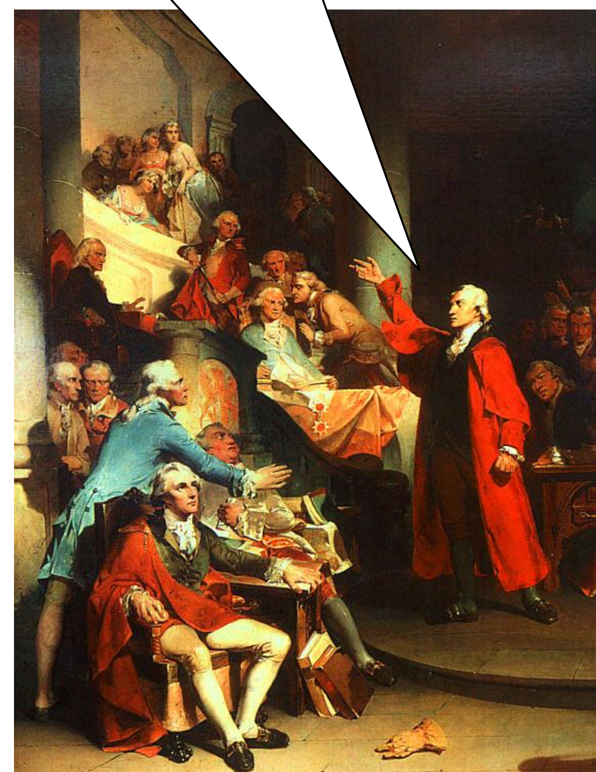
i.e., $P(A \wedge B) = P(A)P(B)$

Why is independence useful?



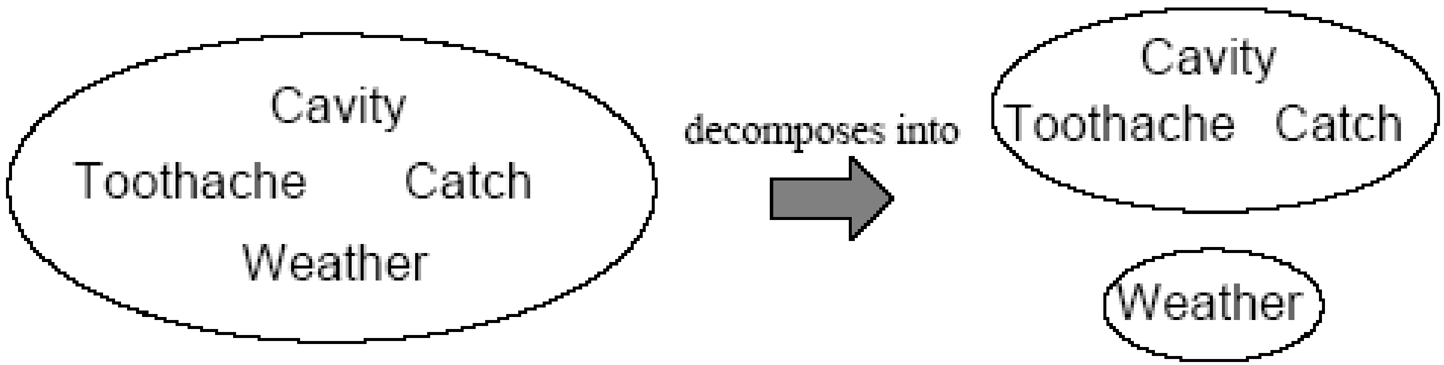
IN CONGRESS, JULY 4, 1776.

The unanimous Declaration of the thirteen united States of America.



Independence

Joint



2*2*2*4=32 values

$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$
$$= \mathbf{P}(Toothache, Catch, Cavity)\mathbf{P}(Weather)$$

Only 2*2*2 + 4 = 12 values needed

32 entries reduced to 12; for n independent biased coins, $2^n \to n$

Complete independence is powerful but rare. What to do if it doesn't hold?

Conditional Independence

Joint distribution:

$\mathbf{P}(Toothache, Cavity, Catch)$ has $2^3 - 1 = 7$ independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

(1) $P(catch|toothache, cavity) = P(catch|cavity)$

The same independence holds if I haven't got a cavity:

(2) $P(catch|toothache, \neg cavity) = P(catch|\neg cavity)$

$Catch$ is *conditionally independent* of $Toothache$ given $Cavity$:

$\mathbf{P}(Catch|Toothache, Cavity) = \mathbf{P}(Catch|Cavity)$

Now in the joint distribution, instead of 7 entries, only need 5 (why?)

Conditional Independence II

Given:

$\mathbf{P}(Catch \mid Toothache,\ Cavity) = \mathbf{P}(Catch \mid\ Cavity)$

Joint probability distribution:

$$\mathbf{P}(Catch, Toothache, Cavity)$$

$$= \mathbf{P}(Catch \mid Toothache, Cavity)\mathbf{P}(Toothache, Cavity)$$

$$= \mathbf{P}(Catch \mid Toothache, Cavity)\mathbf{P}(Toothache \mid Cavity)\mathbf{P}(Cavity)$$

$$= \mathbf{P}(Catch \mid Cavity)\mathbf{P}(Toothache \mid Cavity)\mathbf{P}(Cavity)$$

2 + 2 + 1

= 5 independent numbers

Power of Cond. Independence

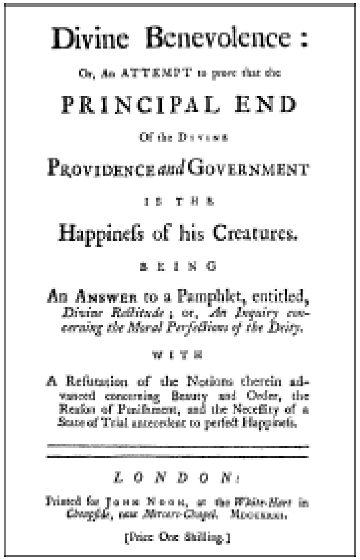
- Often, conditional independence can reduce the storage complexity of the joint distribution from exponential to linear!!

- Conditional independence is the most basic & robust form of knowledge in uncertain environments.

Thomas Bayes

Reverand Thomas Bayes
Nonconformist minister
(1702-1761)

- Publications:
- *Divine Benevolence, or an Attempt to Prove That the Principal End of the Divine Providence and Government is the Happiness of His Creatures* (1731)
- *An Introduction to the Doctrine of Fluxions* (1736)
- *An Essay Towards Solving a Problem in the Doctrine of Chances* (1764)



Divine Benevolence:

Or, An ATTEMPT to prove that the

PRINCIPAL END

Of the DIVINE

PROVIDENCE *and* GOVERNMENT

IS THE

Happiness of his Creatures.

BEING

An ANSWER to a Pamphlet, entitled, *Divine Rectitude; or, An Inquiry concerning the Moral Perfections of the Deity.*

WITH

A Refutation of the Notions therein advanced concerning Beauty and Order, the Reason of Punishment, and the Necessity of a *State of Trial* antecedent to perfect Happiness.

LONDON:

Printed for JOHN NOON, at the *White-Hart* in *Cheapside*, near *Mercers-Chapel*. MDCCXXXI.

[Price One Shilling.]

Next Time

- Bayesian Networks
- To Do
  - Project 3
  - Read Chapter 14