Counting and Probability Summary

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1 Permutations

A sequence of r elements from a set of n elements is called a r-*permutation*. The number of r-permutations of n elements is

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

2 Combinations

A subset of r elements from a set of n elements is called a r-*combination*. The number of r-combinations of n elements is

$$\left(\begin{array}{c} n \\ r \end{array}\right) = \frac{n!}{(n-r)!r!}.$$

3 Strings and Cartesian Products

The number of strings of length r where each letter of the alphabet is taken from a set of size n is

$$n^r.$$

More generally, the number of strings of length r where the i-th letter is taken from a set of size n_i is

$$n_1 \cdot n_2 \cdots n_r = \prod_{i=1}^{r} n_i.$$

Another way to think of this is that $\prod_{i=1}^{r} n_i$ is the number of ordered r-tuples in the Cartesian product $A_1 \times A_2 \times \cdots \times A_r$ where the number of elements of A_i is n_i.

4 Finite Probability

Finite probability and counting are very closely related. A *sample space* is the set of all possibilities under consideration. An *event* is a subset of the possibilities, that is, just a subset of the sample space. The probability of an event E is simply the ratio

$$p(E) = \frac{|E|}{|S|}$$

where $|X|$ denotes the number of elements in X. Here are some basic facts about probabilities.

$$\begin{aligned} p(S-E) &= 1 - p(E) \\ p(E_1 \cup E_2) &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \end{aligned}$$

Two events, E_1 and E_2 are mutually exclusive if $p(E_1 \cap E_2) = 0$. If E_1, E_2, ... , E_n are mutually exclusive events and exhaust all the possibilities then

$$\sum_{i=1}^{n} p(E_i) = 1.$$

If E_1 and E_2 are independent events, that is the events E and F come from unrelated samples, then

$$p(E \cap F) = p(E)p(F).$$

Another way to say this is that if two events occur independent of each other then their probabilities multiply. For example, take two flips of a coin. The probability of two heads is the probability of one head, 1/2, times the probability of a second head, 1/2, which is 1/4.

5 Probabilities with Replacement

Suppose we have a bucket with n balls in it with r red and w white. Hence $n = r + w$. Choose k balls from the bucket in succession, but after picking a ball record its color, throw it back into the bucket and shake it before choosing the next ball. What is the probability of chosing exactly i red balls. The sample space is the set of all sequences of length k taken from the n balls. Thus, the sample space has size n^k. The number of ways to select the k balls so that i are red is

$$\binom{k}{i} r^i w^{k-i}.$$

To see this imagine the k positions of the balls chosen. First choose i positions for the red balls. Now, in those positions choose an i-string of red balls, and in the remaining $k - i$ positions choose a $(k - i)$-string of white balls. These three choices are done independently, yielding the product above. The final probability is

$$\frac{\binom{k}{i} r^i w^{k-i}}{n^k} = \binom{k}{i} (\frac{r}{n})^i (\frac{w}{n})^{k-i}.$$

6 Probabilities without Replacement

Suppose we have a bucket with n balls in it with r red and w white. Hence $n = r + w$. Choose k balls from the bucket in succession, but in this case don't replace the balls. The sample space in this case is k-permutations of n balls, so it has size $n!/(n-k)!$. Again imagine the k positions of the balls chosen. Again there are $\binom{k}{i}$ ways to select the i positions of the red balls. Once these positions are chosen there are $r!/(r-i)!$ ways to choose the red balls and $w!/(w-(k-i))!$ ways to choose the white balls. Altogether there are

$$\binom{k}{i}\frac{r!}{(r-i)!}\frac{w!}{(w-(k-i))!}$$

ways to pick k balls, i of which are red. Thus the probability of picking k balls (without replacement), i of which are red is

$$\frac{\binom{k}{i}\frac{r!}{(r-i)!}\frac{w!}{(w-(k-i))!}}{n!/(n-k)!}$$

which simplifies to

$$\frac{\binom{r}{i}\binom{w}{k-i}}{\binom{n}{k}}.$$

This suggests another sample space for the problem. The sample space would be all subsets of size k.

7 Expectations

Assume that the possible events are E_1, E_2,..., E_n and that they are mutually exclusive and exhaustive. That is the sum of their probabilities is 1 and $p(E_i \cap E_j) = 0$ if $i \neq j$. Suppose that each event E_i has a cost $c(E_i)$. The expected cost of an event in this set of possible events is

$$\sum_{i=1}^{n} c(E_i)p(E_i).$$