Parametric surfaces

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CSE 457
Spring 2020

Reading

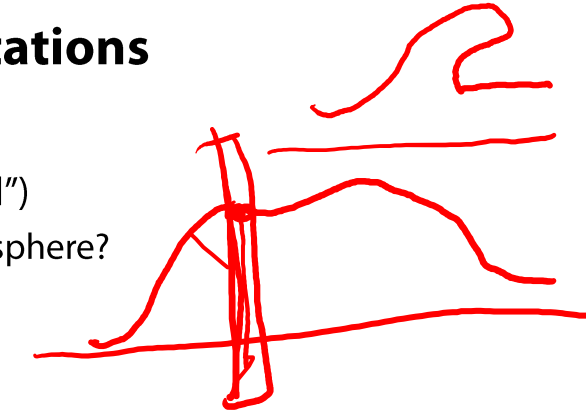
Optional reading:

- Angel and Shreiner readings for "Parametric Curves" lecture, with emphasis on 10.1.2, 10.1.3, 10.1.5, 10.6.2, 10.7.3, 10.9.4.
- Marschner and Shirley, 2.5.

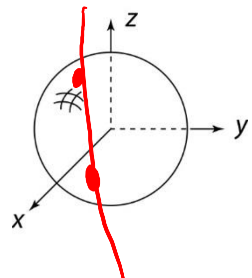
Further reading

- Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling,* 1987.

Mathematical surface representations



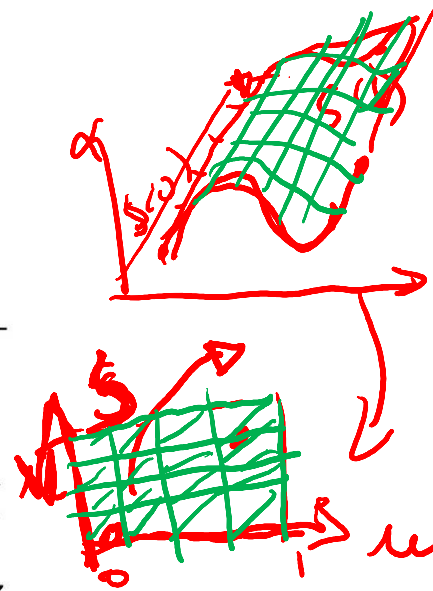
- Explicit $z = f(x, y)$ (a.k.a., a "height field")
  - what if the curve isn't a function, like a sphere?

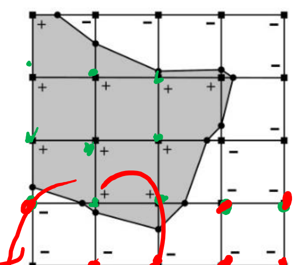


$g(x,y,z)$

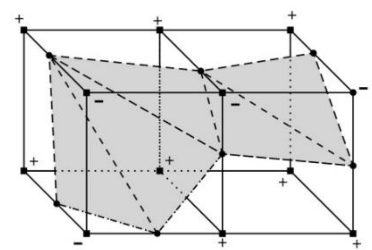
$\underbrace{x^2+y^2+z^2+r^2}=0$

- Implicit $g(x, y, z) = 0$

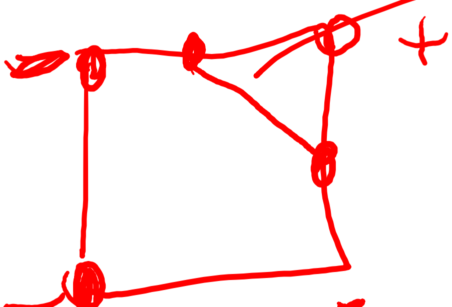




Isocontour from "marching squares"

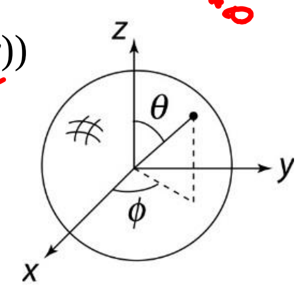


Isocontour from "marching cubes"



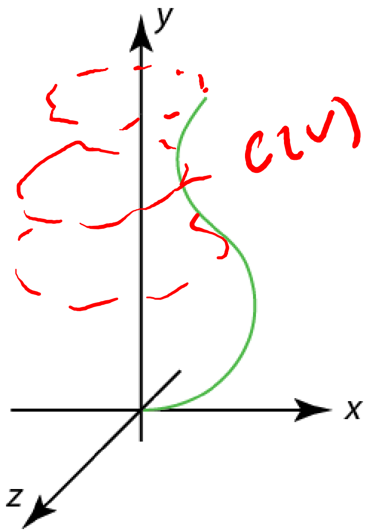
- Parametric $S(u, v) = (x(u, v), y(u, v), z(u, v))$
  - For the sphere:

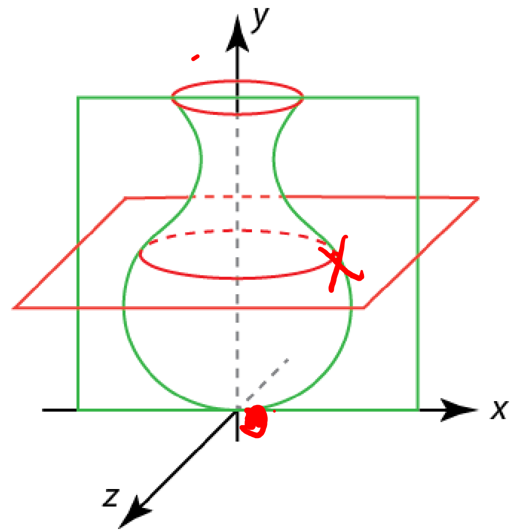
$$x(u, v) = r\ \cos(2\pi v)\ \sin(pu)$$
$$y(u, v) = r\ \sin(2\pi v)\ \sin(\pi u)$$
$$z(u, v) = r\ \cos(\pi u)$$



As with curves, we'll focus on parametric surfaces.

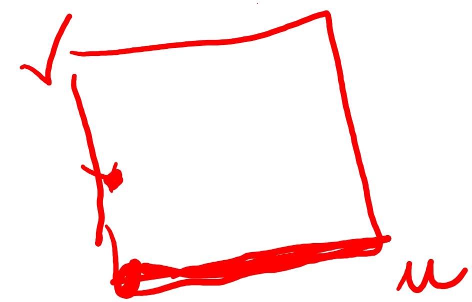
Constructing surfaces of revolution





Given: A curve $C(v)$ in the xy-plane:

$$C(v) = \begin{bmatrix} C_x(v) \\ C_y(v) \\ 0 \\ 1 \end{bmatrix}$$



Let $R_y(\theta)$ be a rotation about the y-axis.

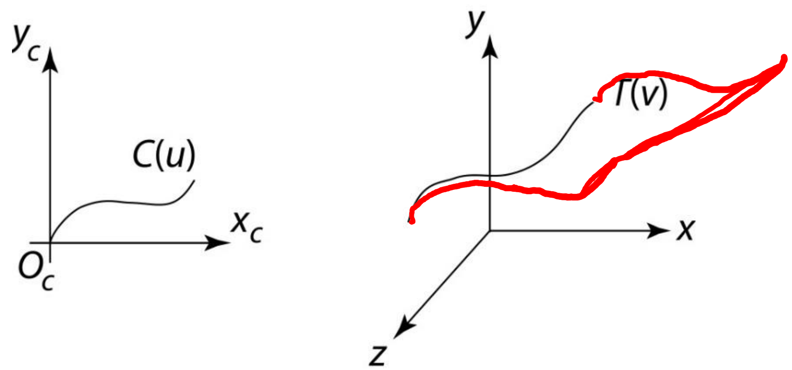
Find: A surface $S(u,v)$ which is $C(v)$ rotated about the y-axis, where $u,v \in [0, 1]$.

Solution: $S(u,v) = R_y(2\pi u)\, C(v)$

General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u, v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



More specifically:

- Suppose that $C(u)$ lies in an (x_c, y_c) coordinate system with origin O_c.
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

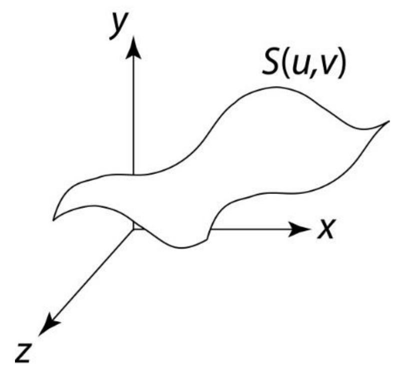
Orientation

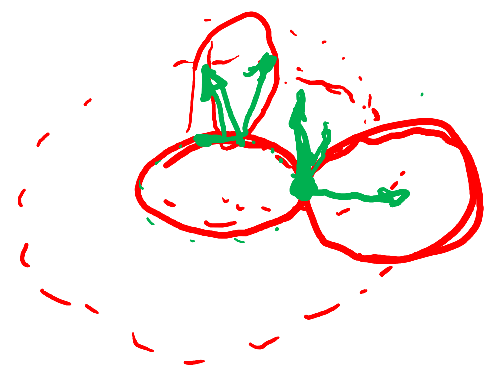
The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

1. **Fixed** (or **static**): Just translate O_c along $T(v)$.

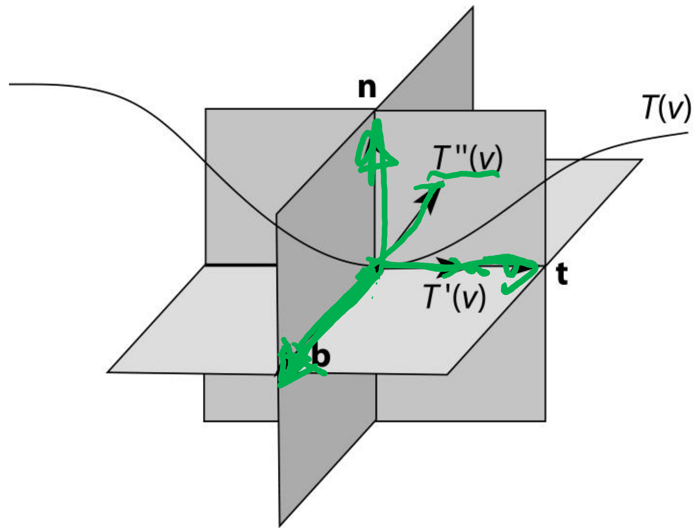




2. Moving. Use the **Frenet frame** of $T(v)$.
   - Allows smoothly varying orientation.
   - Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

Tangent: $\mathbf{t}(v) = \text{normalize}[T'(v)]$

Binormal: $\mathbf{b}(v) = \text{normalize}[T'(v) \times T''(v)]$

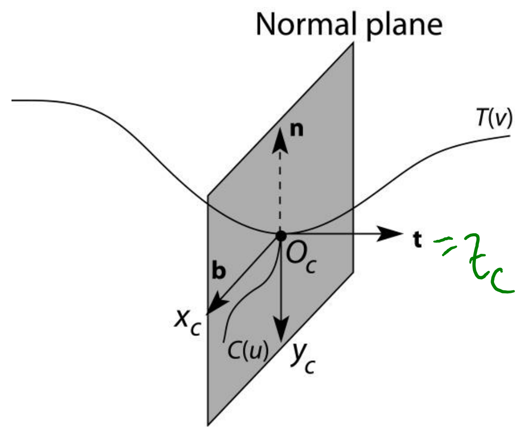
Normal: $\mathbf{n}(v) = \mathbf{b}(v)$´ $\mathbf{t}(v)$

As we move along $T(v)$, the Frenet frame ($\mathbf{t}$, $\mathbf{b}$, $\mathbf{n}$) varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

- Put $C(u)$ in the **normal plane** .
- Place O_c on $T(v)$.
- Align x_c for $C(u)$ with $\mathbf{b}$.
- Align y_c for $C(u)$ with $-\mathbf{n}$.



If $T(v)$ is a circle, you get a surface of revolution exactly!

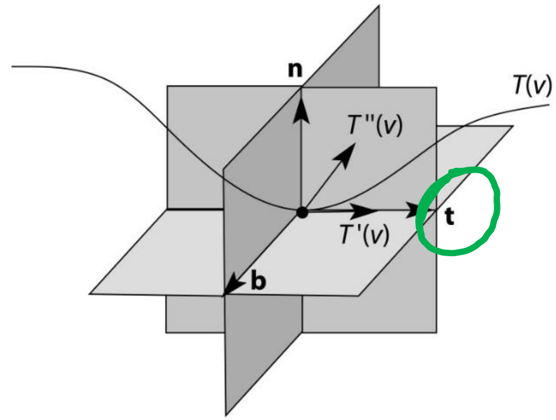
Degenerate frames

$\langle T', T' \rangle = \|T'\| = 1$

$\langle T', T' \rangle' = 0$

$0 = \langle T'', T' \rangle + \langle T', T'' \rangle$

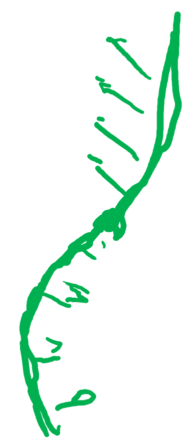
Let's look back at where we computed the coordinate frames from curve derivatives:



Where might these frames be ambiguous or undetermined?

$$\begin{cases} T'(x) = 0 \\ T''(x) = 0 \end{cases}$$

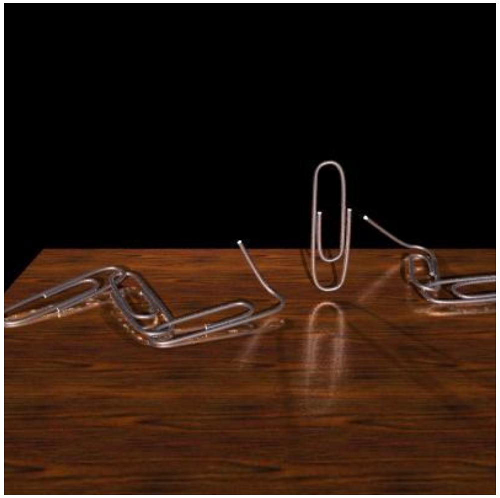
$\Rightarrow$ ARC LENGTH

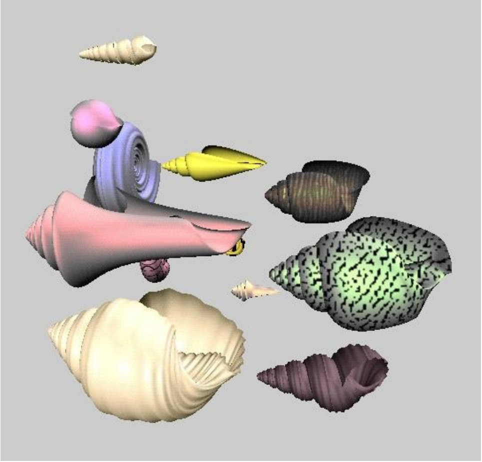


Variations

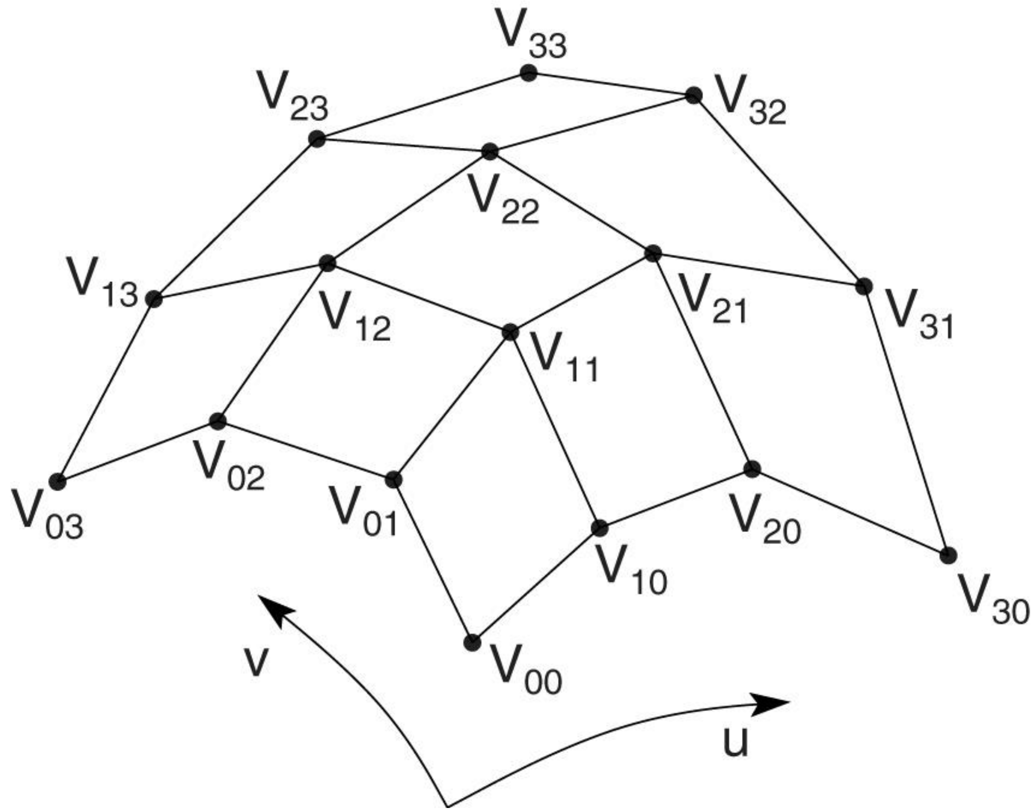
Several variations are possible:

- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- …





Tensor product Bézier surfaces

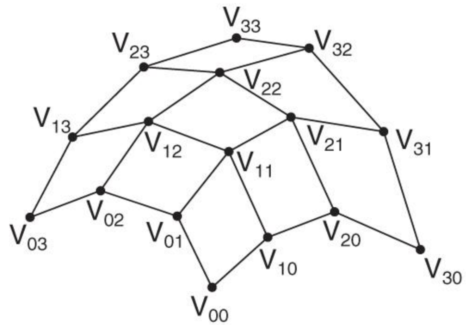


Given a grid of control points V_{ij}, forming a **control net**, construct a surface $S(u, v)$ by:

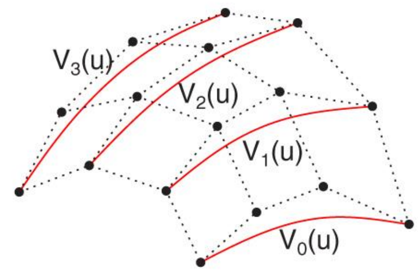
- treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0(u),\ldots, V_n(u)$.
- treating $V_0(u),\ldots,V_n(u)$ as control points for a curve parameterized by v.

Tensor product Bézier surfaces, cont.

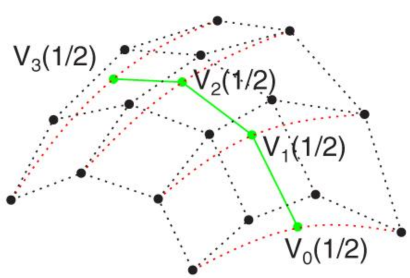
Let’s walk through the steps:



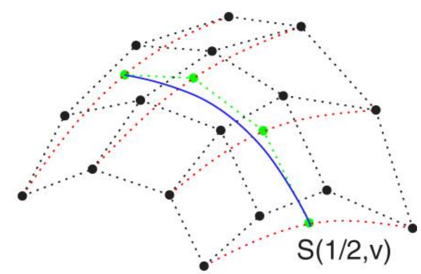
Control net



Control curves in *u*



Control polygon at *u*=1/2



Curve at *S*(1/2,*v*)

Which control points are always interpolated by the surface?

Polynomial form of Bézier surfaces

Recall that cubic Bézier *curves* can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^{n} V_i b_i(u)$$

A tensor product Bézier surface can be written as:

$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n} V_{ij} b_i(u) b_j(v)$$

In the previous slide, we constructed curves along u, and then along v. This corresponds to re-grouping the terms like so:

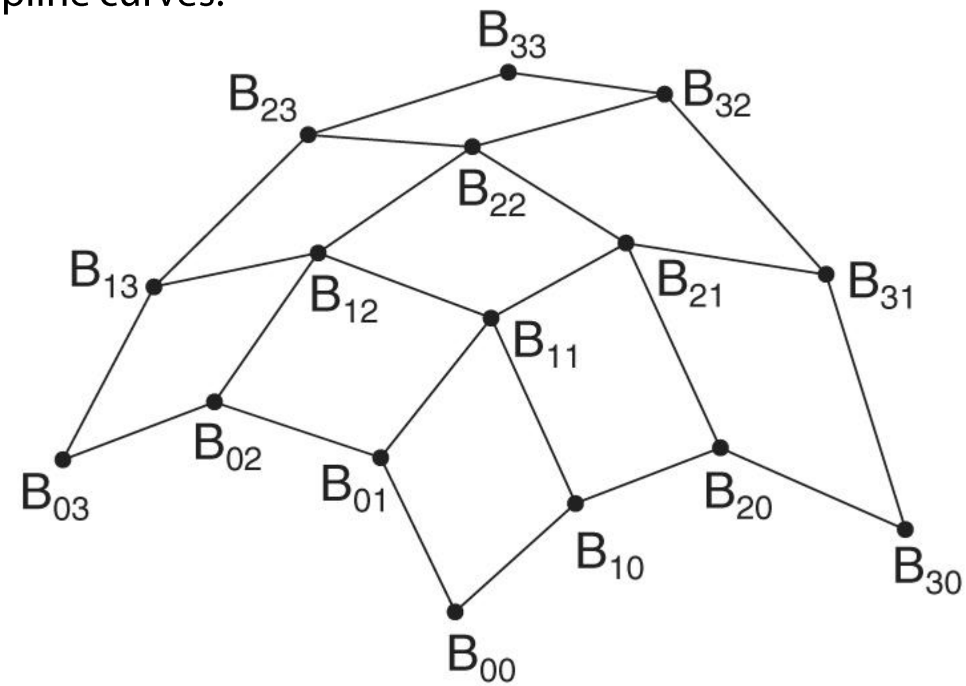
$$S(u,v) = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} V_{ij} b_i(u) \right) b_j(v)$$

But, we could have constructed them along v, then u:

$$S(u,v) = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} V_{ij} b_j(v) \right) b_i(u)$$

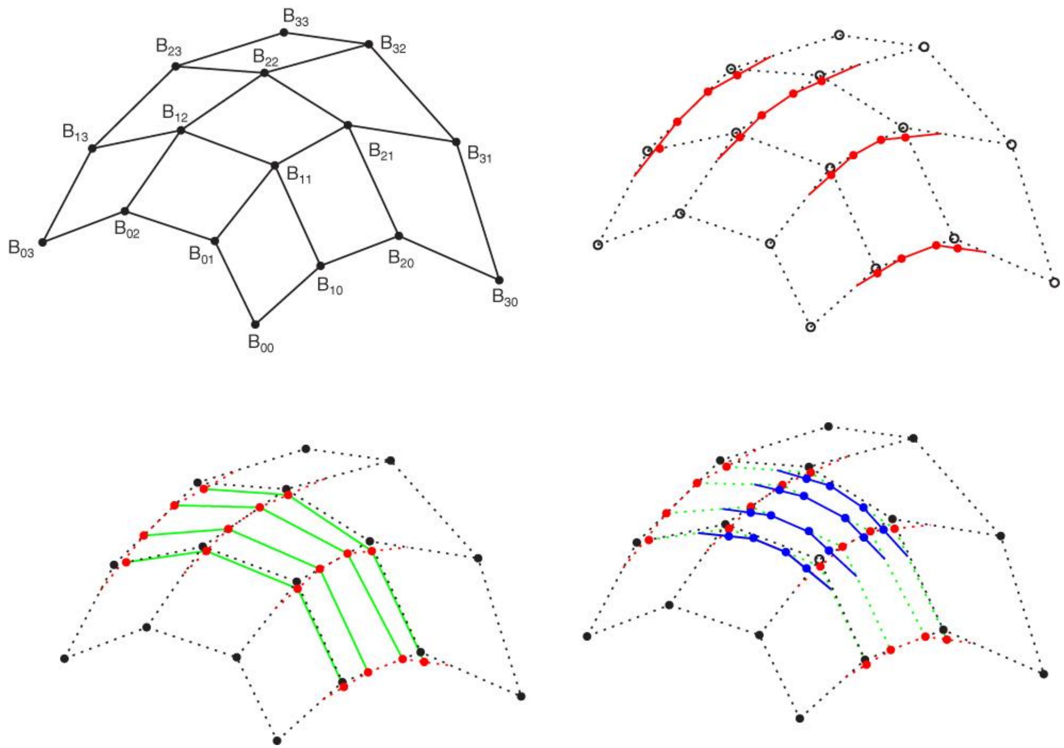
Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- treat rows of B as control points to generate Bézier control points in u.
- treat Bézier control points in u as B-spline control points in v.
- treat B-spline control points in v to generate Bézier control points in u.

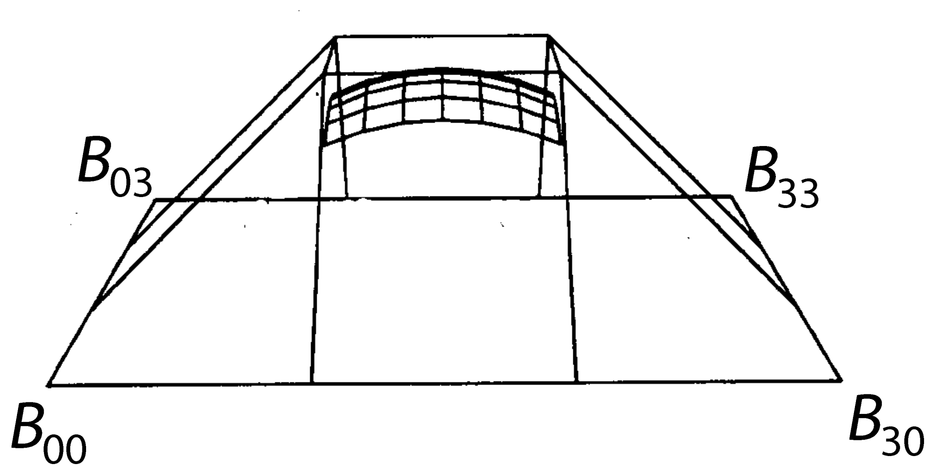
Tensor product B-spline surfaces, cont.



Which B-spline control points are always interpolated by the surface?

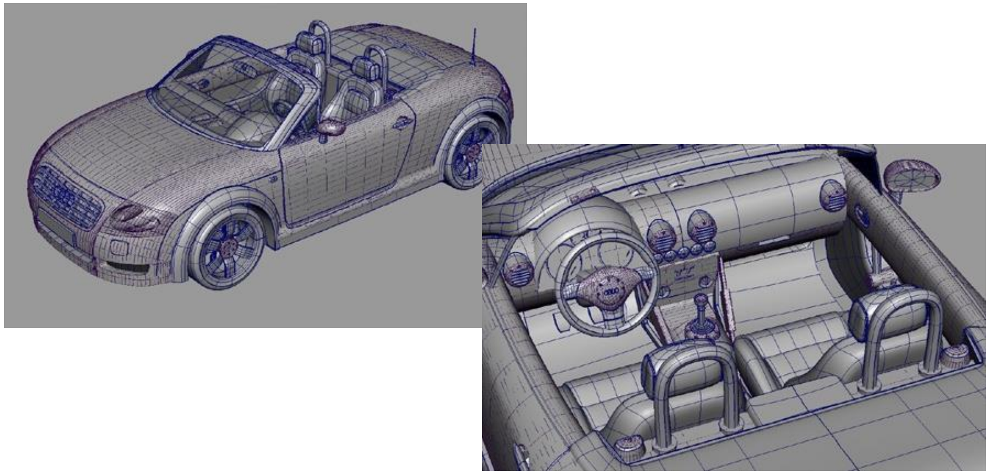
Tensor product B-splines, cont.

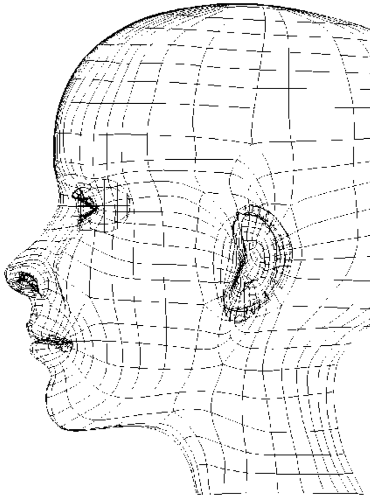
Another example:

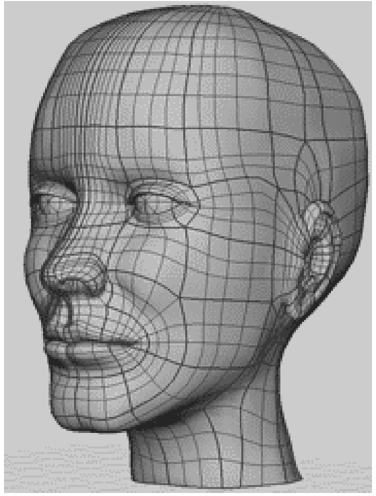


NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.



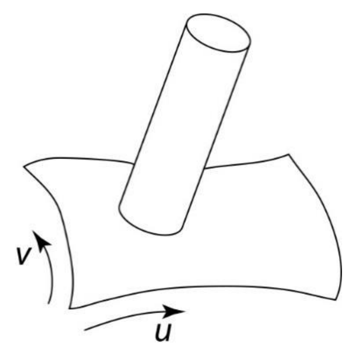




Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:



We can do this by **trimming** the *u*-*v* domain.

- Define a closed curve in the *u*-*v* domain (a **trim curve**)
- Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

Summary

What to take home:

- How to construct swept surfaces from a profile and trajectory curve:
  - with a fixed frame
  - with a Frenet frame
- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces