

# Introduction to Artificial Intelligence

## Uncertainty

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### Chapter 14

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# Outline

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- ◊ Uncertainty
- ◊ Probability
- ◊ Syntax
- ◊ Semantics
- ◊ Inference rules

# Uncertainty

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Let action  $A_t$  = leave for airport  $t$  minutes before flight  
Will  $A_t$  get me there on time?

Problems:

- 1) partial observability (road state, other drivers' plans, etc.)
- 2) noisy sensors (KUOW traffic reports)
- 3) uncertainty in action outcomes (flat tire, etc.)
- 4) immense complexity of modelling and predicting traffic

Hence a purely logical approach either

- 1) risks falsehood: " $A_{25}$  will get me there on time"
- or 2) leads to conclusions that are too weak for decision making:  
    " $A_{25}$  will get me there on time if there's no accident on the bridge  
    and it doesn't rain and my tires remain intact etc etc."

( $A_{1440}$  might reasonably be said to get me there on time  
but I'd have to stay overnight in the airport . . .)

# Methods for handling uncertainty

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**Default or nonmonotonic** logic:

Assume my car does not have a flat tire

Assume  $A_{25}$  works unless contradicted by evidence

Issues: What assumptions are reasonable? How to handle contradiction?

**Rules with fudge factors:**

$A_{25} \mapsto_{0.3} \text{get there on time}$

$\text{Sprinkler} \mapsto_{0.99} \text{WetGrass}$

$\text{WetGrass} \mapsto_{0.7} \text{Rain}$

Issues: Problems with combination, e.g.,  $\text{Sprinkler}$  causes  $\text{Rain}??$

**Probability**

Given the available evidence,

$A_{25}$  will get me there on time with probability 0.04

Mahaviracarya (9th C.), Cardano (1565) theory of gambling

(**Fuzzy logic** handles *degree of truth* NOT uncertainty e.g.,

$\text{WetGrass}$  is true to degree 0.2)

# Probability

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Probabilistic assertions summarize effects of

**laziness**: failure to enumerate exceptions, qualifications, etc.

**ignorance**: lack of relevant facts, initial conditions, etc.

**Subjective** or **Bayesian** probability:

Probabilities relate propositions to one's own state of knowledge

e.g.,  $P(A_{25}|\text{no reported accidents}) = 0.06$

These are **not** assertions about the world

Probabilities of propositions change with new evidence:

e.g.,  $P(A_{25}|\text{no reported accidents, 5 a.m.}) = 0.15$

(Analogous to logical entailment status  $KB \models \alpha$ , not truth.)

# Making decisions under uncertainty

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Suppose I believe the following:

$$\begin{aligned} P(A_{25} \text{ gets me there on time} | \dots) &= 0.04 \\ P(A_{90} \text{ gets me there on time} | \dots) &= 0.70 \\ P(A_{120} \text{ gets me there on time} | \dots) &= 0.95 \\ P(A_{1440} \text{ gets me there on time} | \dots) &= 0.9999 \end{aligned}$$

Which action to choose?

# Making decisions under uncertainty

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Which action to choose?

Depends on my **preferences** for missing flight vs. airport cuisine, etc.

**Utility theory** is used to represent and infer preferences

**Decision theory** = utility theory + probability theory

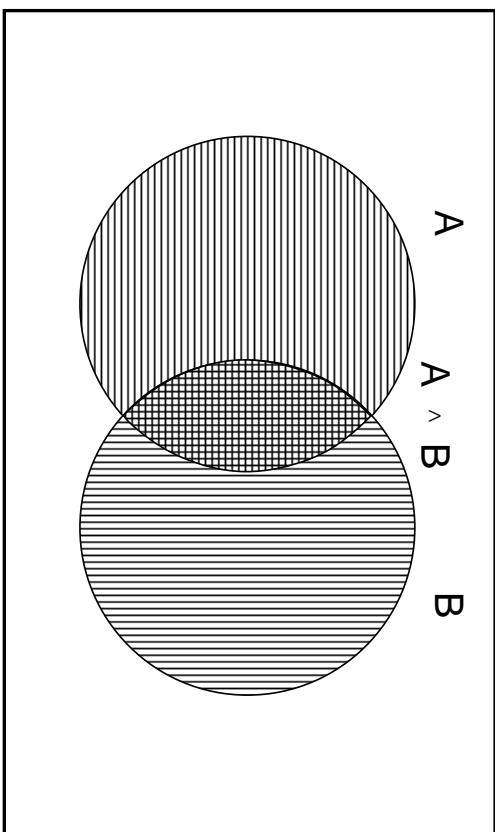
# Axioms of probability

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For any propositions  $A, B$

1.  $0 \leq P(A) \leq 1$
2.  $P(\text{True}) = 1$  and  $P(\text{False}) = 0$
3.  $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

True



de Finetti (1931): an agent who bets according to probabilities that violate these axioms can be forced to bet so as to lose money regardless of outcome.

# Syntax

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Similar to propositional logic: possible worlds defined by assignment of values to **random variables**.

**Propositional** or **Boolean** random variables

e.g., *Cavity* (do I have a cavity?)

Include propositional logic expressions

e.g.,  $\neg \text{Burglary} \vee \text{Earthquake}$

**Multivalued** random variables

e.g., *Weather* is one of  $\langle \text{sunny}, \text{rain}, \text{cloudy}, \text{snow} \rangle$

Values must be exhaustive and mutually exclusive

Proposition constructed by assignment of a value:

e.g.,  $\text{Weather} = \text{sunny}$ ; also  $\text{Cavity} = \text{true}$  for clarity

## Syntax contd.

**Prior or unconditional probabilities** of propositions

e.g.,  $P(Cavity) = 0.1$  and  $P(Weather = sunny) = 0.72$  correspond to belief prior to arrival of any (new) evidence

**Probability distribution** gives values for all possible assignments:

$$P(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle \text{ (normalized, i.e., sums to 1)}$$

**Joint probability distribution** for a set of variables gives values for each possible assignment to all the variables

$P(Weather, Cavity) =$  a  $4 \times 2$  matrix of values:

$Weather =$	<i>sunny</i>	<i>rain</i>	<i>cloudy</i>	<i>snow</i>
$Cavity = true$				
$Cavity = false$				

## Syntax contd.

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### Conditional or posterior probabilities

e.g.,  $P(Cavity|Toothache) = 0.8$   
i.e., **given that *Toothache* is all I know**

Notation for conditional distributions:

$P(Weather|Earthquake) = 2\text{-element vector of } 4\text{-element vectors}$

If we know more, e.g., *Cavity* is also given, then we have

$$P(Cavity|Toothache, Cavity) = 1$$

Note: the less specific belief *remains valid* after more evidence arrives, but is not always *useful*

New evidence may be irrelevant, allowing simplification, e.g.,

$$P(Cavity|Toothache, 49ersWin) = P(Cavity|Toothache) = 0.8$$

This kind of inference, sanctioned by domain knowledge, is crucial

# Conditional probability

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Definition of conditional probability:

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \text{ if } P(B) \neq 0$$

**Product rule** gives an alternative formulation:

$$P(A \wedge B) = P(A|B)P(B) = P(B|A)P(A)$$

A general version holds for whole distributions, e.g.,

$$\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather|Cavity)\mathbf{P}(Cavity)$$

(View as a  $4 \times 2$  set of equations, *not* matrix mult.)

**Chain rule** is derived by successive application of product rule:

$$\begin{aligned} \mathbf{P}(X_1, \dots, X_n) &= \mathbf{P}(X_1, \dots, X_{n-1}) \mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \mathbf{P}(X_1, \dots, X_{n-2}) \mathbf{P}(X_{n-1}|X_1, \dots, X_{n-2}) \mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \ddots \\ &= \prod_{i=1}^n \mathbf{P}(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$

# Bayes' Rule

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Product rule  $P(A \wedge B) = P(A|B)P(B) = P(B|A)P(A)$

$$\Rightarrow \text{Bayes' rule } P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Why is this useful???

For assessing **diagnostic** probability from **causal** probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g., let  $M$  be meningitis,  $S$  be stiff neck:

$$P(M|S) = \frac{P(S|M)P(M)}{P(S)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

Note: posterior probability of meningitis still very small!

# Normalization

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Suppose we wish to compute a posterior distribution over  $A$  given  $B = b$ , and suppose  $A$  has possible values  $a_1 \dots a_m$

We can apply Bayes' rule for each value of  $A$ :

$$P(A = a_1 | B = b) = P(B = b | A = a_1)P(A = a_1) / P(B = b)$$

⋮

$$P(A = a_m | B = b) = P(B = b | A = a_m)P(A = a_m) / P(B = b)$$

Adding these up, and noting that  $\sum_i P(A = a_i | B = b) = 1$ :

$$1 / P(B = b) = 1 / \sum_i P(B = b | A = a_i)P(A = a_i)$$

This is the **normalization factor**, constant w.r.t.  $i$ , denoted  $\alpha$ :

$$P(A | B = b) = \alpha P(B = b | A)P(A)$$

Typically compute an unnormalized distribution, normalize at end

e.g., suppose  $P(B = b | A)P(A) = \langle 0.4, 0.2, 0.2 \rangle$   
then  $P(A | B = b) = \alpha \langle 0.4, 0.2, 0.2 \rangle = \frac{\langle 0.4, 0.2, 0.2 \rangle}{0.4+0.2+0.2} = \langle 0.5, 0.25, 0.25 \rangle$

# Conditioning

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Introducing a variable as an extra condition:

$$P(X|Y) = \sum_z P(X|Y, Z=z)P(Z=z|Y)$$

Intuition: often easier to assess each specific circumstance, e.g.,

$$\begin{aligned} P(\text{RunOver}|C\text{ross}) \\ &= P(\text{RunOver}|C\text{ross}, \text{Light}=green)P(\text{Light}=green|C\text{ross}) \\ &\quad + P(\text{RunOver}|C\text{ross}, \text{Light}=yellow)P(\text{Light}=yellow|C\text{ross}) \\ &\quad + P(\text{RunOver}|C\text{ross}, \text{Light}=red)P(\text{Light}=red|C\text{ross}) \end{aligned}$$

When  $Y$  is absent, we have **summing out** or **marginalization**:

$$P(X) = \sum_z P(X|Z=z)P(Z=z) = \sum_z P(X, Z=z)$$

In general, given a joint distribution over a set of variables, the distribution over any subset (called a **marginal** distribution) can be calculated by summing out the other variables.

# Full joint distributions

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A **complete probability model** specifies every entry in the joint distribution for all the variables  $\mathbf{X} = X_1, \dots, X_n$   
i.e., a probability for each possible world  $X_1 = x_1, \dots, X_n = x_n$

E.g., suppose  $Toothache$  and  $Cavity$  are the random variables:

	$Toothache = true$	$Toothache = false$
$Cavity = true$	0.04	0.06
$Cavity = false$	0.01	0.89

Possible worlds are mutually exclusive  $\Rightarrow P(w_1 \wedge w_2) = 0$

Possible worlds are exhaustive  $\Rightarrow w_1 \vee \dots \vee w_n$  is True

$$\text{hence } \sum_i P(w_i) = 1$$

## Full joint distributions contd.

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1) For any proposition  $\phi$  defined on the random variables  $\phi(w_i)$  is true or false

2)  $\phi$  is equivalent to the disjunction of  $w_i$ 's where  $\phi(w_i)$  is true

$$\text{Hence } P(\phi) = \sum_{\{w_i : \phi(w_i)\}} P(w_i)$$

i.e., the unconditional probability of any proposition is computable as the sum of entries from the full joint distribution

Conditional probabilities can be computed in the same way as a ratio:

$$P(\phi|\xi) = \frac{P(\phi \wedge \xi)}{P(\xi)}$$

E.g.,

$$P(Cavity|Toothache) = \frac{P(Cavity \wedge Toothache)}{P(Toothache)} = \frac{0.04}{0.04 + 0.01} = 0.8$$