13a. C2 Interpolating Curves

Introduction

In the previous lectures on curves, we described a number of cubic curves.

The C^2 interpolating curves had the properties of:

- C^2 continuity
- Interpolation of control points
- No local control

The various C^2 interpolating curves we considered differed in their end conditions.

In this lecture, we derive the curve corresponding to "natural" end conditions.

Closed C^2 interpolating curves

We begin with the closed C^2 interpolating curve.

Consider a closed B-sline curve:

The interpolated points (joints) are easy to compute.
We already did this in class:

$$
\begin{aligned}
X_i &= \frac{1}{6}B_{i-1} + \frac{2}{3}B_i + \frac{1}{6}B_{i+1} \\
&= \frac{1}{6}(B_{i-1} + 4B_i + B_{i+1})
\end{aligned}
$$

What about the first and the last joints, X_0 and X_{m-1}?

Closed C^2 interpolating curves, cont'd

The equations for the first and last joints just rap around:

$$
\begin{aligned}
X_0 &= \underline{\hspace{3cm}} \\
X_{m-1} &= \underline{\hspace{3cm}}
\end{aligned}
$$

Closed C^2 interpolating curves, cont'd

This leads to a matrix formulation:

$$\frac{1}{6}\begin{bmatrix} 4 & 1 & & & & & 1 \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & 1 & 4 & 1 & & \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & 1 & 4 & 1 \\ 1 & & & & & 1 & 4 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B_{m-1} \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_{m-1} \end{bmatrix}$$

which we can re-write as:

$$\mathbf{MB} = \mathbf{X}$$

So, all we have to do is solve for $\mathbf{B} = \mathbf{M}^{-1}\mathbf{X}$. The resulting B-spline curve is our answer!

More on solving this later...

Natural C^2 interpolating curves

Now consider an open B-spline curve:

How would we place the de Boor points $B_{-1} \cdots B_m$ to satisfy the natural endpoint conditions?

Natural C^2 interpolating curves, cont'd

First, remember that the natural end condition means:

$$Q''(0) = Q''(u_{\max}) = \underline{\hspace{3cm}}$$

For the bezier control polygon, this means:

$$Q''(0) = (V_0 - V_1) + (V_2 - V_1) = 0$$

or

$$(V_0 - V_1) = -(V_2 - V_1)$$

Natural C^2 interpolating curves, cont'd

What does this mean geometrically?

This comes for free with endpoint interpolating B-splines:

No need for B_{-1} and B_m...

Natural C^2 interpolating curves, cont'd

The equations for X_i in terms of B_i are the same as before:

$$X_i = \frac{1}{6}(B_{i-1} + 4B_i + B_{i+1})$$

except at the endpoints, which are now:

$$\begin{aligned} X_0 &= \underline{\hspace{2cm}} \\ X_{m-1} &= \underline{\hspace{2cm}} \end{aligned}$$

Natural C^2 interpolating curves, cont'd

This leads to a matrix formulation:

$$\frac{1}{6}\begin{bmatrix} 6 & & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & 1 & 4 & 1 & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \\ & & & & & 1 & 4 & 1 \\ & & & & & & & 6 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B_{m-1} \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_{m-1} \end{bmatrix}$$

which we can re-write as:

$$\mathbf{MB} = \mathbf{X}$$

We could solve for $\mathbf{M}^{-1}$ and then ocompute $\mathbf{B} = \mathbf{M}^{-1}\mathbf{X}$, but we're better off using forward elimination and back substitution.

Note: $\mathbf{M}$ is sparse (tri-diagonal), but $\mathbf{M}^{-1}$ is dense, thus *global control.*