CSE 473: Artificial Intelligence

Reinforcement Learning

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Many slides adapted from either Dan Klein, Stuart Russell, Luke Zettlemoyer or Andrew Moore

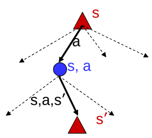
Today's Outline

- Reinforcement Learning
  - Q-value iteration
  - Q-learning
  - Exploration / exploitation
  - Linear function approximation

Unbeknownst to most students of psychology, Pavlov's first experiment was to ring a bell and cause his dog to attack Freud's cat.

Recap: MDPs

- Markov decision processes:
  - States S
  - Actions A
  - Transitions T(s,a,s′) aka P(s′|s,a)
  - Rewards R(s,a,s′) (and discount γ)
  - Start state s_0 (or distribution P_0)
- Algorithms
  - Value Iteration
  - Q-value iteration
- Quantities:
  - Policy = map from states to actions
  - Utility = sum of discounted future rewards
  - Q-Value = expected utility from a q-state
    - Ie. from a state/action pair



Andrey Markov (1856-1922)

Bellman Equations

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(a,s) = \sum_{s' \in \mathcal{S}} \mathcal{P}r(s'|s,a)\left[\mathcal{R}(s,a,s') + \gamma V^*(s')\right]$$

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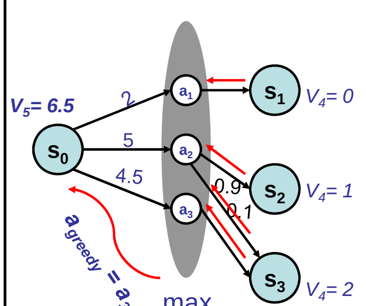
Bellman Backup



$Q_5(s,a_1) = 2 + \gamma\ 0$
$\sim$ **2**

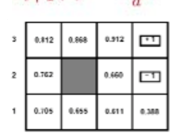
$Q_5(s,a_2) = 5 + \gamma\ 0.9 \sim 1$
$+ \gamma\ 0.1 \sim 2$
$\sim$ **6.1**

$Q_5(s,a_3) = 4.5 + \gamma\ 2$
$\sim$ **6.5**

Q-Value Iteration

- Regular Value iteration: find successive approx optimal values
  - Start with $V_0^*(s) = 0$
  - Given V_i^*, calculate the values for all states for depth i+1:

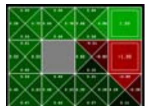
$$V_{i+1}(s) \leftarrow \max_a Q_{i+1}(s,a)$$



- Storing Q-values is more useful!
  - Start with $Q_0^*(s,a) = 0$
  - Given Q_i^*, calculate the q-values for all q-states for depth i+1:

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma\ V_i(s')\right]$$

Q-Value Iteration



Initialize each q-state: $Q_0(s,a) = 0$

Repeat

For all q-states, s,a

Compute $Q_{i+1}(s,a)$ from Q_i by Bellman backup at s,a.

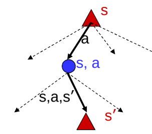
Until $\max_{s,a} |Q_{i+1}(s,a) - Q_i(s,a)| < \varepsilon$

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma\, V_i(s') \right]$$

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q_i(s',a') \right]$$

Reinforcement Learning

- Markov decision processes:
  - States S
  - Actions A
  - Transitions T(s,a,s′) aka P(s′|s,a)
  - Rewards R(s,a,s′) (and discount γ)
  - Start state s_0 (or distribution P_0)
- Algorithms
  - Q-value iteration → Q-learning
  - Approaches for mixing exploration & exploitation
    - ε-greedy
    - Exploration functions



Applications

- Robotic control
  - helicopter maneuvering, autonomous vehicles
  - Mars rover - path planning, oversubscription planning
  - elevator planning
- Game playing - backgammon, tetris, checkers
- Neuroscience
- Computational Finance, Sequential Auctions
- Assisting elderly in simple tasks
- Spoken dialog management
- Communication Networks – switching, routing, flow control
- War planning, evacuation planning

Stanford Autonomous Helicopter

http://heli.stanford.edu/

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Two main reinforcement learning approaches

- Model-based approaches:
  - explore environment & learn model, T=P(**s**′|**s**,**a**) and R(**s**,**a**), (almost) everywhere
  - use model to plan policy, MDP-style
  - approach leads to strongest theoretical results
  - often works well when state-space is manageable
- Model-free approach:
  - don′t learn a model; learn value function or policy directly
  - weaker theoretical results
  - often works better when state space is large

Two main reinforcement learning approaches

- Model-based approaches:

Learn T + R

$|S|^2|A| + |S||A|$ parameters (40,000)

- Model-free approach:

Learn Q

$|S||A|$ parameters (400)

Recap: Sampling Expectations

- Want to compute an expectation weighted by P(x):

$$E[f(x)] = \sum_x P(x) f(x)$$

- Model-based: estimate P(x) from samples, compute expectation

$$x_i \sim P(x)$$
$$\hat{P}(x) = \text{count}(x)/k \qquad E[f(x)] \approx \sum_x \hat{P}(x) f(x)$$

- Model-free: estimate expectation directly from samples

$$x_i \sim P(x) \qquad E[f(x)] \approx \frac{1}{k}\sum_i f(x_i)$$

- Why does this work? Because samples appear with the right frequencies!

Recap: Exp. Moving Average

- Exponential moving average
  - Makes recent samples more important

$$\bar{x}_n = \frac{x_n + (1-\alpha)\cdot x_{n-1} + (1-\alpha)^2 \cdot x_{n-2} + \dots}{1 + (1-\alpha) + (1-\alpha)^2 + \dots}$$

  - Forgets about the past (distant past values were wrong anyway)
  - Easy to compute from the running average

$$\bar{x}_n = (1-\alpha)\cdot \bar{x}_{n-1} + \alpha \cdot x_n$$

- Decreasing learning rate can give converging averages

Q-Learning Update

- Q-Learning = ***sample-based*** Q-value iteration

$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma \max_{a'} Q^*(s',a')\right]$$

- How learn Q*(s,a) values?
  - Receive a sample (s,a,s′,r)
  - Consider your old estimate: $Q(s,a)$
  - Consider your new sample estimate:

$$sample = R(s,a,s') + \gamma \max_{a'} Q(s',a')$$

  - Incorporate the new estimate into a running average:

$$Q(s,a) \leftarrow (1-\alpha)Q(s,a) + (\alpha)\,[sample]$$

Exploration-Exploitation tradeoff

- You have visited part of the state space and found a reward of 100
  - is this the best you can hope for???
- **Exploitation**: should I stick with what I know and find a good policy w.r.t. this knowledge?
  - at risk of missing out on a better reward somewhere
- **Exploration**: should I look for states w/ more reward?
  - at risk of wasting time & getting some negative reward

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Exploration / Exploitation

- Several schemes for action selection
  - Simplest: random actions (*ε* ***greedy***)
    - Every time step, flip a coin
    - With probability ε , act randomly
    - With probability 1- ε, act according to current policy
  - Problems with random actions?
    - You do explore the space, but keep thrashing around once learning is done
    - One solution: lower ε over time
    - Another solution: ***exploration functions***

Q-Learning: ε Greedy

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

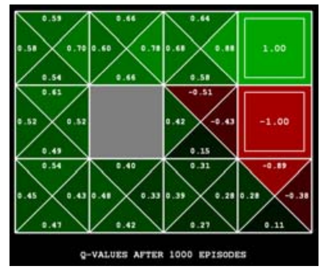
Exploration Functions

- When to explore
  - Random actions: explore a fixed amount
  - Better idea: explore areas whose badness is not (yet) established
- Exploration function
  - Takes a value estimate and a count, and returns an ***optimistic utility***, e.g. $f(u,n) = u + k/n$ (exact form not important)
  - Exploration policy $\pi(s')$=

$$\max_{a'} Q_i(s',a') \quad \text{vs.} \quad \max_{a'} f(Q_i(s',a'), N(s',a'))$$

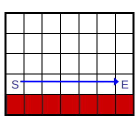
Q-Learning Final Solution

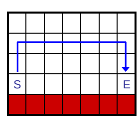
- Q-learning produces tables of q-values:



Q-Learning Properties

- Amazing result: Q-learning converges to optimal policy
  - If you explore enough
  - If you make the learning rate small enough
  - … but not decrease it too quickly!
  - Not too sensitive to how you select actions (!)
- Neat property: off-policy learning
  - learn optimal policy without following it (some caveats)



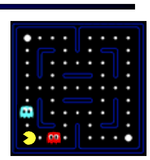


Q-Learning – Small Problem

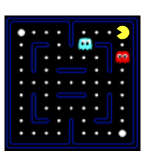
- Doesn't work
- In realistic situations, we can't possibly learn about every single state!
  - Too many states to visit them all in training
  - Too many states to hold the q-tables in memory
- Instead, we need to ***generalize***:
  - Learn about a few states from experience
  - Generalize that experience to new, ***similar*** states (Fundamental idea in machine learning)

Example: Pacman

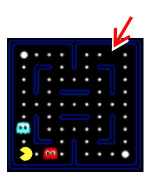
- Let's say we discover through experience that this state is bad:



- In naïve Q learning, we know nothing about related states and their Q values:

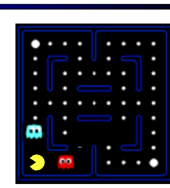


- Or even this third one!



Feature-Based Representations

- Solution: describe a state using a vector of features (properties)
  - Features are functions from states to real numbers (often 0/1) that capture important properties of the state



  - Example features:
    - Distance to closest ghost
    - Distance to closest dot
    - Number of ghosts
    - 1 / (dist to dot)^2
    - Is Pacman in a tunnel? (0/1)
    - …… etc.
  - Can also describe a q-state (s, a) with features (e.g. action moves closer to food)

Linear Feature Functions

- Using a feature representation, we can write a q function (or value function) for any state using a linear combination of a few weights:

$$Q(s,a) = w_1 f_1(s,a) + w_2 f_2(s,a) + \ldots + w_n f_n(s,a)$$

- Advantage: our experience is summed up in a ***few*** powerful numbers

  $|S|^2|A|$? $|S||A|$?

- Disadvantage: states may share features but actually be very different in value!

Function Approximation

$$Q(s,a) = w_1 f_1(s,a) + w_2 f_2(s,a) + \ldots + w_n f_n(s,a)$$

- Q-learning with linear q-functions:

$$\textit{transition} = (s,a,r,s')$$

$$\text{difference} = \left[r + \gamma \max_{a'} Q(s',a')\right] - Q(s,a)$$

$$Q(s,a) \leftarrow Q(s,a) + \alpha\,[\text{difference}]$$ Exact Q's

$$w_i \leftarrow w_i + \alpha\,[\text{difference}]\, f_i(s,a)$$ Approximate Q's

- Intuitive interpretation:
  - Adjust weights of active features
  - E.g. if something unexpectedly bad happens, disprefer all states with that state's features
- Formal justification: online least squares

Example: Q-Pacman

$$Q(s,a) = 4.0 f_{DOT}(s,a) - 1.0 f_{GST}(s,a)$$

$$f_{DOT}(s, \text{NORTH}) = 0.5$$

$$f_{GST}(s, \text{NORTH}) = 1.0$$

$$Q(s,a) = +1$$

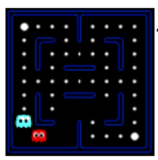
$$R(s,a,s') = -500$$

$$\textit{correction} = -501$$

$a = \text{NORTH}$
$r = -500$

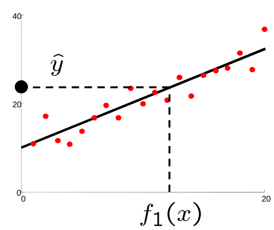
$$w_{DOT} \leftarrow 4.0 + \alpha\,[-501]\,0.5$$

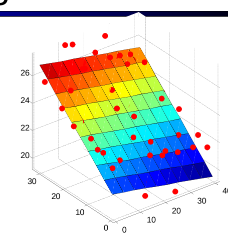
$$w_{GST} \leftarrow -1.0 + \alpha\,[-501]\,1.0$$



$$Q(s,a) = 3.0 f_{DOT}(s,a) - 3.0 f_{GST}(s,a)$$

Linear Regression



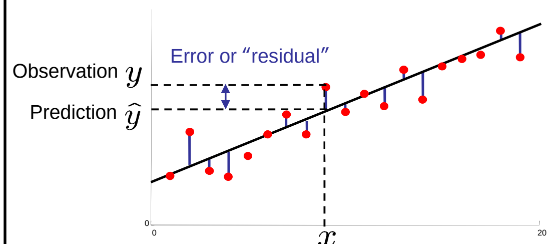


Prediction
$\hat{y} = w_0 + w_1 f_1(x)$

Prediction
$\hat{y}_i = w_0 + w_1 f_1(x) + w_2 f_2(x)$

Ordinary Least Squares (OLS)

$$\text{total error} = \sum_i (y_i - \hat{y}_i)^2 = \sum_i \left(y_i - \sum_k w_k f_k(x_i)\right)^2$$



Minimizing Error

Imagine we had only one point x with features f(x):

$$\text{error}(w) = \frac{1}{2}\left(y - \sum_k w_k f_k(x)\right)^2$$

$$\frac{\partial\, \text{error}(w)}{\partial w_m} = -\left(y - \sum_k w_k f_k(x)\right) f_m(x)$$

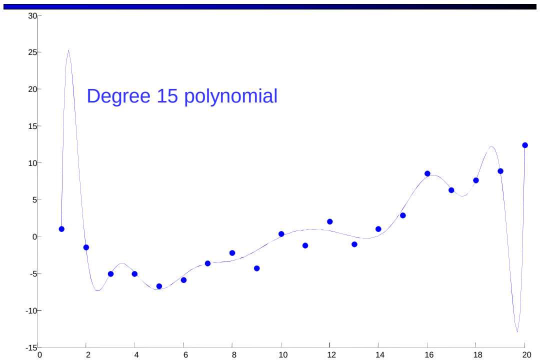
$$w_m \leftarrow w_m + \alpha \left(y - \sum_k w_k f_k(x)\right) f_m(x)$$

Approximate q update:

"target" "prediction"

$$w_m \leftarrow w_m + \alpha\left[r + \gamma \max_a Q(s',a') - Q(s,a)\right] f_m(s,a)$$

Overfitting



Which Algorithm?

Q-learning, no features, 50 learning trials:

QuickTime™ and a
GIF decompressor
are needed to see this picture.

Which Algorithm?

Q-learning, no features, 1000 learning trials:

QuickTime™ and a
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Which Algorithm?

Q-learning, simple features, 50 learning trials:

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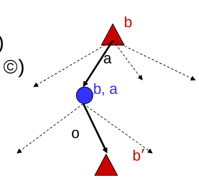
Partially observable MDPs

- Markov decision processes:
  - States S
  - Actions A
  - Transitions P(s′|s,a) (or T(s,a,s′))
  - Rewards R(s,a,s′) (and discount ©)
  - Start state distribution $b_0=P(s_0)$
- POMDPs, just add:
  - Observations O
  - Observation model P(o|s,a) (or O(s,a,o))

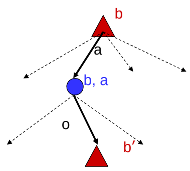


A POMDP: Ghost Hunter

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

POMDP Computations

- Sufficient statistic: belief states
  - $b_o = \Pr(s_o)$
  - [illegible]
- POMDPs search trees
  - max nodes are belief states
  - expectation nodes branch on possible observations
  - (this is motivational; we will not discuss in detail)



Types of Planning Problems

	State	Action Model
Classical Planning	observable	Deterministic, accurate
MDPs	observable	stochastic
POMDPs	partially observable	stochastic

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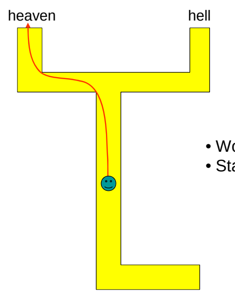
Classical Planning



- World deterministic
- State observable

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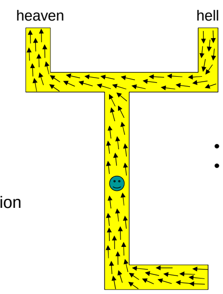
MDP-Style Planning



- Policy
- Universal Plan
- Navigation function

- World stochastic
- State observable

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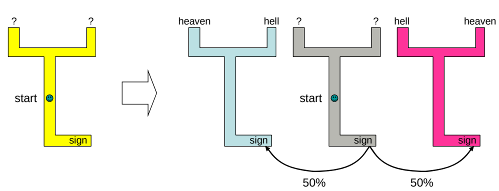
Stochastic, Partially Observable



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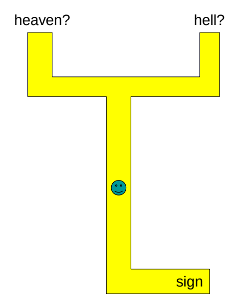
Stochastic, Partially Observable



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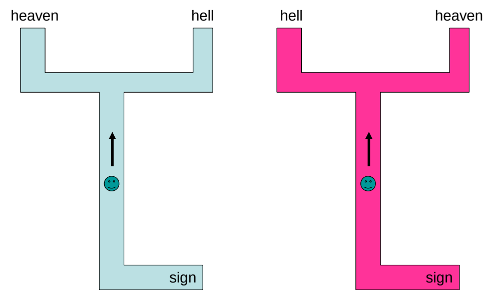
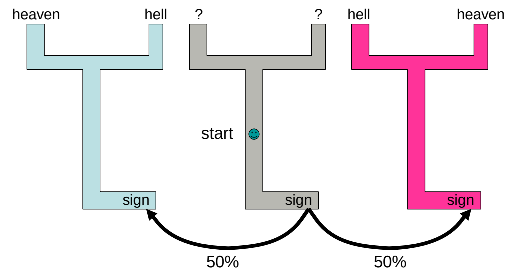
Stochastic, Partially Observable



Stochastic, Partially Observable



Notation (1)

- Recall the Bellman optimality equation:

$$V^*(s) = \max_{a \in A(s)} \sum_{s'} P^a_{ss'} \left[R^a_{ss'} + \gamma V^*(s') \right]$$

- Throughout this section we assume

$$R^a_{ss'} = \frac{1}{\gamma} R^a_s = \frac{1}{\gamma} r(s,a)$$

is independent of s' so that the Bellman optimality equation turns into

$$V^*(s) = \gamma \max_{a \in A(s)} \left[R^a_s + \sum_{s'} V^*(s') P^a_{ss'} \right] = \gamma \max_{a \in A(s)} \left[r(s,a) + \sum_{s'} V^*(s') P^a_{ss'} \right]$$

Notation (2)

- In the remainder we will use a slightly different notation for this equation:

$$V(x) \;=\; \gamma \max_u \left[r(x,u) + \int V(x')\, p(x' \mid u, x)\, dx' \right]$$

- According to the previously used notation we would write

$$V^*(s) = \gamma \max_{a \in A(s)} \left[r(s,a) + \sum_{s'} V^*(s') P^a_{ss'} \right]$$

- We replaced s by x and a by u, and turned the sum into an integral.

Value Iteration

- Given this notation the value iteration formula is

$$V_T(x) \;=\; \gamma \max_u \left[r(x,u) + \int V_{T-1}(x')\, p(x' \mid u, x)\, dx' \right]$$

with

$$V_1(b) \;=\; \gamma \max_u r(x,u)$$

POMDPs

- In POMDPs we apply the very same idea as in MDPs.
- **Since the state is not observable**, the agent has to **make its decisions based on** the belief state which is a **posterior distribution over states**.
- Let b be the belief of the agent about the state under consideration.
- POMDPs compute a **value function over belief spaces**:

$$V_T(b) \;=\; \gamma \max_u \left[r(b,u) + \int V_{T-1}(b') p(b' \mid u, b)\, db' \right]$$

Problems

- Each belief is a probability distribution, thus, **each value in a POMDP is a function of an entire probability distribution**.
- **This is problematic, since probability distributions are continuous**.
- Additionally, we have to deal with the **huge complexity of belief spaces**.
- For **finite worlds** with finite state, action, and measurement spaces and finite horizons, however, we can **effectively represent the value functions by piecewise linear functions**.

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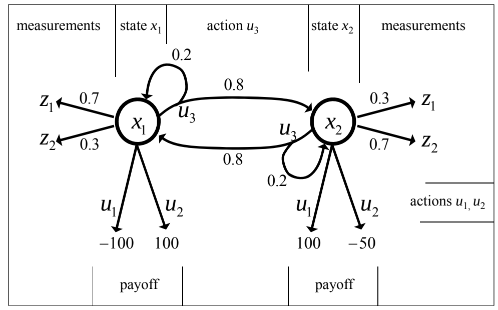
An Illustrative Example



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The Parameters of the Example

- The actions u_1 and u_2 are terminal actions.
- The action u_3 is a sensing action that potentially leads to a state transition.
- The horizon is finite and γ=1.

$$
\begin{aligned}
r(x_1,u_1) &= -100 & r(x_2,u_1) &= +100\\
r(x_1,u_2) &= +100 & r(x_2,u_2) &= -50\\
r(x_1,u_3) &= -1 & r(x_2,u_3) &= -1
\end{aligned}
$$

$$
\begin{aligned}
p(x_1'|x_1,u_3) &= 0.2 & p(x_2'|x_1,u_3) &= 0.8\\
p(x_1'|x_2,u_3) &= 0.8 & p(z_2'|x_2,u_3) &= 0.2
\end{aligned}
$$

$$
\begin{aligned}
p(z_1|x_1) &= 0.7 & p(z_2|x_1) &= 0.3\\
p(z_1|x_2) &= 0.3 & p(z_2|x_2) &= 0.7
\end{aligned}
$$

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Payoff in POMDPs

- In MDPs, the payoff (or return) depended on the state of the system.
- In POMDPs, however, the true state is not exactly known.
- Therefore, we compute the **expected payoff** by **integrating over all states**:

$$
\begin{aligned}
r(b,u) &= E_x[r(x,u)]\\
&= \int r(x,u)p(x)\ dx\\
&= p_1\ r(x_1,u) + p_2\ r(x_2,u)
\end{aligned}
$$

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Payoffs in Our Example (1)

- If we are totally certain that we are in state x_1 and execute action u_1, we receive a reward of -100
- If, on the other hand, we definitely know that we are in x_2 and execute u_1, the reward is +100.
- In between it is the linear combination of the extreme values weighted by their probabilities

$$
\begin{aligned}
r(b,u_1) &= -100\ p_1 + 100\ p_2\\
&= -100\ p_1 + 100\ (1-p_1)
\end{aligned}
$$

$$
r(b,u_2) = 100\ p_1 - 50\ (1-p_1)
$$

$$
r(b,u_3) = -1
$$

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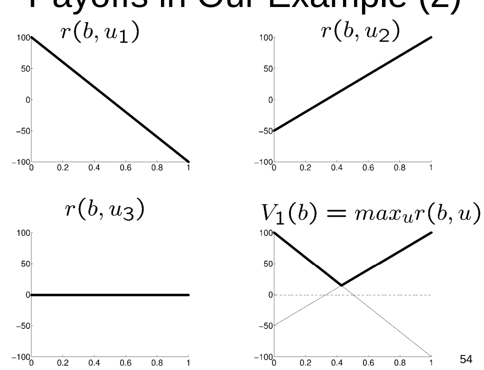
Payoffs in Our Example (2)



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The Resulting Policy for T=1

- Given we have a finite POMDP with T=1, we would use $V_1(b)$ to determine the optimal policy.
- In our example, the optimal policy for T=1 is

$$\pi_1(b) = \begin{cases} u_1 & \text{if } p_1 \le \frac{3}{7} \\ u_2 & \text{if } p_1 > \frac{3}{7} \end{cases}$$

- This is the upper thick graph in the diagram.

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Piecewise Linearity, Convexity

- The resulting value function $V_1(b)$ is the maximum of the three functions at each point

$$\begin{aligned} V_1(b) &= \max_u r(b,u) \\ &= \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ -1 & \end{array} \right\} \end{aligned}$$

- It is piecewise linear and convex.

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Pruning

- If we carefully consider $V_1(b)$, we see that only the first two components contribute.
- The third component can therefore safely be pruned away from $V_1(b)$.

$$V_1(b) = \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \end{array} \right\}$$

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Increasing the Time Horizon

- If we go over to a time horizon of T=2, the agent can also consider the sensing action u_3.
- Suppose we perceive z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.
- Thus $V_1(b \mid z_1)$ is given by

$$\begin{aligned} V_1(b \mid z_1) &= \max \left\{ \begin{array}{ll} -100 \cdot \frac{0.7\, p_1}{p(z_1)} & +100 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \\ 100 \cdot \frac{0.7\, p_1}{p(z_1)} & -50 \cdot \frac{0.3\,(1-p_1)}{p(z_1)} \end{array} \right\} \\ &= \frac{1}{p(z_1)} \max \left\{ \begin{array}{ll} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\} \end{aligned}$$

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Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$\begin{aligned} \bar{V}_1(b) &= E_z[V_1(b \mid z)] \\ &= \sum_{i=1}^{2} p(z_i)\, V_1(b \mid z_i) \\ &= \max \left\{ \begin{array}{ll} -70\, p_1 & +30\,(1-p_1) \\ 70\, p_1 & -15\,(1-p_1) \end{array} \right\} \\ &\quad + \max \left\{ \begin{array}{ll} -30\, p_1 & +70\,(1-p_1) \\ 30\, p_1 & -35\,(1-p_1) \end{array} \right\} \end{aligned}$$

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Resulting Value Function

- The four possible combinations yield the following function which again can be simplified and pruned.

$$\begin{aligned} \bar{V}_1(b) &= \max \left\{ \begin{array}{llll} -70\, p_1 & +30\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ -70\, p_1 & +30\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & -30\, p_1 & +70\,(1-p_1) \\ +70\, p_1 & -15\,(1-p_1) & +30\, p_1 & -35\,(1-p_1) \end{array} \right\} \\ &= \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ +40\, p_1 & +55\,(1-p_1) \\ +100\, p_1 & -50\,(1-p_1) \end{array} \right\} \end{aligned}$$

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State Transitions (Prediction)

- When the agent selects u_3 its state potentially changes.
- When computing the value function, we have to take these potential state changes into account.

$$\begin{aligned} p'_1 &= E_x[p(x_1 \mid x, u_3)] \\ &= \sum_{i=1}^{2} p(x_1 \mid x_i, u_3) p_i \\ &= 0.2 p_1 + 0.8(1 - p_1) \\ &= 0.8 - 0.6 p_1 \end{aligned}$$

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Resulting Value Function after executing u_3

- Taking also the state transitions into account, we finally obtain.

$$\bar{V}_1(b \mid u_3) = \max \left\{ \begin{array}{rr} 60\, p_1 & -60\,(1-p_1) \\ 52\, p_1 & +43\,(1-p_1) \\ -20\, p_1 & +70\,(1-p_1) \end{array} \right\}$$

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Value Function for T=2

- Taking into account that the agent can either directly perform u_1 or u_2, or first u_3 and then u_1 or u_2, we obtain (after pruning)

$$\bar{V}_2(b) = \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ 51\, p_1 & +42\,(1-p_1) \end{array} \right\}$$

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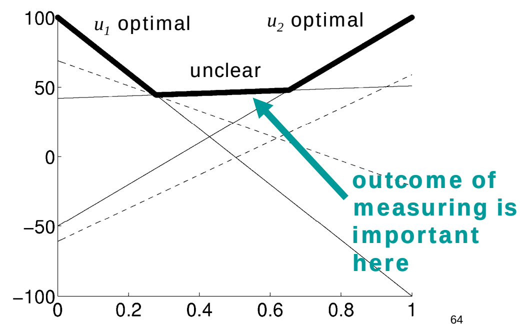
Graphical Representation of $V_2(b)$



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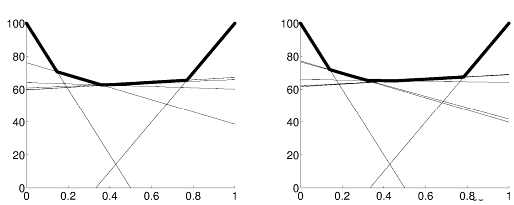
Deep Horizons and Pruning

- We have now completed a full backup in belief space.
- This process can be applied recursively.
- The value functions for T=10 and T=20 are



Why Pruning is Essential

- Each **update introduces additional linear components** to *V*.
- Each **measurement squares the number of linear components**.
- Thus, an unpruned value function for T=20 includes more than $10^{547,864}$ linear functions.
- At T=30 we have $10^{561,012,337}$ linear functions.
- The pruned value functions at T=20, in comparison, contains only 12 linear components.
- The combinatorial explosion of linear components in the value function are the major reason why **POMDPs are impractical for most applications**.

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A Summary on POMDPs

- POMDPs compute the optimal action in partially observable, stochastic domains.
- For finite horizon problems, the resulting value functions are piecewise linear and convex.
- In each iteration the number of linear constraints grows exponentially.
- POMDPs so far have only been applied successfully to very small state spaces with small numbers of possible observations and actions.

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