Affine transformations

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CSE 457

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Reading

Optional reading:

- Angel and Shreiner: 3.1, 3.7-3.11
- Marschner and Shirley: 2.3, 2.4.1-2.4.4, 6.1.1-6.1.4, 6.2.1, 6.3

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

Vector representation

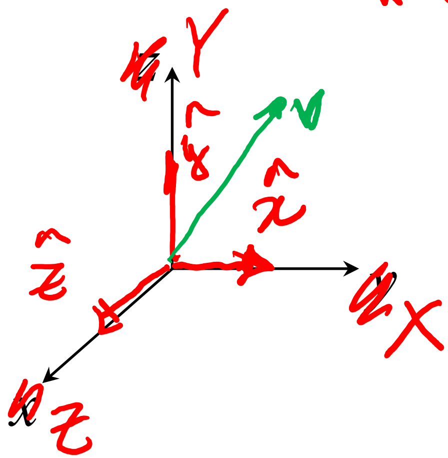
We can represent a **point**, $\mathbf{p} = (x, y)$, in the plane or $\mathbf{p} = (x, y, z)$ in 3D space:

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- ~~as row vectors~~ $\cancel{\begin{bmatrix} x & y \end{bmatrix}}$ $\cancel{\begin{bmatrix} x & y & z \end{bmatrix}}$

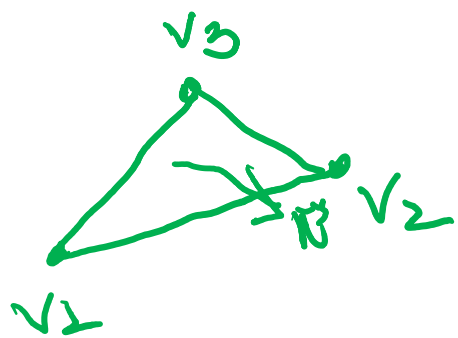
Canonical axes

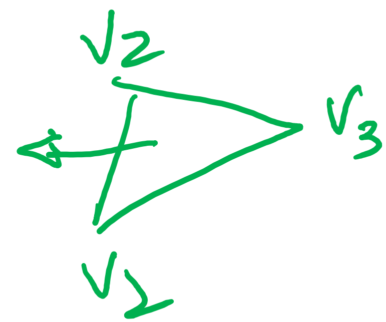
RIGHT-HANDED COORD SYSTEM



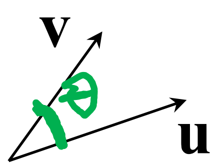
$$V = v_x \underbrace{\hat{x}}_{\begin{bmatrix}1\\0\\0\end{bmatrix}} + v_y \underbrace{\hat{y}}_{\begin{bmatrix}0\\1\\0\end{bmatrix}} + v_z \cdot \underbrace{\hat{z}}_{\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

$$= \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$





Vector length and dot products



$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$u \cdot v = v_x \cdot u_x + v_y u_y + v_z \cdot u_z$$

$$u \cdot v \overset{?}{=} v \cdot u \quad \checkmark$$

$$u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$u \cdot v = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = u^T v$$

$$\|\hat{u}\| = 1$$

$$v \cdot v = \|v\|^2$$

$$\hat{u} = \frac{u}{\|u\|}$$

$$\hat{v} = \frac{v}{\|v\|}$$

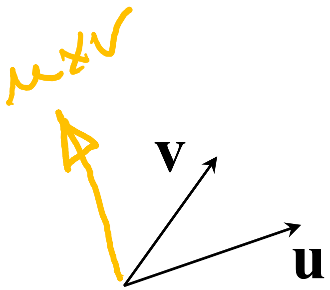
$$\hat{u} \cdot \hat{v} = \cos(\theta)$$

$$u \cdot v = \|u\| \cdot \|v\| \cdot \underline{\cos(\theta)}$$

$$u \cdot v = 0 \Rightarrow u \perp v$$

$$\|u\|, \|v\| = 0$$

Vector cross products



$$u \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_y v_z - u_z v_y)\hat{x} + (u_z v_x - u_x v_z)\hat{y} + (u_x v_y - u_y v_x)\hat{z}$$

$$u \times v \text{ (in x-y plane)} = \begin{bmatrix} 0 \\ 0 \\ u_x v_y - u_y v_x \end{bmatrix}$$

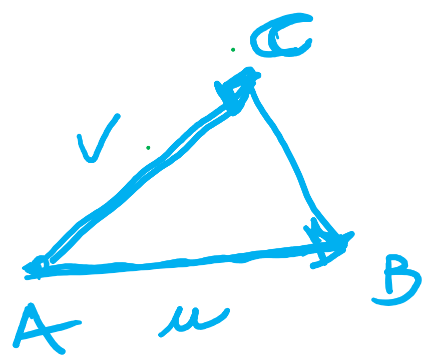
$$u \times v \stackrel{?}{=} -v \times u$$

$$(u \times v) \cdot u = 0$$

$$(u \times v) \cdot v = 0$$

$$\|u \times v\| = \|u\| \cdot \|v\| \cdot |\sin(\theta)| = \text{Area}(\square_{u,v})$$

$$\text{Area}(\Delta_{u,v}) = \frac{1}{2}\|u \times v\|$$



Representation, cont.

$(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1} A^{-1}$

We can represent a **2-D transformation** M by a matrix

$(AB)^{-1} AB = I$

$(AB)^{-1} A = B^{-1}$

$AB^{-1} = B^{-1} A^{-1}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, M goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If **p** is a row vector, M^{T} goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ax + by & cx + dy \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix M:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements a, b, c, d…

Identity

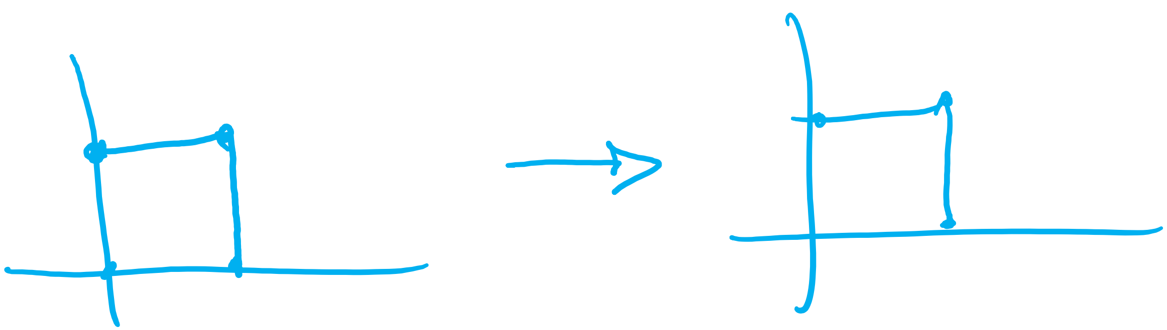
Suppose we choose $a = d = 1, b = c = 0$:

- Gives the **identity** matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Doesn't move the points at all

$$x' = x$$
$$y' = y$$



Scaling

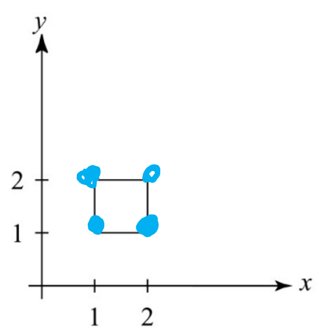
Suppose we set $b = c = 0$, but let a and d take on any *positive* value:

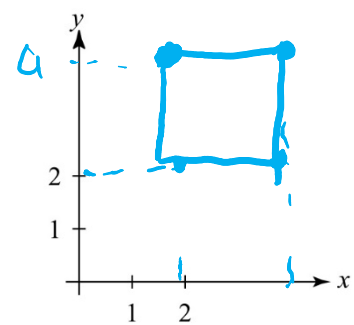
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

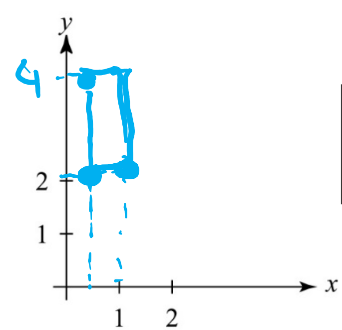
- Provides **differential (non-uniform) scaling** in x and y:

$$x' = ax$$
$$y' = dy$$





$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



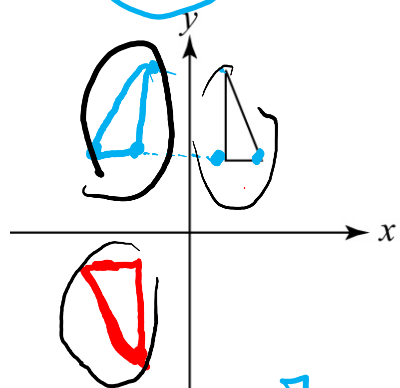
$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

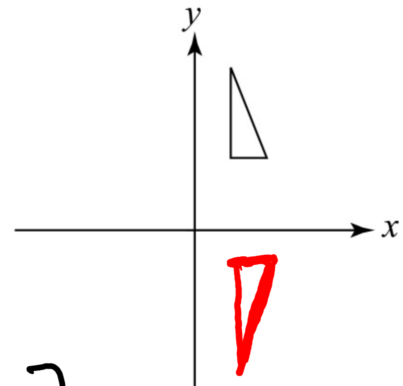
Reflection, Mirror

Suppose we keep $b = c = 0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$





$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}}}$$

Shear

Now let's leave $a = d = 1$ and experiment with $b \ldots$

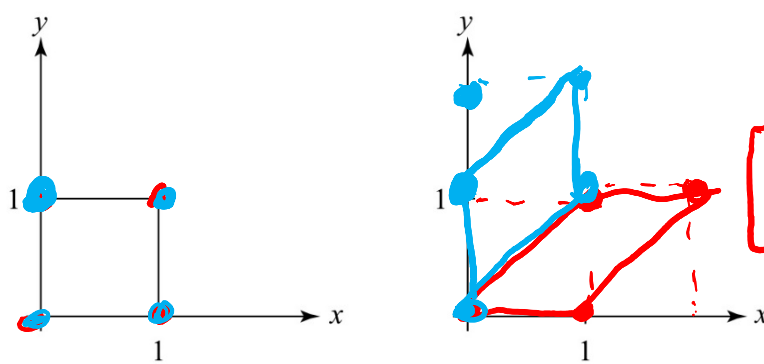
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



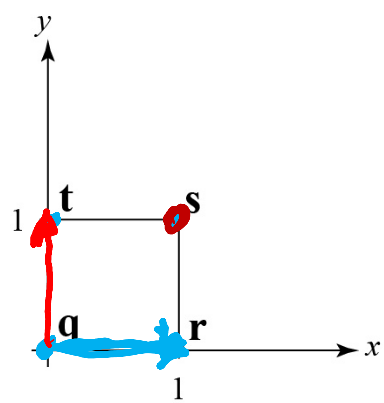
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

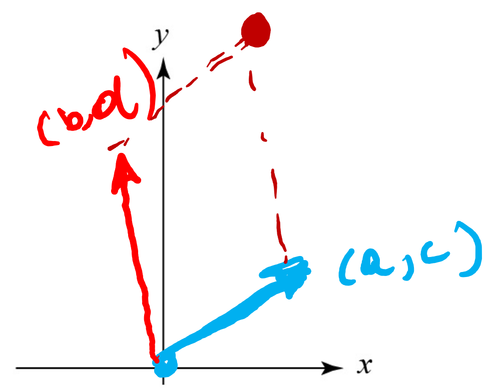
Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathbf{q} & \mathbf{r} & \mathbf{s} & \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{q'} & \mathbf{r'} & \mathbf{s'} & \mathbf{t'} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$





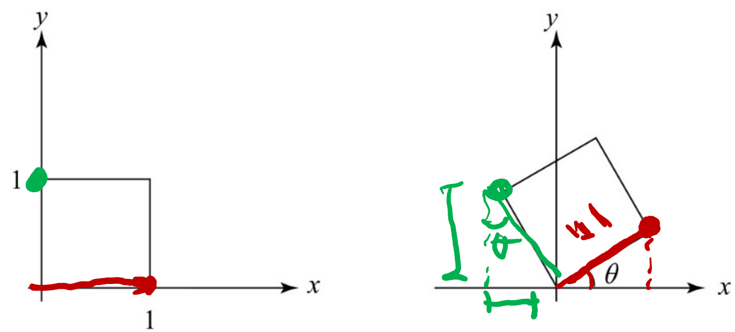
Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners $(1,0)$ and $(0,1)$ are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



$$\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

Thus,

$$M = R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

TRANSLATION

Affine transformations

In order to incorporate the idea that both the basis and the origin can change, we augment the linear space **A** with an origin **t**

An affine transformation then is expressed as:

$$\mathbf{p'} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} + \mathbf{t}$$

How can we write an affine transformation with matrices?

Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

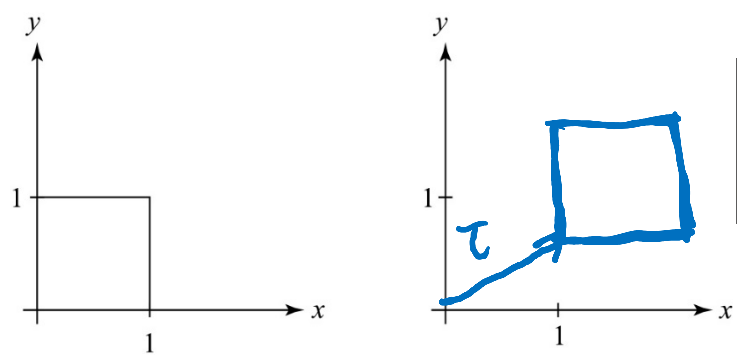
$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "*w*" component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} ax + by + t_x \\ cx + dy + t_y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} A \cdot \binom{x}{y} + t \\ 1 \end{bmatrix}$$

2D affine transformations always have a bottom row of $[0\ 0\ 1]$.

An "affine point" is a "linear point" with an added *w*-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

Rotation about arbitrary points

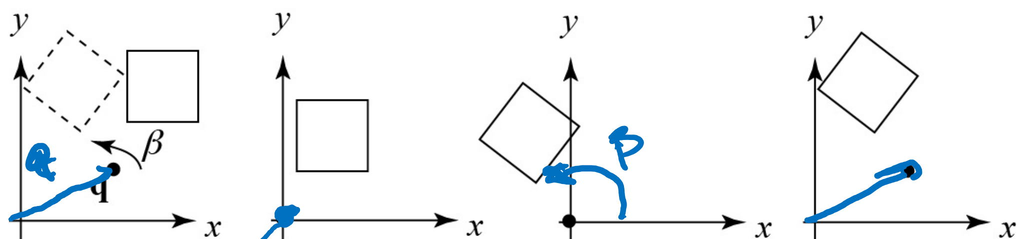
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Until now, we have only considered rotation about the origin.

$$T(t) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

With homogeneous coordinates, you can specify a rotation by β, about any point $\mathbf{q} = [q_x \ q_y]^T$ with a matrix.

Let's do this with rotation and translation matrices of the form $\mathrm{R}(\theta)$ and $\mathrm{T}(\mathbf{t})$, respectively.



$$M = T(-q) \cdot R(\beta) \ T(q)$$

1. Translate $\mathbf{q}$ to origin
2. Rotate
3. Translate back

$$M = T(q) \cdot R(\beta) \ T(-q) \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

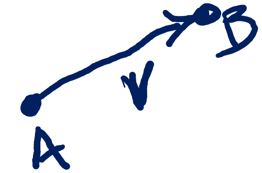
Points and vectors

$v = B - A$

$$\begin{pmatrix} B_x \\ B_y \\ 1 \end{pmatrix} - \begin{pmatrix} A_x \\ A_y \\ 1 \end{pmatrix}$$

$$v = \begin{pmatrix} B_x - A_x \\ B_y - A_y \\ 0 \end{pmatrix}$$

Vectors have an additional coordinate of $w = 0$. Thus, a change of origin has no effect on vectors.



Q: What happens if we multiply a vector by an affine matrix?

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} a v_x + b v_y \\ c v_x + d v_y \\ 0 \end{bmatrix}$$

These representations reflect some of the rules of affine operations on points and vectors:

vector + vector	→	VECTOR
scalar · vector	→	VECTOR
point - point	→	VECTOR
point + vector	→	POINT
point + point	→	CHAOS
scalar · vector + scalar · vector	→	VECTOR
scalar · point + scalar · point	→	IT DEPENDS
point + scalar · vector	→	POINT

$$\begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix} = \begin{bmatrix} v_x + u_x \\ v_y + u_y \\ 0 \end{bmatrix}$$

$$\alpha \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha v_x \\ \alpha v_y \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ 1 \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} A_x + v_x \\ A_y + v_y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \\ 2 \end{bmatrix}$$

$$\alpha \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} + \beta \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha A_x + \beta B_x \\ \alpha A_y + \beta B_y \\ \alpha + \beta \end{bmatrix}$$

$\alpha + \beta = 0 \rightarrow$ VECTOR

$\alpha + \beta = 1 \rightarrow$ POINT

$\alpha + \beta =$ OTHERWISE $\rightarrow$ CHAOS

One useful combination of affine operations is: $P(t) = P_o + t\mathbf{u}$

Q: What does this describe?

$t \in (-\infty, \infty) \rightarrow$ LINE

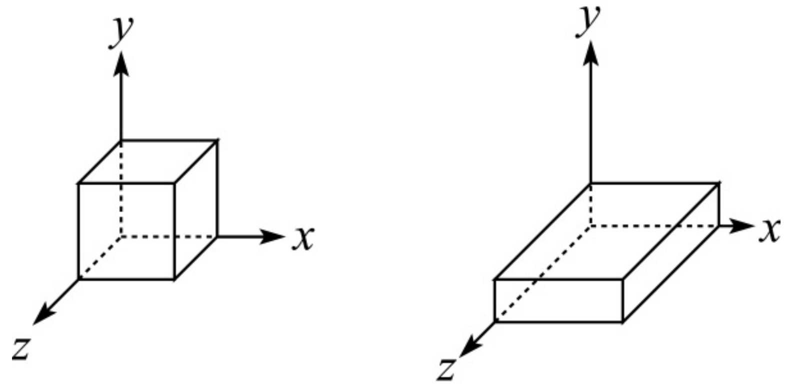
$t \in [0, \infty) \rightarrow$ RAY

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

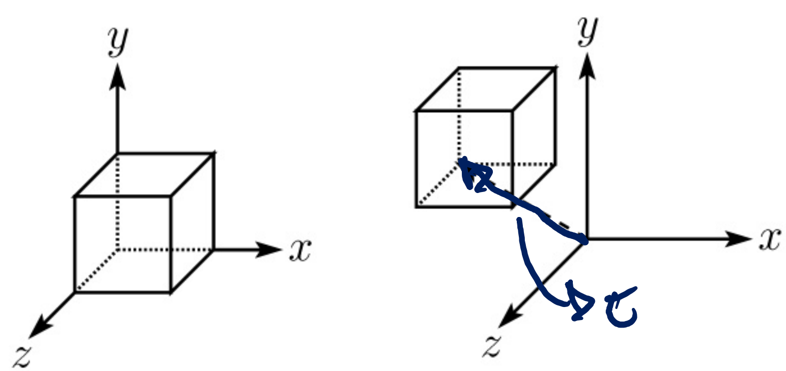
For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



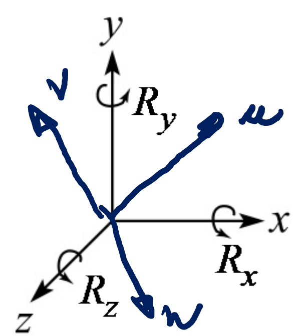
Rotation in 3D

These are the rotations about the canonical axes:

$$R_x(\alpha)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha & 0\\ 0 & \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$R_y(\beta)=\begin{bmatrix}\cos\beta & 0 & \sin\beta & 0\\ 0 & 1 & 0 & 0\\ -\sin\beta & 0 & \cos\beta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$R_z(\gamma)=\begin{bmatrix}\cos\gamma & -\sin\gamma & 0 & 0\\ \sin\gamma & \cos\gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Use right hand rule

A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

$R = [u\ v\ w]$

$u\cdot u=1$ $\quad u\cdot v=0$

$v\cdot v=1$ $\quad v\cdot w=0$

$w\cdot w=1$ $\quad v\cdot v=0$

$R^TR = \begin{bmatrix}u^T\\ v^T\\ w^T\end{bmatrix}\cdot[u\ v\ w]$

$= \begin{bmatrix}u^Tu & u^Tv & u^Tw\\ v^Tu & v^Tv & v^Tw\\ w^Tu & w^Tv & w^Tw\end{bmatrix}$

$= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

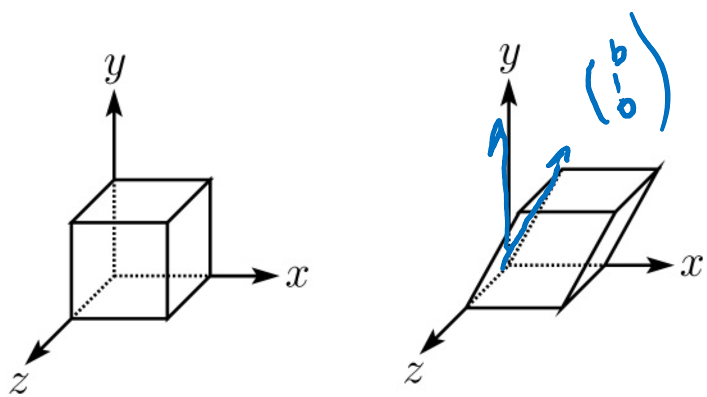
$R^TR = I$

$R^{-1} = R^T$

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



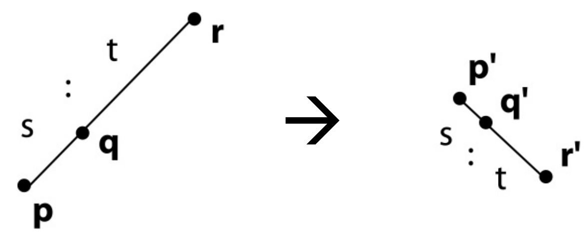
$\begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix}$

We call this a shear with respect to the x-z plane.

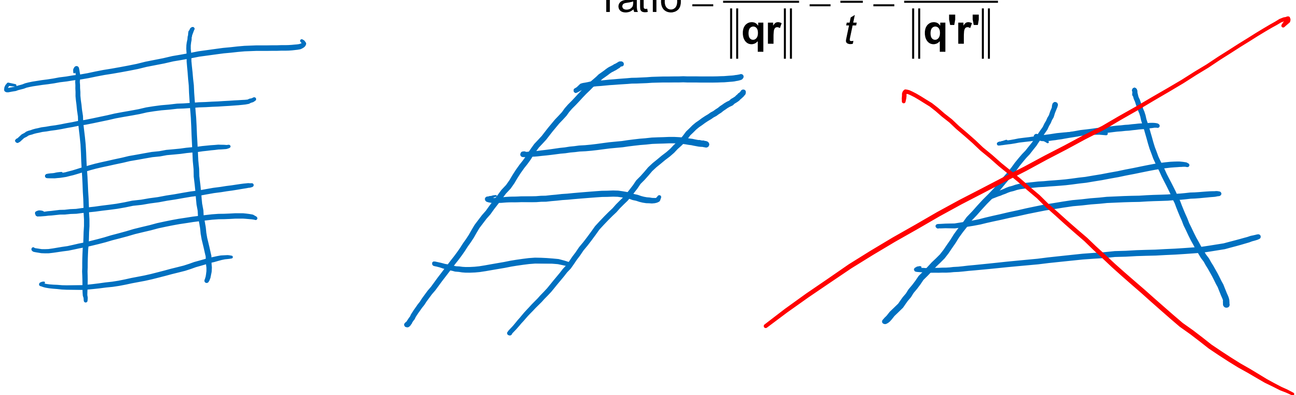
Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$



Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.