Affine transformations

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Reading

Optional reading:

- Angel and Shreiner: 3.1, 3.7-3.11
- Marschner and Shirley: 2.3, 2.4.1-2.4.4, 6.1.1-6.1.4, 6.2.1, 6.3

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
Geometric transformations

Geometric transformations will map points in one space to points in another: \((x', y', z') = f(x, y, z)\).

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.
Vector representation

We can represent a point, \( p = (x, y) \), in the plane or \( p = (x, y, z) \) in 3D space:

- as column vectors

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

- as row vectors

\[
\begin{bmatrix}
x & y \\
x & y & z
\end{bmatrix}
\]
Canonical axes
Vector length and dot products
Vector cross products
Representation, cont.

We can represent a 2-D transformation $M$ by a matrix

$$
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
$$

If $\mathbf{p}$ is a column vector, $M$ goes on the left:

$$
\mathbf{p}' = M\mathbf{p}
$$

$$
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
$$

If $\mathbf{p}$ is a row vector, $M^T$ goes on the right:

$$
\mathbf{p}' = \mathbf{p}M^T
$$

$$
\begin{bmatrix}
  x' & y'
\end{bmatrix} =
\begin{bmatrix}
  x & y
\end{bmatrix}
\begin{bmatrix}
  a & c \\
  b & d
\end{bmatrix}
$$

We will use column vectors.
Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix \( M \):

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

So:

\[
x' = ax + by
\]
\[
y' = cx + dy
\]

We will develop some intimacy with the elements \( a, b, c, d \ldots \)
Identity

Suppose we choose \( a = d = 1, b = c = 0 \):

- Gives the **identity** matrix:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

- Doesn't move the points at all
Scaling

Suppose we set $b = c = 0$, but let $a$ and $d$ take on any positive value:

- Gives a **scaling** matrix:
  \[
  \begin{bmatrix}
  a & 0 \\
  0 & d
  \end{bmatrix}
  \]

- Provides **differential (non-uniform) scaling** in $x$ and $y$:
  \[
  x' = ax \\
  y' = dy
  \]
Suppose we keep $b = c = 0$, but let either $a$ or $d$ go negative.

Examples:
Now let's leave $a = d = 1$ and experiment with $b \ldots$

The matrix

$$\begin{bmatrix}
1 & b \\
0 & 1
\end{bmatrix}$$

gives:

$$x' = x + by$$

$$y' = y$$
Effect on unit square

Let's see how a general 2 x 2 transformation $M$ affects the unit square:

$$
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  q & r & s & t
\end{bmatrix}
= 
\begin{bmatrix}
  q' & r' & s' & t'
\end{bmatrix}
$$

$$
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  0 & 1 & 1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
  0 & a & a+b & b
\end{bmatrix}
$$

$$
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  0 & 0 & 1 & 1
\end{bmatrix}
= 
\begin{bmatrix}
  0 & c & c+d & d
\end{bmatrix}
$$
Effect on unit square, cont.

Observe:

- Origin invariant under $M$
- $M$ can be determined just by knowing how the corners $(1,0)$ and $(0,1)$ are mapped
- $a$ and $d$ give $x$- and $y$-scaling
- $b$ and $c$ give $x$- and $y$-shearing
Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for “rotation about the origin”:

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} \rightarrow
\]

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix} \rightarrow
\]

Thus,

\[
M = R(\theta) =
\]

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]
Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

**Q:** What important operation does that leave out?
Affine transformations

In order to incorporate the idea that both the basis and the origin can change, we augment the linear space $\mathbf{u}, \mathbf{v}$ with an origin $\mathbf{t}$

An affine transformation then is expressed as:

$$\mathbf{p'} = x \cdot \mathbf{u} + y \cdot \mathbf{v} + \mathbf{t}$$

How can we write an affine transformation with matrices?
Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

\[
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix} \rightarrow 
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
\]

Adding the third “w” component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

\[
\begin{bmatrix}
    x' \\
    y' \\
    w'
\end{bmatrix}
= T(t) \begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
= \begin{bmatrix}
    1 & 0 & t_x \\
    0 & 1 & t_y \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
\]

\[\begin{bmatrix}
    1 & 0 & 1 \\
    0 & 1 & 1/2 \\
    0 & 0 & 1
\end{bmatrix}\]

... gives **translation**!
Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

\[
M = \begin{bmatrix}
    a & b & t_x \\
    c & d & t_y \\
    0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
    A & \mathbf{t} \\
    0 & 0 & 1
\end{bmatrix}
\]

2D affine transformations always have a bottom row of [0 0 1].

An “affine point” is a “linear point” with an added \(w\)-coordinate which is always 1:

\[
\mathbf{p}_{\text{aff}} = \begin{bmatrix}
    \mathbf{p}_{\text{lin}} \\
    1
\end{bmatrix} = \begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
\]

Applying an affine transformation gives another affine point:

\[
M\mathbf{p}_{\text{aff}} = \begin{bmatrix}
    A\mathbf{p}_{\text{lin}} + \mathbf{t} \\
    1
\end{bmatrix}
\]
Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation by $\beta$, about any point $\mathbf{q} = [q_x \ q_y]^T$ with a matrix.

Let’s do this with rotation and translation matrices of the form $R(\theta)$ and $T(t)$, respectively.

1. Translate $\mathbf{q}$ to origin
2. Rotate
3. Translate back
Points and vectors

Vectors have an additional coordinate of \( w = 0 \). Thus, a change of origin has no effect on vectors.

**Q:** What happens if we multiply a vector by an affine matrix?

These representations reflect some of the rules of affine operations on points and vectors:

- vector + vector →
- scalar \( \cdot \) vector →
- point - point →
- point + vector →
- point + point →
- scalar \( \cdot \) point + scalar \( \cdot \) vector →
- scalar \( \cdot \) vector + scalar \( \cdot \) vector →
- scalar \( \cdot \) point + scalar \( \cdot \) point →

One useful combination of affine operations is: \( P(t) = P_o + tu \)

**Q:** What does this describe?
Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, scaling:

\[
\begin{bmatrix}
    x'
    y'
    z'
    1
\end{bmatrix}
= 
\begin{bmatrix}
    s_x & 0 & 0 & 0 \\
    0 & s_y & 0 & 0 \\
    0 & 0 & s_z & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix}
\]
Translation in 3D

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & t_x \\
0 & 1 & 0 & t_y \\
0 & 0 & 1 & t_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
Rotation in 3D

These are the rotations about the canonical axes:

\[ R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?
Shearing in 3D

Shearing is also more complicated. Here is one example:

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  1
\end{bmatrix} =
\begin{bmatrix}
  1 & b & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]

We call this a shear with respect to the x-z plane.
Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)

\[
\begin{align*}
\text{ratio} &= \frac{\|pq\|}{\|qr\|} = \frac{s}{t} = \frac{\|p'q'\|}{\|q'r'\|}
\end{align*}
\]
Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.