INFERENCE IN BAYESIAN NETWORKS

CHAPTER 14.4–5

Outline

◇ Exact inference by enumeration

◇ Exact inference by variable elimination

◇ Approximate inference by stochastic simulation

◇ Approximate inference by Markov chain Monte Carlo

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(outcome|action, evidence)$

Value of information: which evidence to seek next?

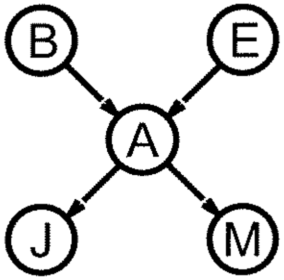
Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:



$\mathbf{P}(B|j,m)$
$= \mathbf{P}(B,j,m)/P(j,m)$
$= \alpha\mathbf{P}(B,j,m)$
$= \alpha\,\Sigma_e\,\Sigma_a\,\mathbf{P}(B,e,a,j,m)$

Rewrite full joint entries using product of CPT entries:
$\mathbf{P}(B|j,m)$
$= \alpha\,\Sigma_e\,\Sigma_a\,\mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$
$= \alpha\mathbf{P}(B)\,\Sigma_e\,P(e)\,\Sigma_a\,\mathbf{P}(a|B,e)P(j|a)P(m|a)$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

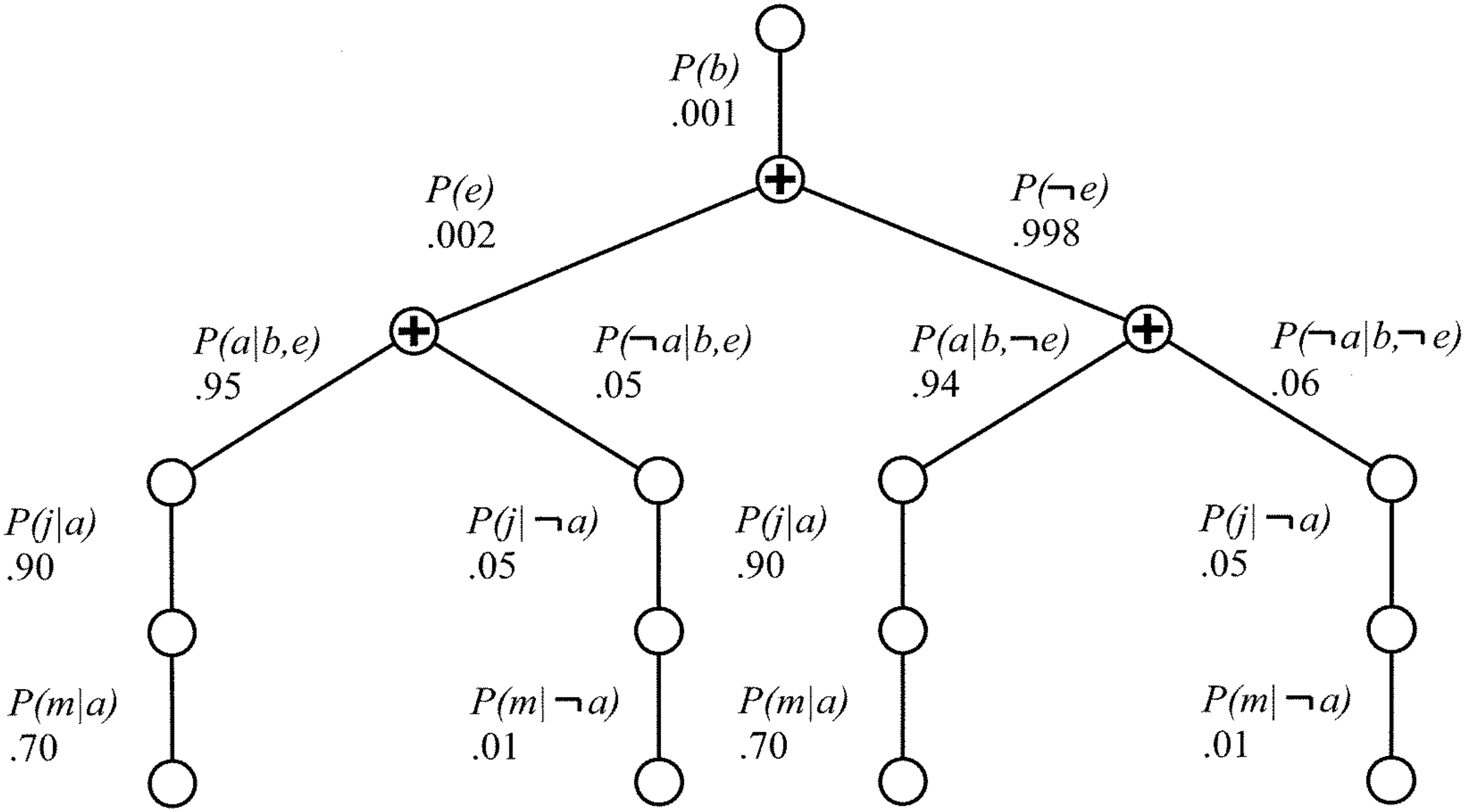
Enumeration algorithm

```
function ENUMERATION-ASK($X$, e, $bn$) returns a distribution over $X$
    inputs: $X$, the query variable
            e, observed values for variables $\mathbf{E}$
            $bn$, a Bayesian network with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$

    $\mathbf{Q}(X) \leftarrow$ a distribution over $X$, initially empty
    for each value $x_i$ of $X$ do
        extend e with value $x_i$ for $X$
        $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(VARS[$bn$], e)
    return NORMALIZE($\mathbf{Q}(X)$)
```

```
function ENUMERATE-ALL($vars$, e) returns a real number
    if EMPTY?($vars$) then return 1.0
    $Y \leftarrow$ FIRST($vars$)
    if $Y$ has value $y$ in e
        then return $P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), e)
        else return $\Sigma_y \; P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathrm{e}_y$)
            where $\mathrm{e}_y$ is e extended with $Y = y$
```

Evaluation tree



Enumeration is inefficient: repeated computation
e.g., computes $P(j|a)P(m|a)$ for each value of e

Inference by variable elimination

Variable elimination: carry out summations right-to-left, storing intermediate results (factors) to avoid recomputation

$$
\begin{aligned}
\mathbf{P}(B|j,m) & \\
&= \alpha \underbrace{\mathbf{P}(B)}_{B} \textstyle\sum_e \underbrace{P(e)}_{E} \textstyle\sum_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M} \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) P(j|a) f_M(a) \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) f_J(a) f_M(a) \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a f_A(a,b,e) f_J(a) f_M(a) \\
&= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A\text{)} \\
&= \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E\text{)} \\
&= \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}
$$

Variable elimination: Basic operations

Summing out a variable from a product of factors:

move any constant factors outside the summation

add up submatrices in pointwise product of remaining factors

$$\Sigma_x f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \, \Sigma_x \, f_{i+1} \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f_{\bar{X}}$$

assuming $f_1, \ldots, f_i$ do not depend on X

Pointwise product of factors f_1 and f_2:

$$f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) \times f_2(y_1, \ldots, y_k, z_1, \ldots, z_l)$$
$$= f(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_l)$$

E.g., $f_1(a, b) \times f_2(b, c) = f(a, b, c)$

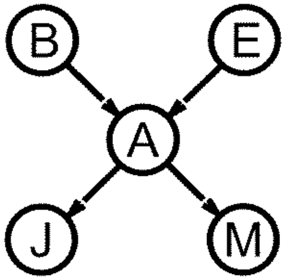
Variable elimination algorithm

```
function ELIMINATION-ASK($X$, e, $bn$) returns a distribution over $X$
    inputs: $X$, the query variable
            e, evidence specified as an event
            $bn$, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

    $factors \leftarrow [\,]$; $vars \leftarrow$ REVERSE(VARS[$bn$])
    for each $var$ in $vars$ do
        $factors \leftarrow$ [MAKE-FACTOR($var$, e)|$factors$]
        if $var$ is a hidden variable then $factors \leftarrow$ SUM-OUT($var$, $factors$)
    return NORMALIZE(POINTWISE-PRODUCT($factors$))
```

Irrelevant variables

Consider the query $P(JohnCalls|Burglary = true)$



$$P(J|b) = \alpha P(b) \sum_{e} P(e) \sum_{a} P(a|b,e) P(J|a) \sum_{m} P(m|a)$$

Sum over m is identically 1; M is **irrelevant** to the query

Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
$Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$
so $MaryCalls$ is irrelevant

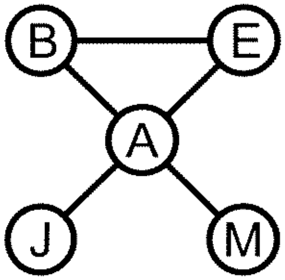
(Compare this to backward chaining from the query in Horn clause KBs)

Irrelevant variables contd.

Defn: moral graph of Bayes net: marry all parents and drop arrows

Defn: $\mathbf{A}$ is m-separated from $\mathbf{B}$ by $\mathbf{C}$ iff separated by $\mathbf{C}$ in the moral graph

Thm 2: Y is irrelevant if m-separated from X by $\mathbf{E}$



For $P(JohnCalls|Alarm = true)$, both $Burglary$ and $Earthquake$ are irrelevant

Complexity of exact inference

Singly connected networks (or polytrees):

- any two nodes are connected by at most one (undirected) path
- time and space cost of variable elimination are $O(d^k n)$

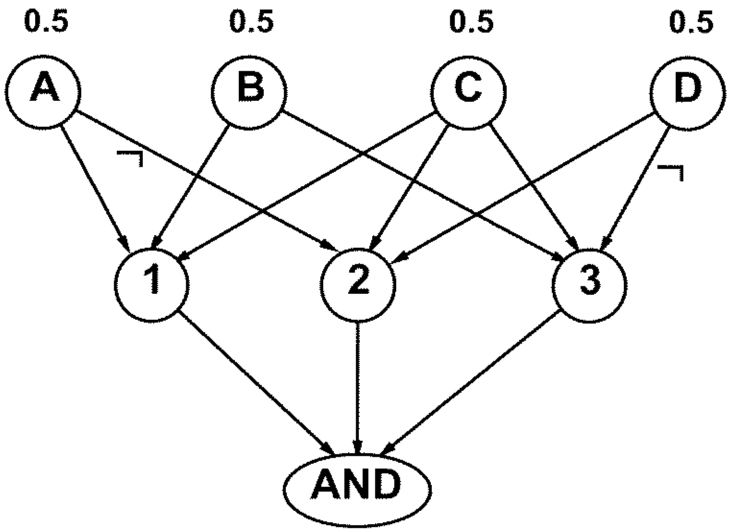
Multiply connected networks:

- can reduce 3SAT to exact inference $\Rightarrow$ NP-hard
- equivalent to counting 3SAT models $\Rightarrow$ #P-complete

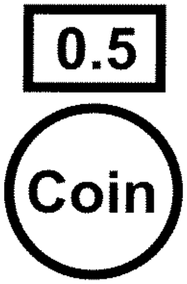
1. A v B v C

2. C v D v ¬A

3. B v C v ¬D



Inference by stochastic simulation



Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P

Outline:

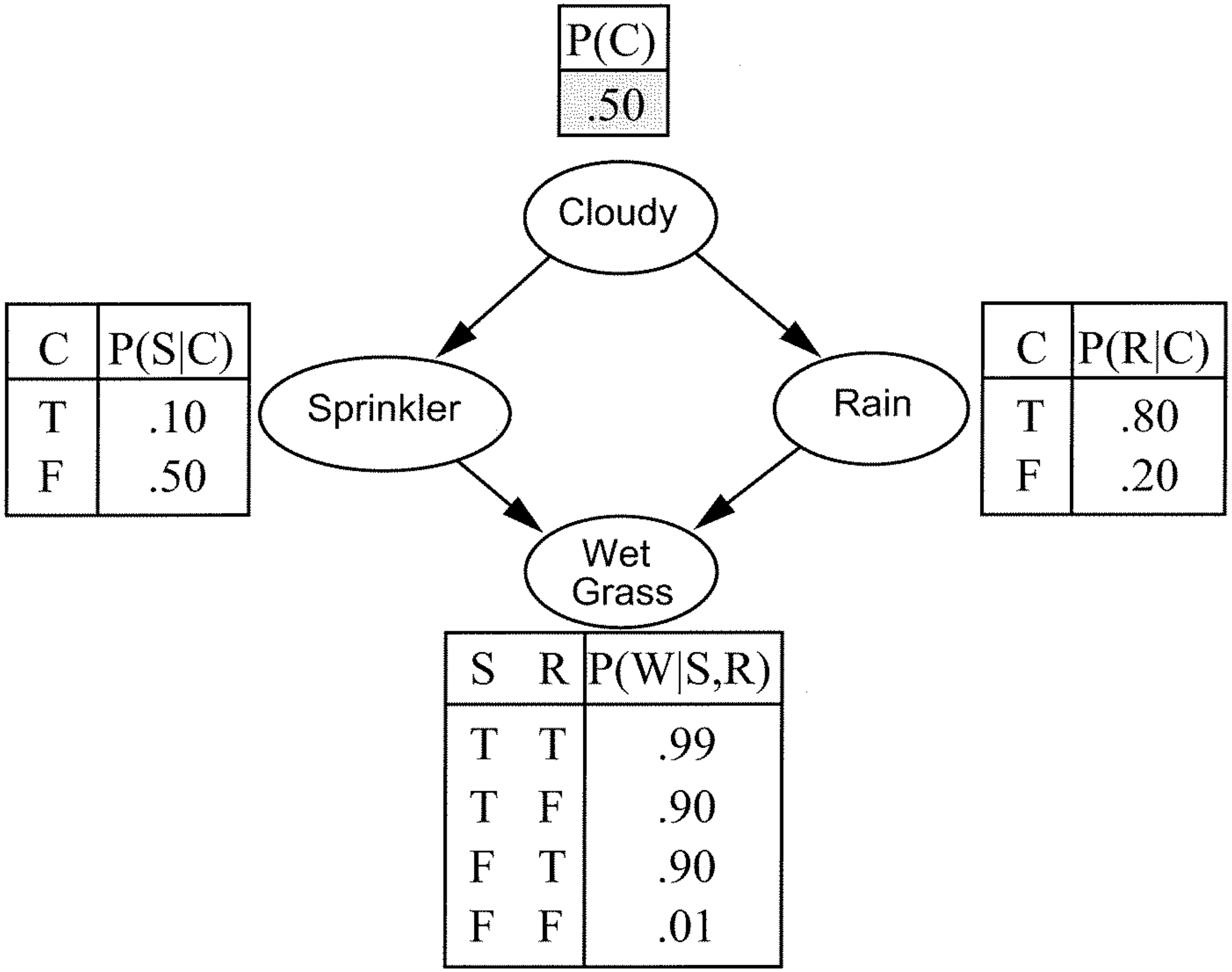
- Sampling from an empty network
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples
- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Sampling from an empty network

```
function PRIOR-SAMPLE($bn$) returns an event sampled from $bn$
   inputs: $bn$, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

   $\mathbf{x} \leftarrow$ an event with $n$ elements
   for $i = 1$ to $n$ do
        $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
             given the values of $Parents(X_i)$ in $\mathbf{x}$
   return $\mathbf{x}$
```

Example



P(C)
.50

C	P(S|C)
T	.10
F	.50

C	P(R|C)
T	.80
F	.20

S	R	P(W|S,R)
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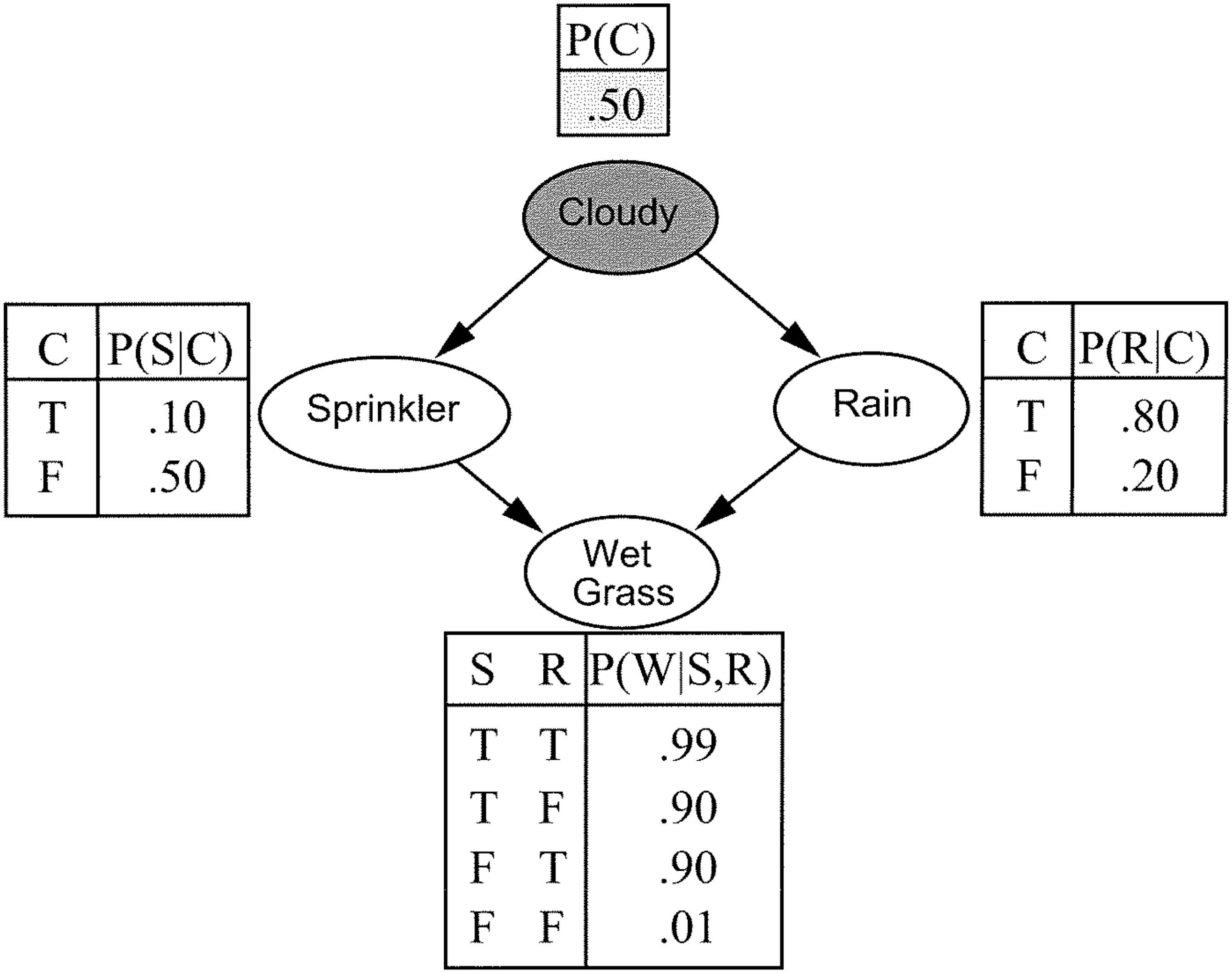
Example



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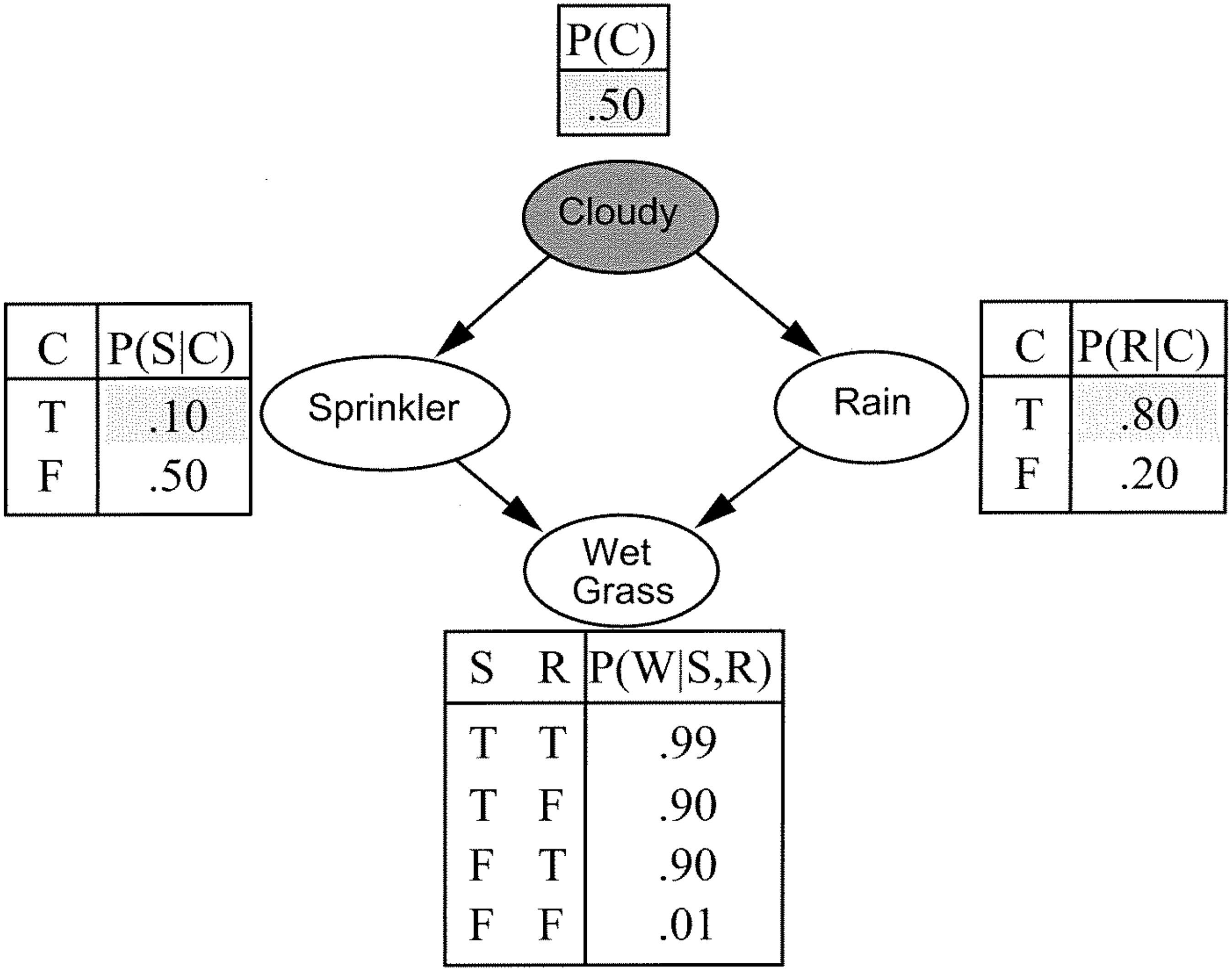

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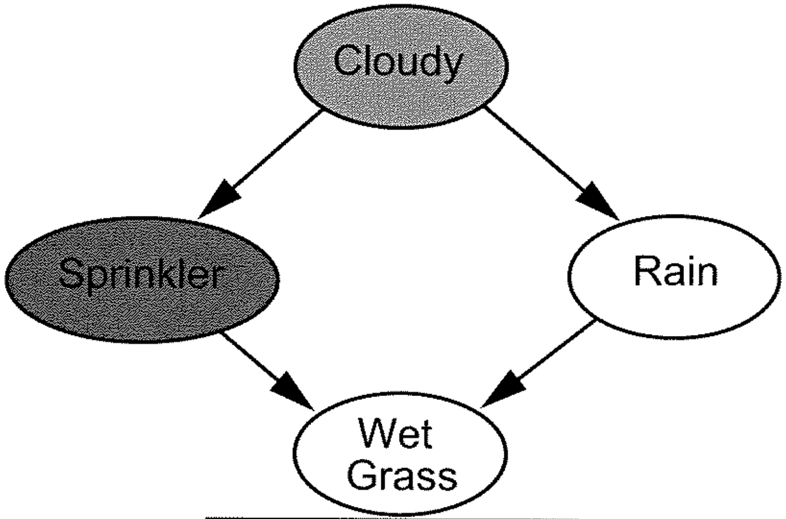

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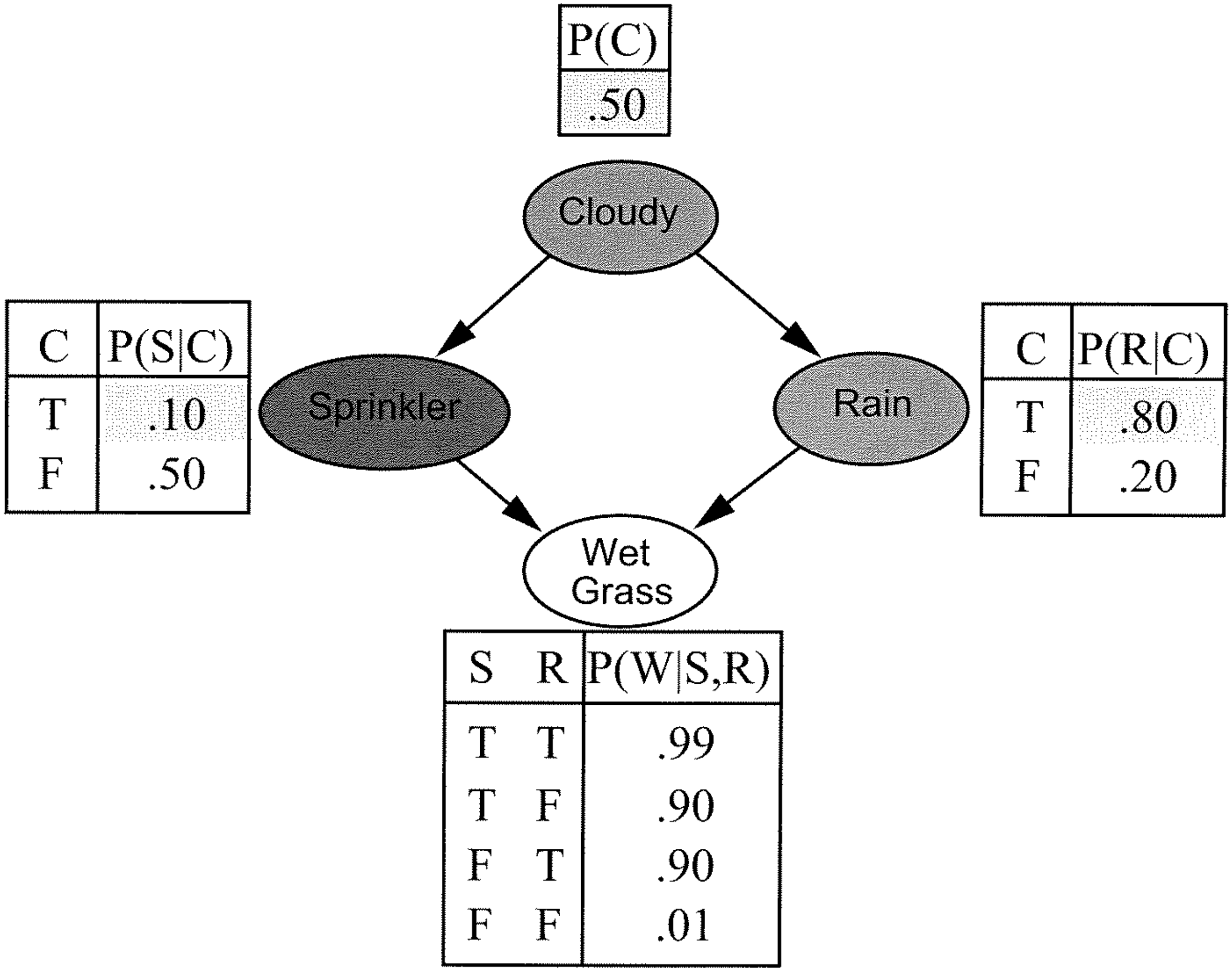

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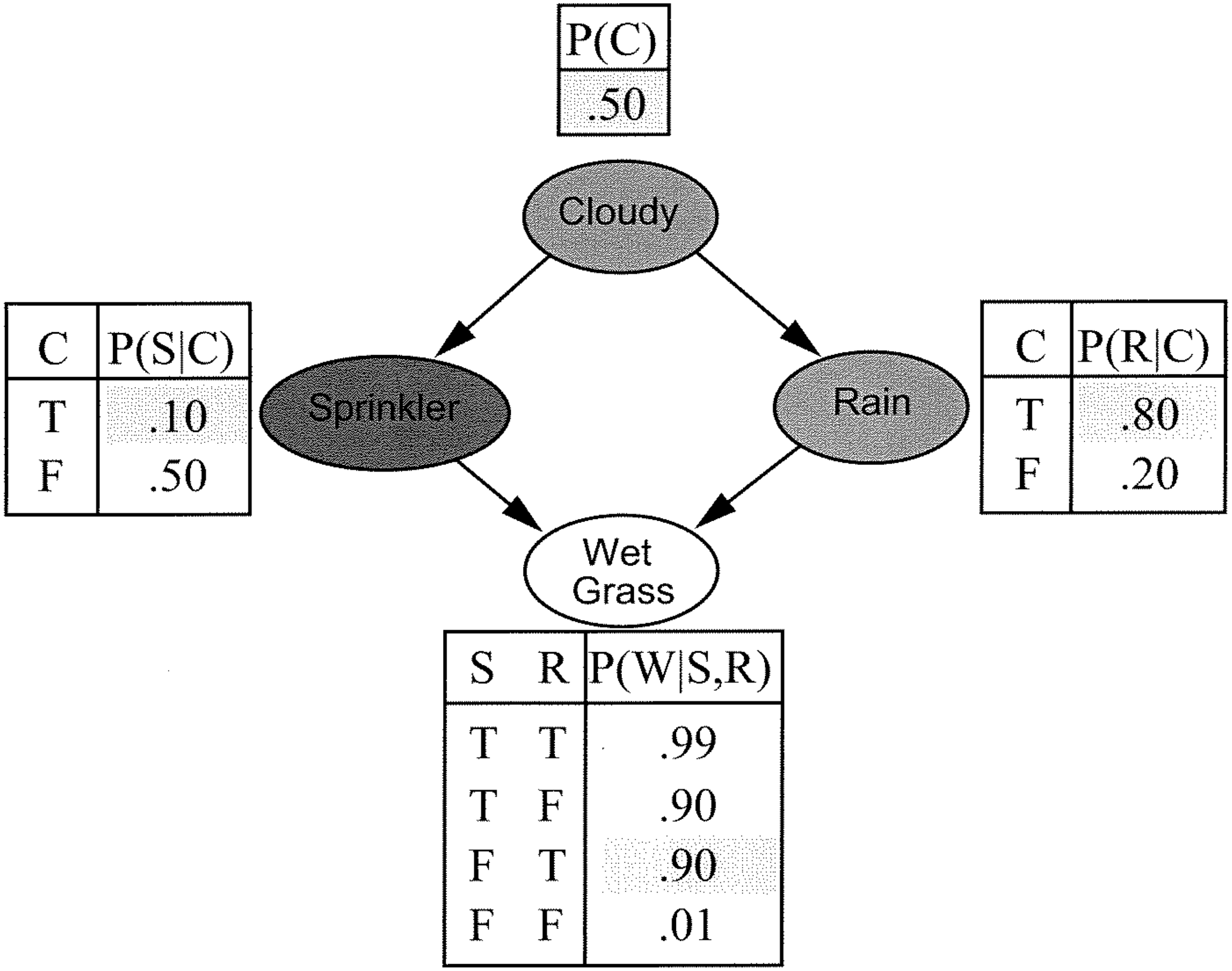

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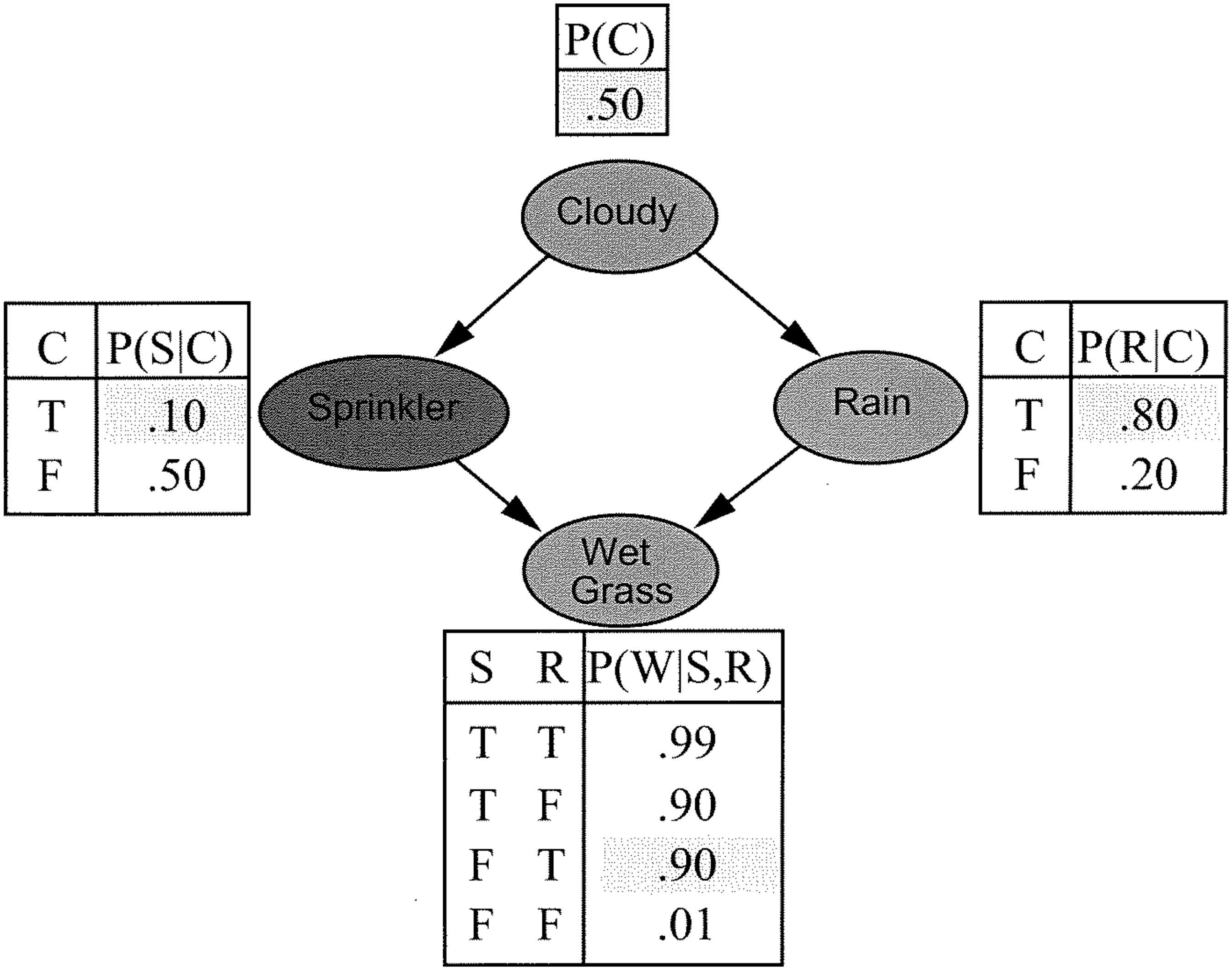

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Sampling from an empty network contd.

Probability that PRIORSAMPLE generates a particular event

$$S_{PS}(x_1 \ldots x_n) = \Pi_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N \to \infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N \to \infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PRIORSAMPLE are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function REJECTION-SAMPLING($X$, $\mathbf{e}$, $bn$, $N$) returns an estimate of $P(X|\mathbf{e})$
    local variables: $\mathbf{N}$, a vector of counts over $X$, initially zero

    for $j = 1$ to $N$ do
        $\mathbf{x} \leftarrow$ PRIOR-SAMPLE($bn$)
        if $\mathbf{x}$ is consistent with $\mathbf{e}$ then
            $\mathbf{N}[x] \leftarrow \mathbf{N}[x]+1$ where $x$ is the value of $X$ in $\mathbf{x}$
    return NORMALIZE($\mathbf{N}[X]$)
```

E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples
  27 samples have $Sprinkler = true$
    Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19 \rangle) = \langle 0.296, 0.704 \rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$\hat{\mathbf{P}}(X|\mathrm{e}) = \alpha \mathbf{N}_{PS}(X, \mathrm{e})$ (algorithm defn.)

$= \mathbf{N}_{PS}(X, \mathrm{e})/N_{PS}(\mathrm{e})$ (normalized by $N_{PS}(\mathrm{e})$)

$\approx \mathbf{P}(X, \mathrm{e})/P(\mathrm{e})$ (property of PRIORSAMPLE)

$= \mathbf{P}(X|\mathrm{e})$ (defn. of conditional probability)

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathrm{e})$ is small

$P(\mathrm{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting

Idea: fix evidence variables, sample only nonevidence variables,
and weight each sample by the likelihood it accords the evidence

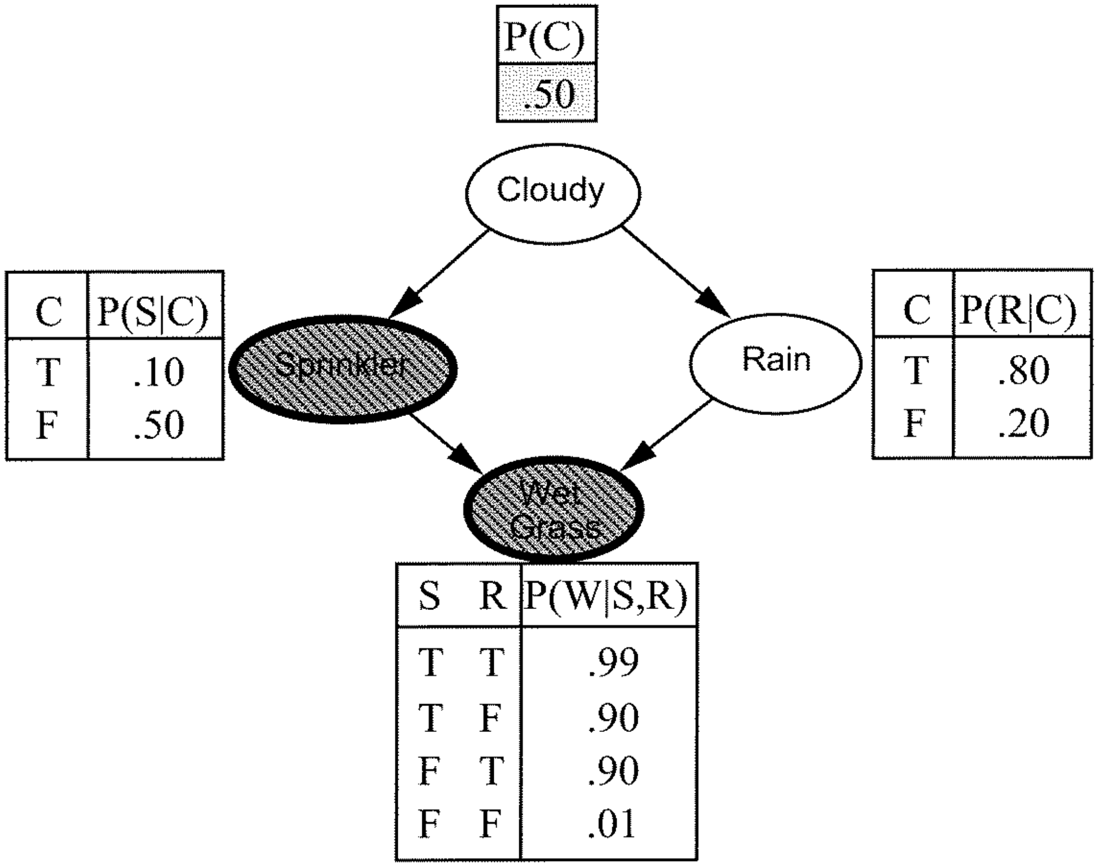
```
function LIKELIHOOD-WEIGHTING($X$, e, $bn$, $N$) returns an estimate of $P(X|\mathbf{e})$
    local variables: $\mathbf{W}$, a vector of weighted counts over $X$, initially zero

    for $j = 1$ to $N$ do
        x, $w \leftarrow$ WEIGHTED-SAMPLE($bn$)
        $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where $x$ is the value of $X$ in x
    return NORMALIZE($\mathbf{W}[X]$)
```

```
function WEIGHTED-SAMPLE($bn$, e) returns an event and a weight

    x $\leftarrow$ an event with $n$ elements; $w \leftarrow 1$
    for $i = 1$ to $n$ do
        if $X_i$ has a value $x_i$ in e
            then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$
            else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
    return x, $w$
```

Likelihood weighting example



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$w = 1.0$

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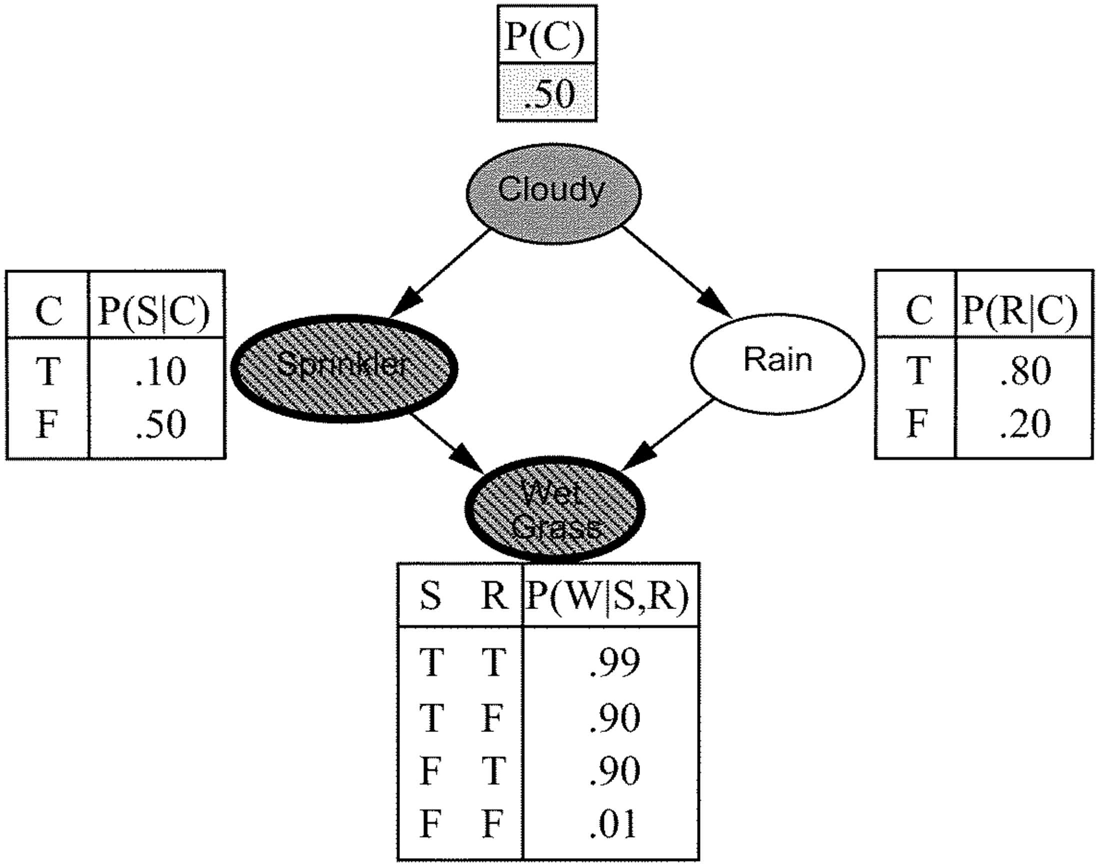

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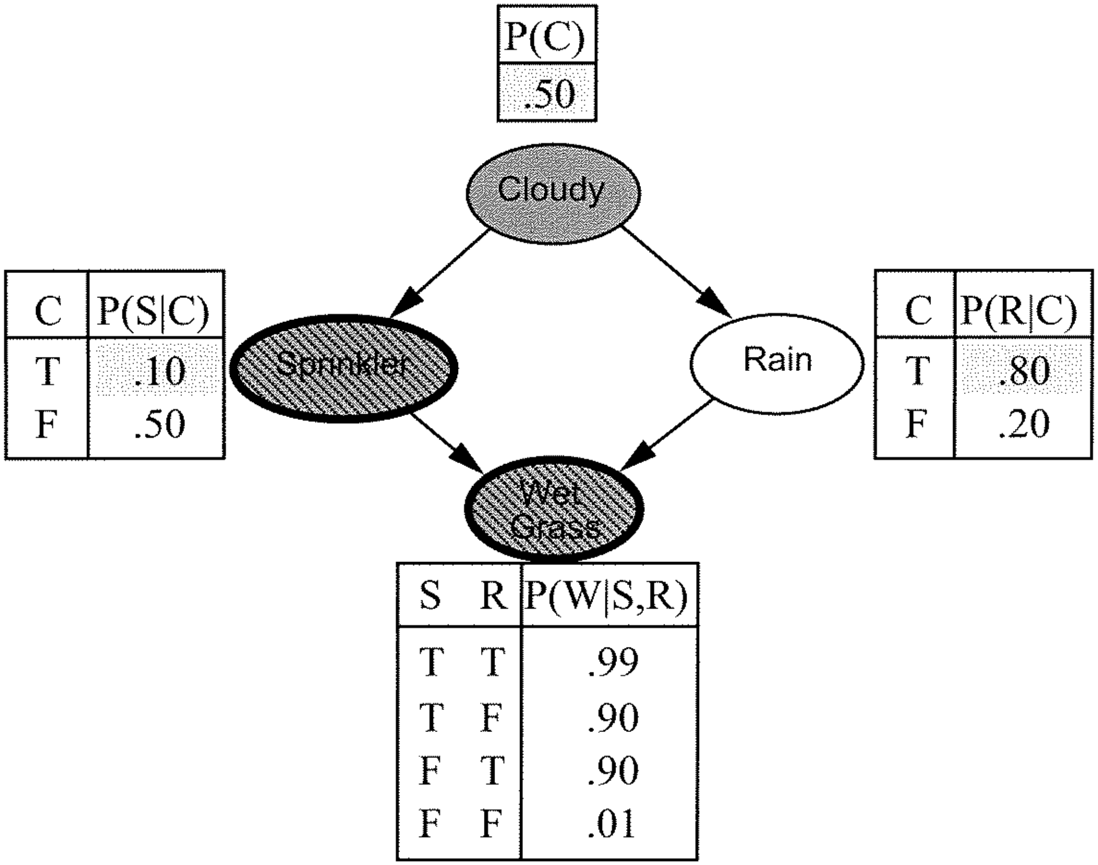

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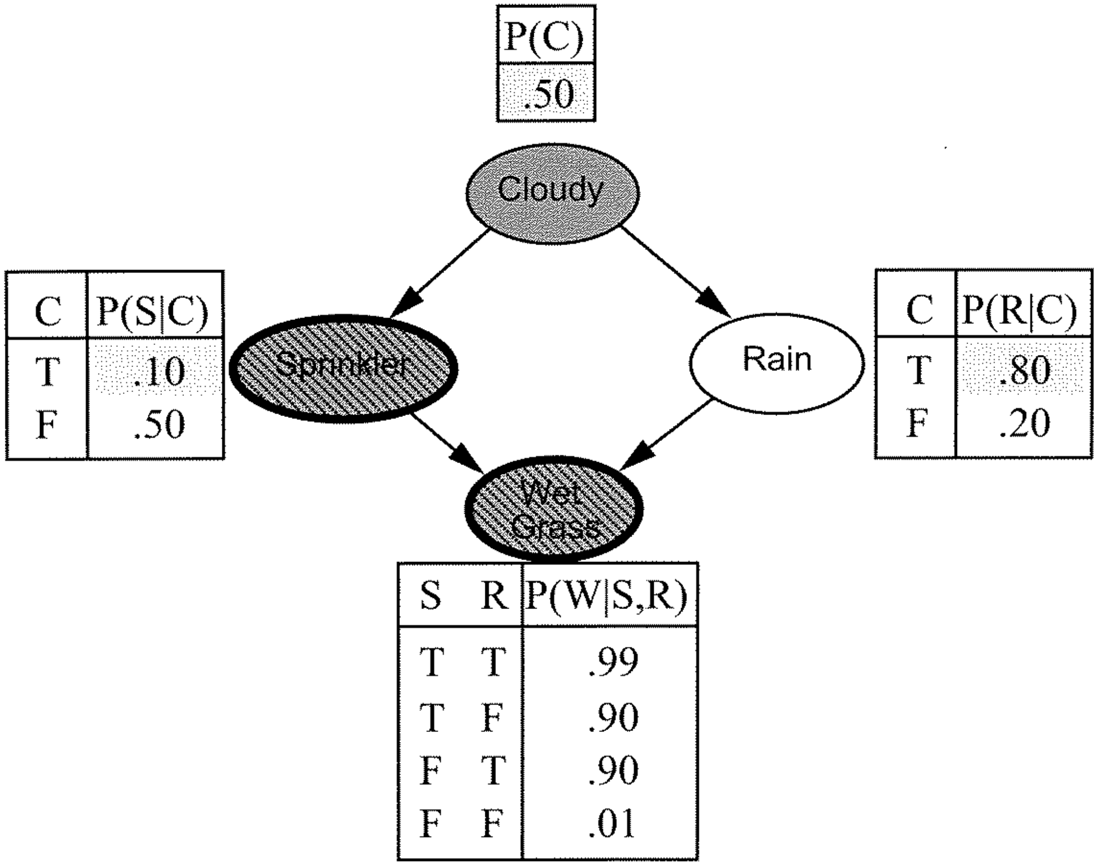

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$w = 1.0 \times 0.1$

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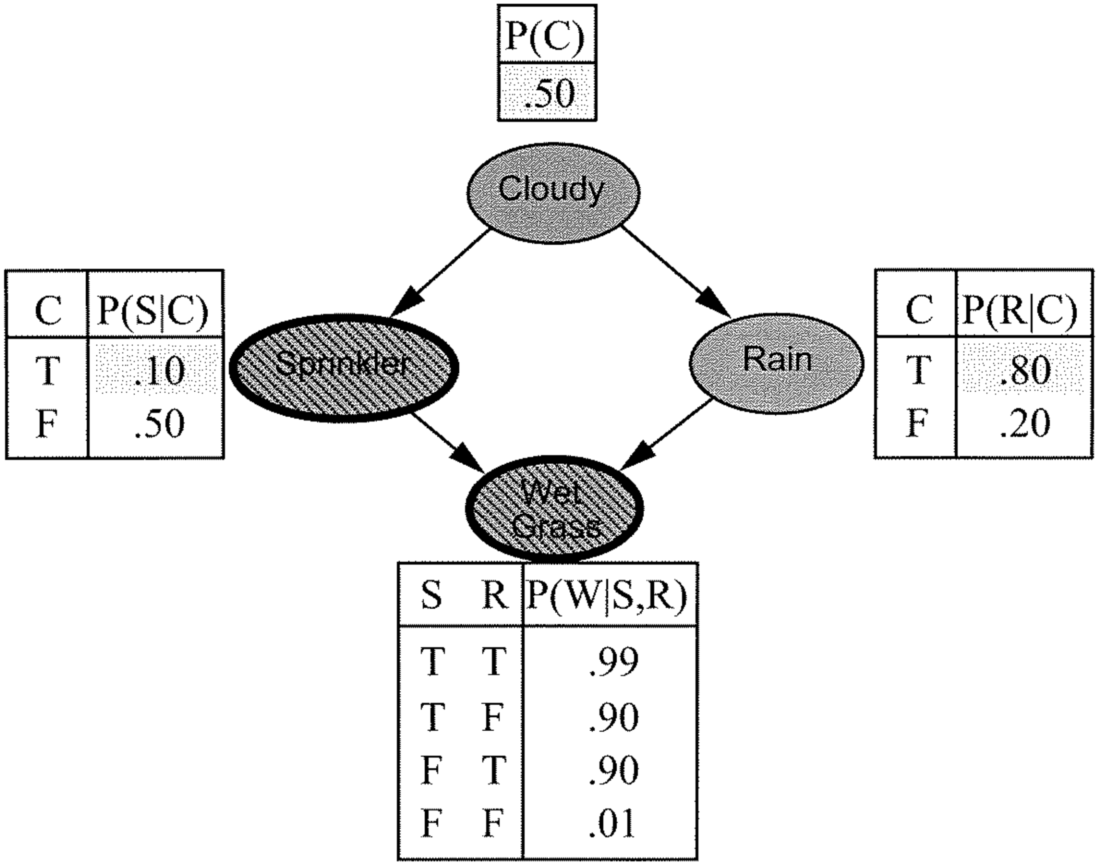

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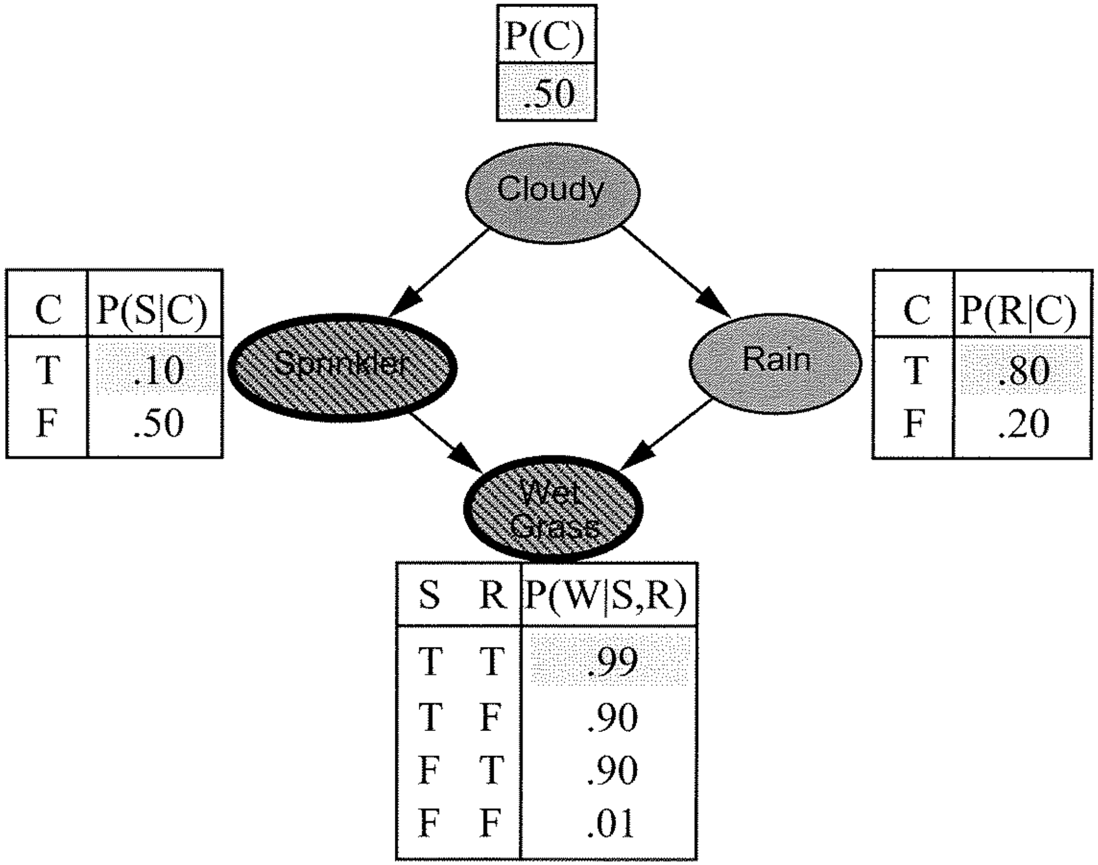

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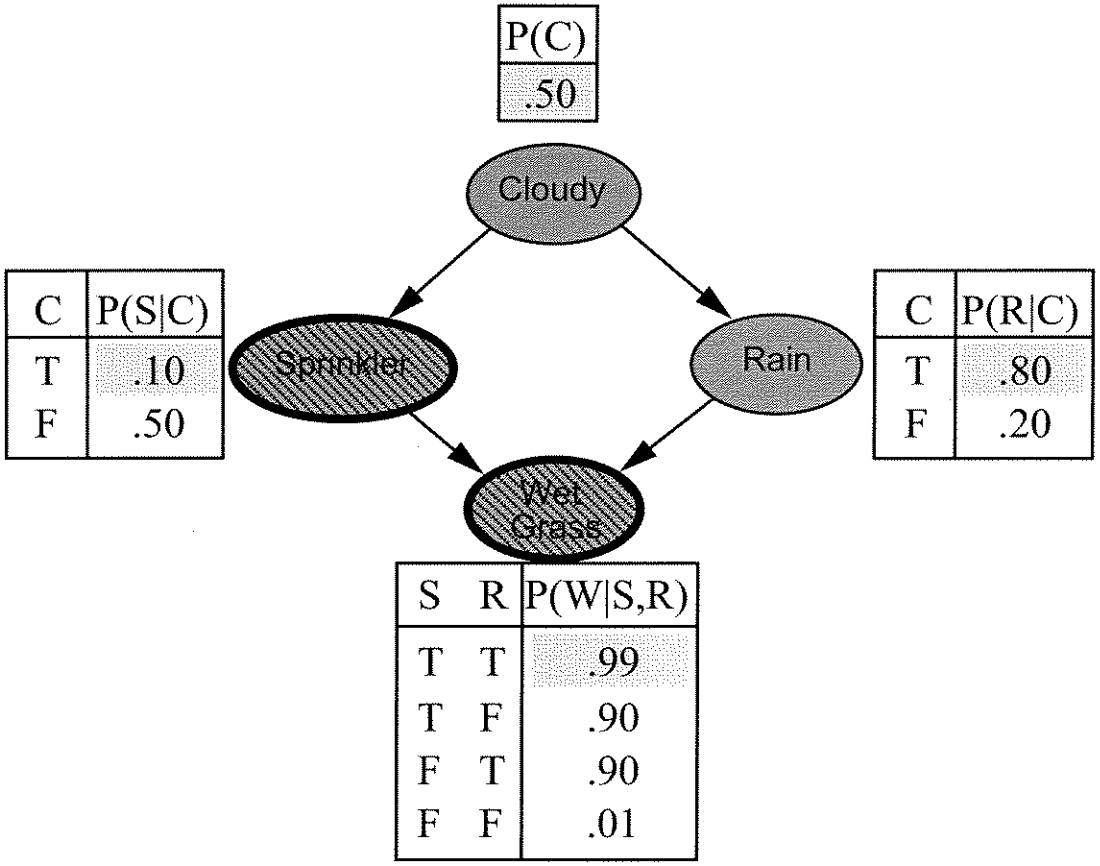
Likelihood weighting example



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$w = 1.0 \times 0.1 \times 0.99 = 0.099$

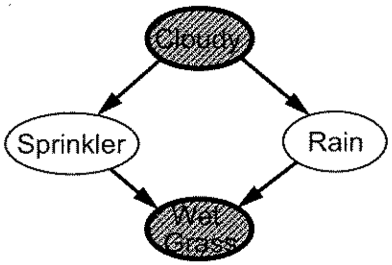
Likelihood weighting analysis

Sampling probability for WEIGHTEDSAMPLE is

$$S_{WS}(\mathbf{z},\mathbf{e}) = \Pi_{i=1}^{l} P(z_i|parents(Z_i))$$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z},\mathbf{e}$ is

$$w(\mathbf{z},\mathbf{e}) = \Pi_{i=1}^{m} P(e_i|parents(E_i))$$

Weighted sampling probability is

$$\begin{aligned} & S_{WS}(\mathbf{z},\mathbf{e})w(\mathbf{z},\mathbf{e}) \\ & = \Pi_{i=1}^{l} P(z_i|parents(Z_i)) \ \ \Pi_{i=1}^{m} P(e_i|parents(E_i)) \\ & = P(\mathbf{z},\mathbf{e}) \text{ (by standard global semantics of network)} \end{aligned}$$

Hence likelihood weighting returns consistent estimates but performance still degrades with many evidence variables because a few samples have nearly all the total weight

Approximate inference using MCMC

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

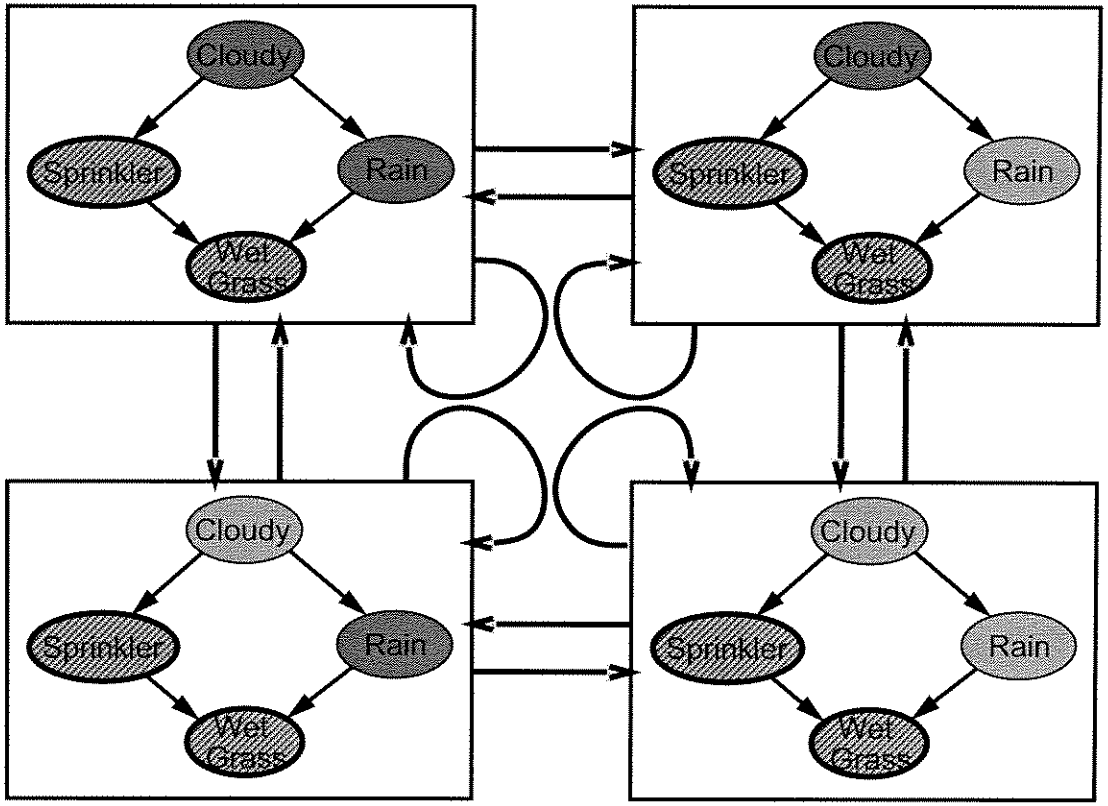
```
function MCMC-Ask($X$, $\mathbf{e}$, $bn$, $N$) returns an estimate of $P(X|\mathbf{e})$
    local variables: $\mathbf{N}[X]$, a vector of counts over $X$, initially zero
                     $\mathbf{Z}$, the nonevidence variables in $bn$
                     $\mathbf{x}$, the current state of the network, initially copied from $\mathbf{e}$

    initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Y}$
    for $j = 1$ to $N$ do
        for each $Z_i$ in $\mathbf{Z}$ do
            sample the value of $Z_i$ in $\mathbf{x}$ from $\mathbf{P}(Z_i|mb(Z_i))$
                given the values of $MB(Z_i)$ in $\mathbf{x}$
            $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where $x$ is the value of $X$ in $\mathbf{x}$
    return Normalize($\mathbf{N}[X]$)
```

Can also choose a variable to sample at random each time

The Markov chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

MCMC example contd.

Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

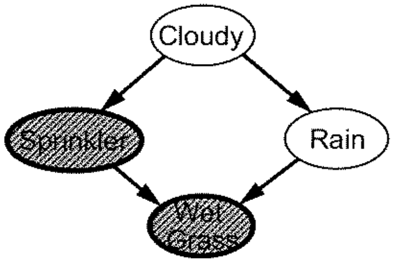
Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

E.g., visit 100 states
    31 have $Rain = true$, 69 have $Rain = false$

$\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$
    $= \textsc{Normalize}(\langle 31, 69\rangle) = \langle 0.31, 0.69\rangle$

Theorem: chain approaches stationary distribution:
    long-run fraction of time spent in each state is exactly
    proportional to its posterior probability

Markov blanket sampling



Markov blanket of $Cloudy$ is
    $Sprinkler$ and $Rain$
Markov blanket of $Rain$ is
    $Cloudy$, $Sprinkler$, and $WetGrass$

Probability given the Markov blanket is calculated as follows:

$$P(x_i'|mb(X_i)) = P(x_i'|parents(X_i))\Pi_{Z_j \in Children(X_i)} P(z_j|parents(Z_j))$$

Easily implemented in message-passing parallel systems, brains

Main computational problems:

1) Difficult to tell if convergence has been achieved
2) Can be wasteful if Markov blanket is large:
    $P(X_i|mb(X_i))$ won't change much (law of large numbers)

Summary

Exact inference by variable elimination:
- polytime on polytrees, NP-hard on general graphs
- space = time, very sensitive to topology

Approximate inference by LW, MCMC:
- LW does poorly when there is lots of (downstream) evidence
- LW, MCMC generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables