Affine transformations

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CSE 457
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Reading

Optional reading:

- Angel 3.1, 3.7-3.11
- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \mathbf{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

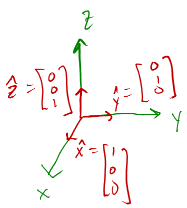
We'll focus on transformations that can be represented easily with matrix operations.

Vector representation

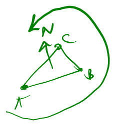
We can represent a **point**, $\mathbf{p} = (x, y)$, in the plane or $\mathbf{p} = (x, y, z)$ in 3D space

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- ~~as row vectors~~ $\cancel{\begin{bmatrix} x & y \end{bmatrix}}$ $\cancel{\begin{bmatrix} x & y & z \end{bmatrix}}$

Canonical axes



right-handed coord. systems



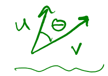
A, B, C

$$v = \begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

$$v_x\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}v_x\\0\\0\end{bmatrix}$$

$$\hat{x} \times \hat{y} = \hat{z}$$

Vector length and dot products



$$v = \begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix} \qquad \|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$u \cdot v \equiv u_x v_x + u_y v_y + u_z v_z$$

$$\|v\|^2 = v \cdot v$$

$$u \cdot v \overset{?}{=} v \cdot u \ \checkmark$$

$$u \cdot v = u^T v = \begin{bmatrix}u_x & u_y & u_z\end{bmatrix}\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}$$

$$v \cdot u = v^T u$$

$$\hat{v} = \frac{v}{\|v\|} \qquad \|\hat{v}\| = 1$$

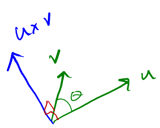
$$u \cdot v = \|u\|\|v\| \cos\theta$$

$$u \cdot v = 0 \Rightarrow u \perp v \quad (\theta = 90^\circ)$$

$\{u \neq 0, v \neq 0\}$

$$\hat{u} \cdot \hat{v} = \cos\theta$$

Vector cross products



$$u \times v \overset{det}{=} \begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ u_x & u_y & u_z \\ v_x & v_y & v_z\end{bmatrix} = (u_y v_z - u_z v_y)\hat{x} + (u_z v_x - u_x v_z)\hat{y} + (u_x v_y - u_y v_x)\hat{z} = \begin{bmatrix}u_y v_z - u_z v_y\\ \\ \\ \end{bmatrix}$$

$$u, v \perp u \times v$$

$$(u \times v) \cdot u = 0$$

$$(u \times v) \cdot v = 0$$

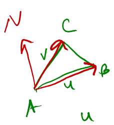
$$u \times v = -v \times u$$

$$u \times v = \|u\|\|v\| \sin\theta \, \hat{N}$$

$$\|u \times v\| = \|u\|\|v\| |\sin\theta|$$

$$u \times v = 0 \Rightarrow u \parallel v$$

$u \neq 0, v \neq 0$ $\qquad u = \alpha v$



$$N = (B - A) \times (C - A)$$

$$\hat{N} = \frac{N}{\|N\|}$$

$\Delta_{u,v}$

$$\|u \times v\| = Area(\square_{u,v})$$

$$\tfrac{1}{2}\|u \times v\| = Area(\Delta_{u,v})$$

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} =$$

$$A \cdot A^{-1} = I$$

If **p** is a column vector, M goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}ax + by\\ cx + dy\end{bmatrix}$$

us

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix}x' & y'\end{bmatrix} = \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}a & c\\b & d\end{bmatrix}$$

We will use **column vectors**.

$$(AB)^{-1}(AB) = I$$

$$(AB)^{-1}AB = I$$

$$(AB)^{-1}A = B^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix M:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements $a, b, c, d\ldots$

Identity

Suppose we choose $a=d=1$, $b=c=0$:

- Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling

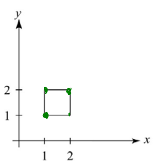
Suppose we set $b=c=0$, but let a and d take on any *positive* value:

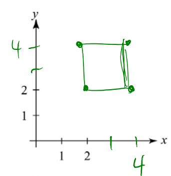
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Provides **differential (non-uniform) scaling** in x and y:

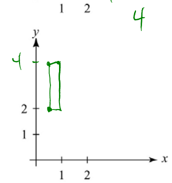
$$x' = ax$$
$$y' = dy$$





$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$$

$x' = 2x$
$y' = 2y$



$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

$x' = \frac{1}{2}x$
$y' = 2\cancel{x}y$

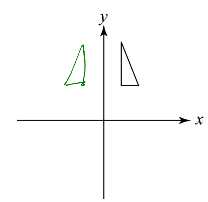
Mirror or reflection

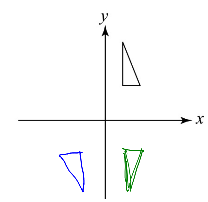
Suppose we keep $b=c=0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{rotation by } 180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$





Shear

Now let's leave $a = d = 1$ and experiment with b...

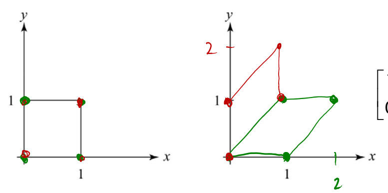
The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x+y \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

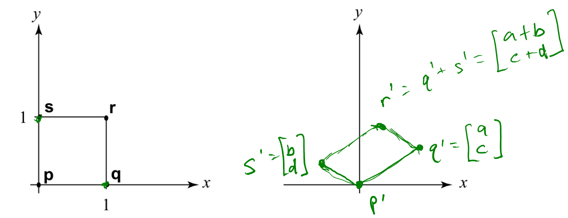
Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \mathbf{p} & \mathbf{q} & \mathbf{r} & \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{p'} & \mathbf{q'} & \mathbf{r'} & \mathbf{s'} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$

p q r s p' q' r' s'



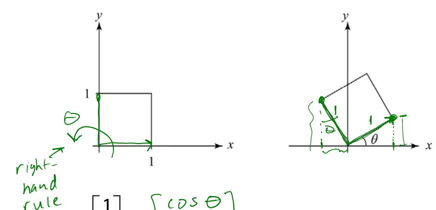
Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

translation

Homogeneous coordinates

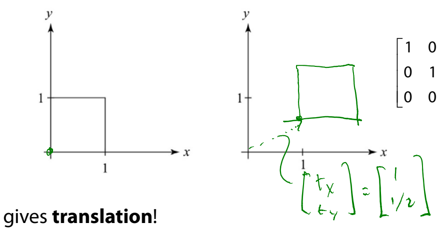
Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\underset{q}{\begin{bmatrix} x \\ y \end{bmatrix}} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "w" component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+1 \\ y+1/2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

. . . gives **translation**!

Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + t_x \\ cx + dy + t_y \\ 1 \end{bmatrix}$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added w-coordinate which is always 1:

$$P_{lin} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P_{aff} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A\,P_{lin} + t \\ 1 \end{bmatrix}$$

Rotation about arbitrary points

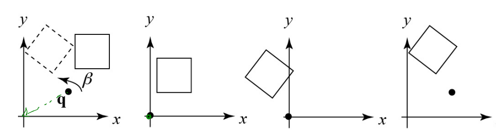
Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation by β, about any point $\mathbf{q} = [q_x \; q_y]^T$ with a matrix.

Let's do this with rotation and translation matrices of the form R(θ) and T($\mathbf{t}$), respectively.

$$T(t) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



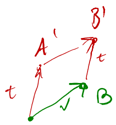
$M \neq T(-q) \quad R(\beta) \quad T(q)$

1. Translate **q** to origin
2. Rotate
3. Translate back

$$\underbrace{T(q)\, R(\beta)\, T(-q)}_{M \checkmark} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

order is important!

Points and vectors



$B = \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} A_x \\ B_y \\ 1 \end{bmatrix}$

Vectors have an additional coordinate of $\underline{w=0}$. Thus, a change of origin has no effect on vectors.

Q: What happens if we multiply a vector by an affine matrix?

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \\ 0 \end{bmatrix}$$

$B - A = \begin{bmatrix} B_x - A_x \\ B_y - A_y \\ 0 \end{bmatrix}$

$//$

$B' - A' = \begin{bmatrix} B_x + t_x \\ B_y + t_y \\ 1 \end{bmatrix} - \begin{bmatrix} A_x + t_x \\ A_y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} B_x - A_x \\ B_y - A_y \\ 0 \end{bmatrix}$

These representations reflect some of the rules of affine operations on points and vectors:

vector + vector	→	vector
scalar · vector	→	vector
point - point	→	vector
point + vector	→	point
point + point	→	chaos
scalar · vector + scalar · vector	→	vector
scalar · point + scalar · point	→	it depends

$\alpha \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} + \beta \begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha A_x + \beta B_x \\ \alpha A_y + \beta B_y \\ \alpha + \beta \end{bmatrix}$

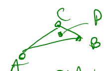
$\alpha + \beta = 1 \Rightarrow$ point

$\alpha + \beta = 0 \Rightarrow$ vector

else $\Rightarrow$ chaos

One useful combination of affine operations is:

$$\mathbf{p}(t) = \mathbf{p}_o + \underline{t}\mathbf{u}$$



$\alpha A + \beta B + \gamma C$

$\alpha + \beta + \gamma = 1 \Rightarrow$ point

Q: What does this describe? $t \in [-\infty, \infty) \Rightarrow$ line

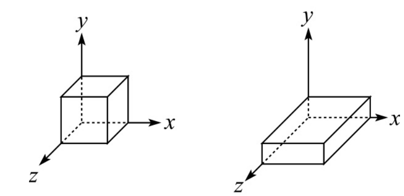
$t \in [0, \infty) \Rightarrow$ half-line ray

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

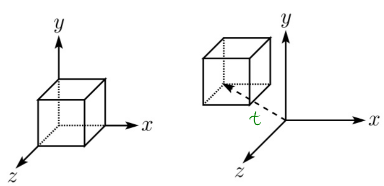
For example, <u>scaling</u>:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} I & t \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



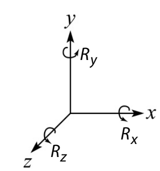
Rotation in 3D (cont'd)

These are the rotations about the canonical axes:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

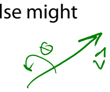
$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

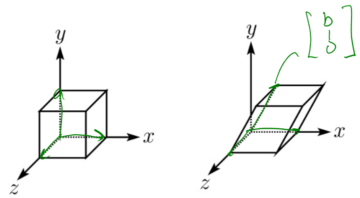


.... quaternions

Shearing in 3D

Shearing is also more complicated. Here is one example:

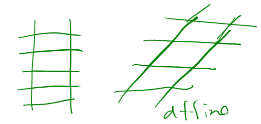
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



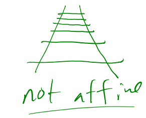
We call this a shear with respect to the x-z plane.

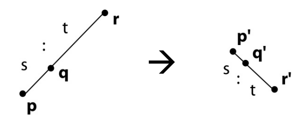
Properties of affine transformations

Here are some useful properties of affine transformations:



- Lines map to lines
- Parallel lines remain parallel
  - (when transforming from N dimensions to N dimensions)
- Midpoints map to midpoints (in fact, ratios are always preserved)





$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Affine transformations in OpenGL

OpenGL maintains a "modelview" matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` **M ← I**
  - set **M** to identity
- `glTranslatef(`t_x`,` t_y`,` t_z`)` **M ← MT**
  - translate by (t_x, t_y, t_z)
- `glRotatef(`θ`, x, y, z)` **M ← MR**
  - rotate by angle θ about axis (x, y, z)
- `glScalef(`s_x`,` s_y`,` s_z`)` **M ← MS**
  - scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.