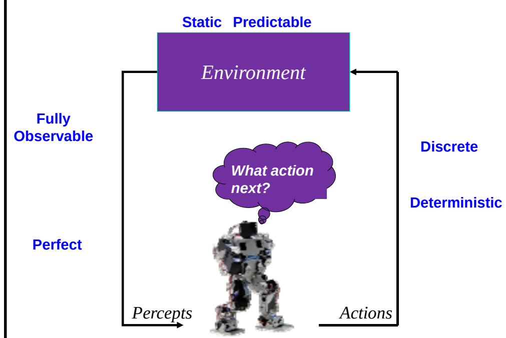
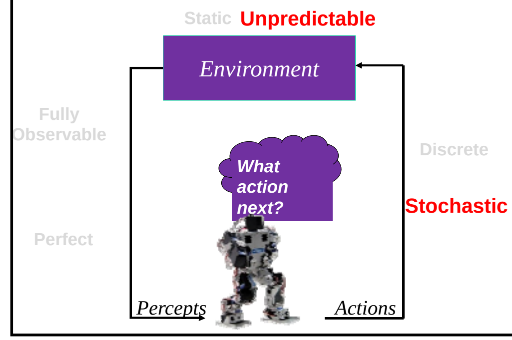
CSE-473 Artificial Intelligence

Partially-Observable MDPS
(POMDPs)

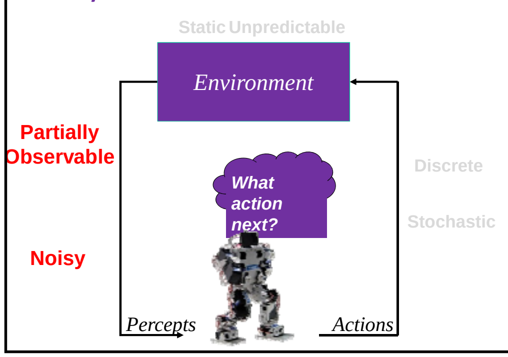
Classical Planning



Stochastic Planning
(MDPs, Reinforcement Learning)



Partially-Observable MDPs



Markov Decision Process (MDP)

- **S**: set of states
- **A**: set of actions
- **P**r(s'|s,a): transition model
- **R**(s,a,s'): reward model
- γ: discount factor
- s_0: start state

Objective of a Fully Observable MDP

- Find a policy π: **S → A**
- which maximizes expected discounted reward
  - given an infinite horizon
  - assuming full observability

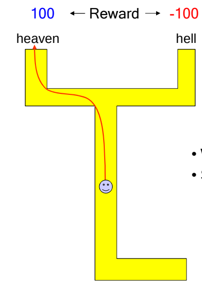
Partially-Observable MDP

- **S**: set of states
- **A**: set of actions
- **Pr**(s' | s,a): transition model
- **R**(s,a,s'): reward model
- γ: discount factor
- s_0: start state
- **E** set of possible evidence (observations)
- **Pr**(e | s)

Objective of a POMDP

- Find a policy
  - π: BeliefStates(**S**) → **A**
  - A belief state is a ***probability distribution*** over states
- which maximizes expected discounted reward
  - given an infinite horizon
  - assuming full observability

Classical Planning



- Sequential Plan
- World deterministic
- State observable

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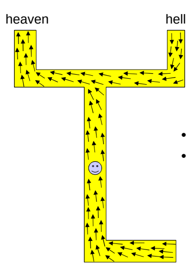
MDP-Style Planning



- Policy
- World stochastic
- State observable

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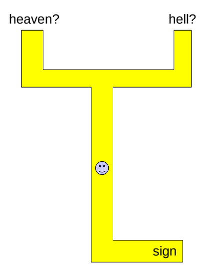
Stochastic, *Partially* Observable



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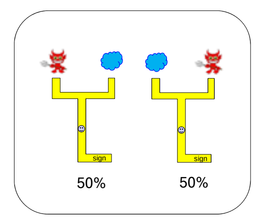
Belief State

- State of agent's mind
- Not just of world



- Note: Distribution: sum of probabilities = 1

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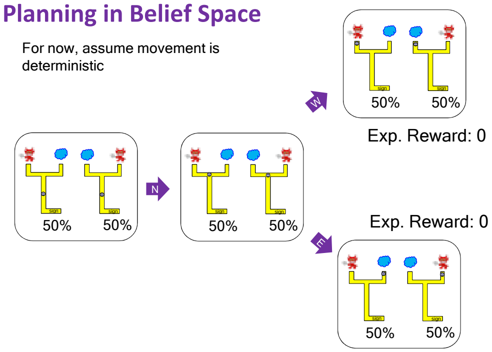
Planning in Belief Space

For now, assume movement is deterministic



Partially-Observable MDP

- **S**: set of states
- **A**: set of actions
- **Pr**(s'|s,a): transition model
- **R**(s,a,s'): reward model
- γ: discount factor
- s_0: start state
- **E** set of possible evidence (observations)
- **Pr**(e|s)

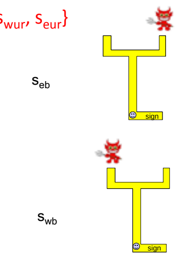
Evidence Model

- **S** = $\{s_{wb}, s_{eb}, s_{wm}, s_{em}\ s_{wul}, s_{eul}\ s_{wur}, s_{eur}\}$
- **E** = {heat}
- **Pr(e|s):**

  Pr(heat | s_{eb}) = 1.0

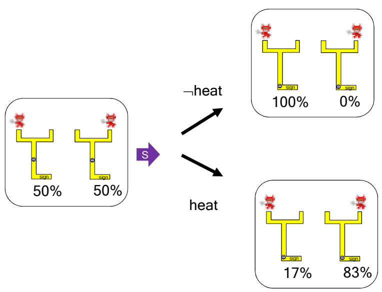
  Pr(heat | s_{wb}) = 0.2

  Pr(heat | s_{other}) = 0.0



Planning in Belief Space

Pr(heat | s_{eb}) = 1.0
Pr(heat | s_{wb}) = 0.2



POMDPs

- In POMDPs we apply the very same idea as in MDPs.
- Since the state is not observable, the agent has to make its decisions based on the belief state which is a posterior distribution over states.
- Let *b* be the belief of the agent about the state under consideration.
- POMDPs compute a value function over belief space:

$$V_T(b) \quad = \quad \max_u \left[r(b,u) + \gamma \int V_{T-1}(b')p(b' \mid u, b)\, db' \right]$$

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Problems

- Each belief is a probability distribution, thus, each value in a **POMDP is a function of an entire probability distribution**.
- **This is problematic, since probability distributions are continuous.**
- How many belief states are there?
- For **finite worlds** with finite state, action, and measurement spaces and finite horizons, however, we can **effectively represent the value functions by piecewise linear functions**.

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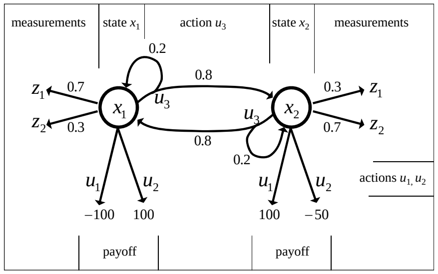
An Illustrative Example



The Parameters of the Example

- The actions u_1 and u_2 are terminal actions.
- The action u_3 is a sensing action that potentially leads to a state transition.
- The horizon is finite and γ=1.

$$\begin{aligned} r(x_1, u_1) &= -100 & r(x_2, u_1) &= +100 \\ r(x_1, u_2) &= +100 & r(x_2, u_2) &= -50 \\ r(x_1, u_3) &= -1 & r(x_2, u_3) &= -1 \end{aligned}$$

$$\begin{aligned} p(x'_1|x_1, u_3) &= 0.2 & p(x'_2|x_1, u_3) &= 0.8 \\ p(x'_1|x_2, u_3) &= 0.8 & p(z'_2|x_2, u_3) &= 0.2 \end{aligned}$$

$$\begin{aligned} p(z_1|x_1) &= 0.7 & p(z_2|x_1) &= 0.3 \\ p(z_1|x_2) &= 0.3 & p(z_2|x_2) &= 0.7 \end{aligned}$$

Payoff in POMDPs

- In MDPs, the payoff (or return) depended on the state of the system.
- In POMDPs, however, the true state is not exactly known.
- Therefore, we compute the **expected payoff** by **integrating over all states**:

$$\begin{aligned} r(b, u) &= E_x[r(x, u)] \\ &= \int r(x, u)p(x)\ dx \\ &= p_1\ r(x_1, u) + p_2\ r(x_2, u) \end{aligned}$$

Payoffs in Our Example (1)

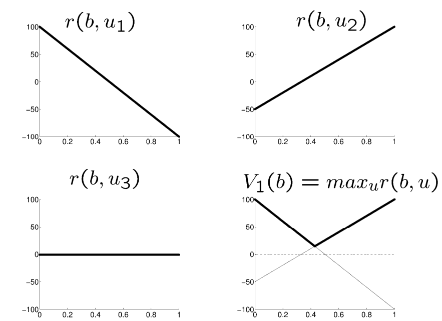
- If we are totally certain that we are in state x_1 and execute action u_1, we receive a reward of -100
- If, on the other hand, we definitely know that we are in x_2 and execute u_1, the reward is +100.
- In between it is the linear combination of the extreme values weighted by the probabilities

$$\begin{aligned} r(b, u_1) &= -100\ p_1 + 100\ p_2 \\ &= -100\ p_1 + 100\ (1 - p_1) \end{aligned}$$

$$r(b, u_2) = 100\ p_1 - 50\ (1 - p_1)$$

$$r(b, u_3) = -1$$

Payoffs in Our Example (2)



The Resulting Policy for T=1

- Given we have a finite POMDP with T=1, we would use $V_1(b)$ to determine the optimal policy.
- In our example, the optimal policy for T=1 is

$$\pi_1(b) = \begin{cases} u_1 & \text{if } p_1 \le \frac{3}{7} \\ u_2 & \text{if } p_1 > \frac{3}{7} \end{cases}$$

- This is the upper thick graph in the diagram.

Piecewise Linearity, Convexity

- The resulting value function $V_1(b)$ is the maximum of the three functions at each point

$$V_1(b) = \max_u r(b,u) = \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ -1 & \end{array} \right\}$$

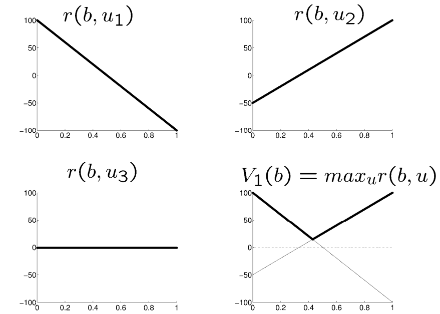
- It is piecewise linear and convex.

Pruning

- If we carefully consider $V_1(b)$, we see that only the first two components contribute.
- The third component can therefore safely be pruned away from $V_1(b)$.

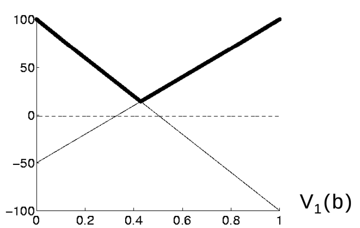
$$V_1(b) = \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \end{array} \right\}$$

Payoffs in Our Example (2)



Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.



Increasing the Time Horizon

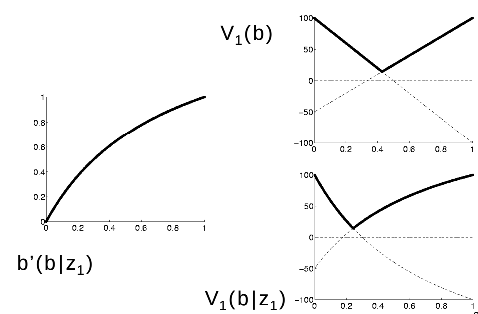
- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.

$$p'_1 = \frac{0.7\, p_1}{p(z_1)}$$

$$p'_2 = \frac{0.3(1-p_1)}{p(z_1)}$$

$$p(z_1) = 0.7\, p_1 + 0.3(1-p_1) = 0.4\, p_1 + 0.3$$

Value Function



Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.
- Thus $V_1(b \mid z_1)$ is given by

$$V_1(b \mid z_1) = \max \left\{ \begin{array}{ll} -100 \cdot \frac{0.7\, p_1}{p(z_1)} & +100 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \\ 100 \cdot \frac{0.7\, p_1}{p(z_1)} & -50 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \end{array} \right\}$$

$$= \frac{1}{p(z_1)} \max \left\{ \begin{array}{ll} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\}$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$\bar{V}_1(b) = E_z[V_1(b \mid z)] = \sum_{i=1}^{2} p(z_i) V_1(b \mid z_i)$$

$$= \sum_{i=1}^{2} p(z_i) V_1\left(\frac{p(z_i \mid x_1) p_1}{p(z_i)}\right)$$

$$= \sum_{i=1}^{2} V_1\left(p(z_i \mid x_1) p_1\right)$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$\begin{aligned} \bar{V}_1(b) &= E_z[V_1(b \mid z)] \\ &= \sum_{i=1}^{2} p(z_i)\, V_1(b \mid z_i) \\ &= \max \left\{ \begin{array}{ll} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\} \\ &\quad + \max \left\{ \begin{array}{ll} -30\, p_1 & +70\,(1-p_1) \\ 30\, p_1 & -35\,(1-p_1) \end{array} \right\} \end{aligned}$$

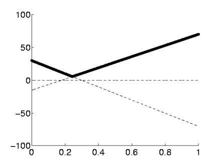
Resulting Value Function

- The four possible combinations yield the following function which then can be simplified and pruned.

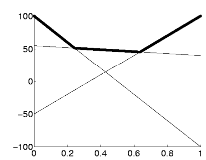
$$\bar{V}_1(b) = \max \left\{ \begin{array}{llll} -70\, p_1 & +30\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ -70\, p_1 & +30\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ +40\, p_1 & +55\,(1-p_1) \\ +100\, p_1 & -50\,(1-p_1) \end{array} \right\}$$

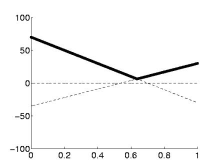
Value Function



p(z$_1$) V$_1$(b|z$_1$)



\bar{V}$_1$(b)

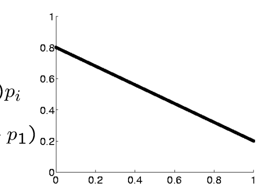


p(z$_2$) V$_2$(b|z$_2$)

State Transitions (Prediction)

- When the agent selects u_3 its state potentially changes.
- When computing the value function, we have to take these potential state changes into account.

$$\begin{aligned} p_1' &= E_x[p(x_1 \mid x, u_3)] \\ &= \sum_{i=1}^{2} p(x_1 \mid x_i, u_3) p_i \\ &= 0.2 p_1 + 0.8(1-p_1) \\ &= 0.8 - 0.6 p_1 \end{aligned}$$



Resulting Value Function after executing u_3

Taking the state transitions into account, we finally obtain.

$$\bar{V}_1(b) = \max \left\{ \begin{array}{llll} -70\,p_1 & +30\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ -70\,p_1 & +30\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{ll} -100\,p_1 & +100\,(1-p_1) \\ +40\,p_1 & +55\,(1-p_1) \\ +100\,p_1 & -50\,(1-p_1) \end{array} \right\}$$

$$\bar{V}_1(b \mid u_3) = \max \left\{ \begin{array}{ll} 60\,p_1 & -60\,(1-p_1) \\ 52\,p_1 & +43\,(1-p_1) \\ -20\,p_1 & +70\,(1-p_1) \end{array} \right\}$$

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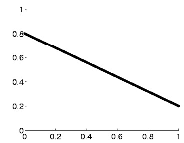
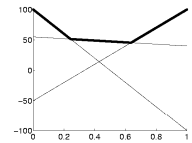
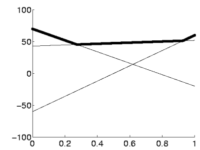
Value Function after executing u_3



\bar{V}_1(b)



\bar{V}_1(b|u_3)



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Value Function for T=2

- Taking into account that the agent can either directly perform u_1 or u_2 or first u_3 and then u_1 or u_2, we obtain (after pruning)

$$\bar{V}_2(b) = \max \left\{ \begin{array}{ll} -100\,p_1 & +100\,(1-p_1) \\ 100\,p_1 & -50\,(1-p_1) \\ 51\,p_1 & +42\,(1-p_1) \end{array} \right\}$$

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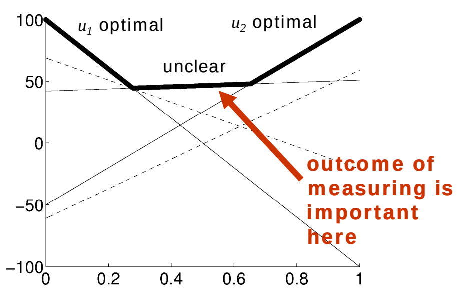
Graphical Representation of $V_2(b)$



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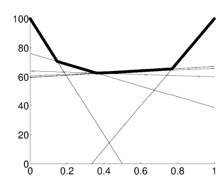
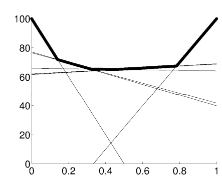
Deep Horizons and Pruning

- We have now completed a full backup in belief space.
- This process can be applied recursively.
- The value functions for T=10 and T=20 are





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Why Pruning is Essential

- Each **update introduces additional linear components** to V.
- Each **measurement squares the number of linear components**.
- Thus, an unpruned value function for T=20 includes more than $10^{547,864}$ linear functions.
- At T=30 we have $10^{561,012,337}$ linear functions.
- The pruned value functions at T=20, in comparison, contains only 12 linear components.
- The combinatorial explosion of linear components in the value function are the major reason why **POMDPs are impractical for most applications**.

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POMDP Summary

- POMDPs compute the optimal action in partially observable, stochastic domains.
- For finite horizon problems, the resulting value functions are piecewise linear and convex.
- In each iteration the number of linear constraints grows exponentially.
- POMDPs so far have only been applied successfully to very small state spaces with small numbers of possible observations and actions.

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