

Parametric surfaces

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CSE 457
Spring 2016

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Reading

Required:

- Angel readings for "Parametric Curves" lecture, with emphasis on 10.1.2, 10.1.3, 10.1.5, 10.6.2, 10.7.3, 10.9.4.

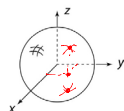
Optional

- Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987.

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Mathematical surface representations

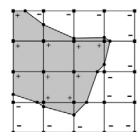
- Explicit $z=f(x, y)$ (a.k.a., a "height field")
 - what if the curve isn't a function, like a sphere?



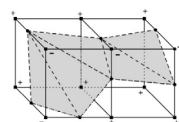
- Implicit $g(x, y, z) = 0$

$$x^2 + y^2 + z^2 = 1$$

$$g(x, y, z) = x^2 + y^2 + z^2 - 1$$



Isocontour from "marching squares"

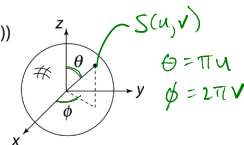


Isocontour from "marching cubes"

- Parametric $S(u, v) = (x(u, v), y(u, v), z(u, v))$

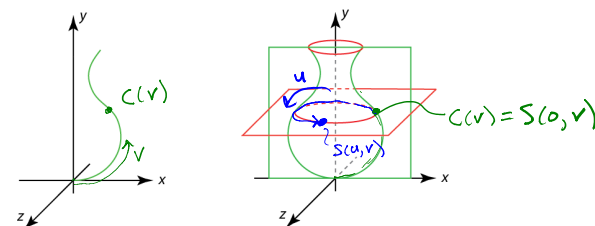
- For the sphere:
 - $x(u, v) = r \cos 2\pi v \sin \pi u$
 - $y(u, v) = r \sin 2\pi v \sin \pi u$
 - $z(u, v) = r \cos \pi u$

As with curves, we'll focus on parametric surfaces.



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Constructing surfaces of revolution



Given: A curve $C(v)$ in the xy -plane:

$$C(v) = \begin{bmatrix} C_x(v) \\ C_y(v) \\ 0 \\ 1 \end{bmatrix}$$

Let $R_y(\theta)$ be a rotation about the y -axis.

Find: A surface $S(u, v)$ which is $C(v)$ rotated about the y -axis, where $u, v \in [0, 1]$.

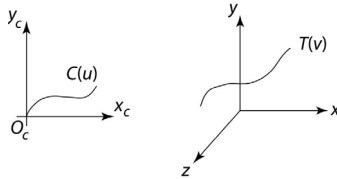
Solution: $S(u, v) = R_y(2\pi u)C(v)$

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General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u, v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



More specifically:

- Suppose that $C(u)$ lies in an (x_c, y_c) coordinate system with origin O_c .
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

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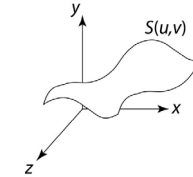
Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

- Fixed (or static):** Just translate O_c along $T(v)$.



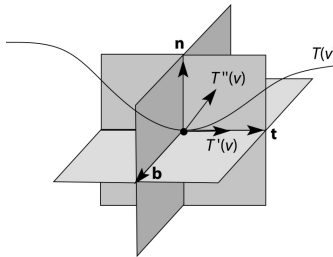
- Moving. Use the **Frenet frame** of $T(v)$.

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.

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Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

Tangent: $\mathbf{t}(v) = \text{normalize}[T'(v)]$

Binormal: $\mathbf{b}(v) = \text{normalize}[T'(v) \times T''(v)]$

Normal: $\mathbf{n}(v) = \mathbf{b}(v) \times \mathbf{t}(v)$

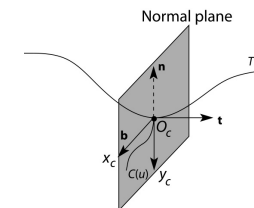
As we move along $T(v)$, the Frenet frame $(\mathbf{t}, \mathbf{b}, \mathbf{n})$ varies smoothly.

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Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

- Put $C(u)$ in the **normal plane**.
- Place O_c on $T(v)$.
- Align x_c for $C(u)$ with \mathbf{b} .
- Align y_c for $C(u)$ with $-\mathbf{n}$.

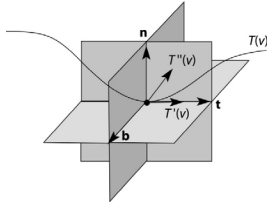


If $T(v)$ is a circle, you get a surface of revolution exactly!

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Degenerate frames

Let's look back at where we computed the coordinate frames from curve derivatives:



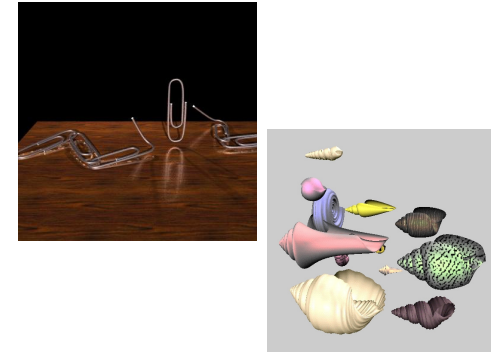
Where might these frames be ambiguous or undetermined?

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Variations

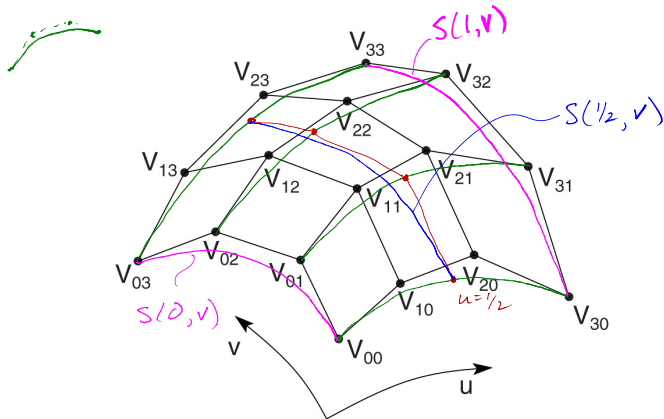
Several variations are possible:

- ♦ Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- ♦ Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- ♦ ...



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Tensor product Bézier surfaces



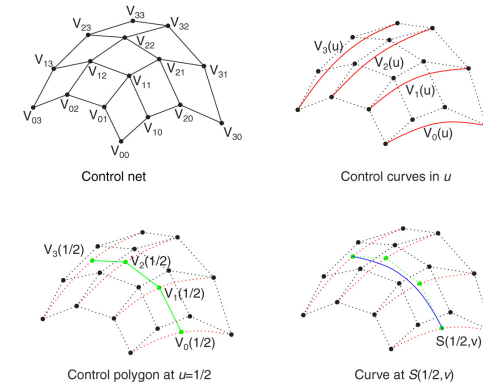
Given a grid of control points V_{ij} , forming a **control net**, construct a surface $S(u, v)$ by:

- ♦ treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0(u), \dots, V_n(u)$.
- ♦ treating $V_0(u), \dots, V_n(u)$ as control points for a curve parameterized by v .

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Tensor product Bézier surfaces, cont.

Let's walk through the steps:



Which control points are always interpolated by the surface?

4 corners

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Polynomial form of Bézier surfaces

Recall that cubic Bézier *curves* can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^n V_i b_i(u)$$

A tensor product Bézier surface can be written as:

$$S(u,v) = \sum_{i=0}^n \sum_{j=0}^n V_{ij} b_i(u) b_j(v)$$

In the previous slide, we constructed curves along u , and then along v . This corresponds to re-grouping the terms like so:

$$S(u,v) = \sum_{j=0}^n \left(\sum_{i=0}^n V_{ij} b_i(u) \right) b_j(v)$$

But, we could have constructed them along v , then u :

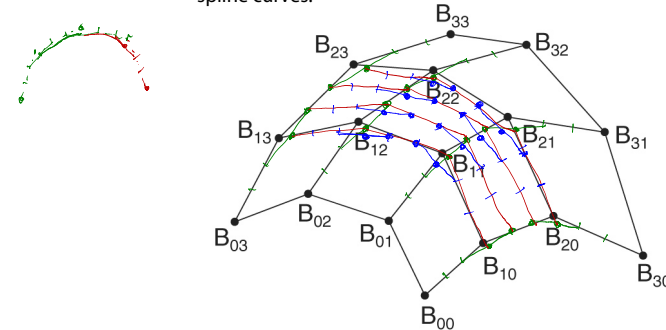
$$S(u,v) = \sum_{i=0}^n \left(\sum_{j=0}^n V_{ij} b_j(v) \right) b_i(u)$$

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Tensor product B-spline surfaces

B-spline → Bézier → curve / surface

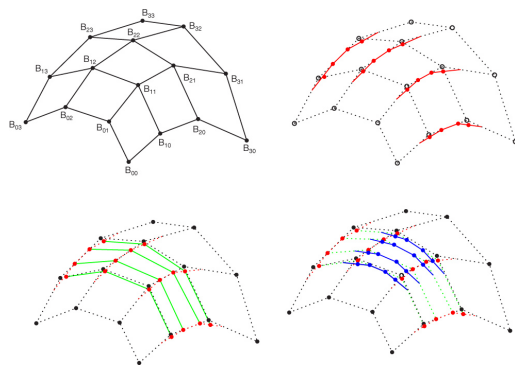
As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- treat rows of B as control points to generate Bézier control points in u .
- treat Bézier control points in u as B-spline control points in v .
- treat B-spline control points in v to generate Bézier control points in u .

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Tensor product B-spline surfaces, cont.



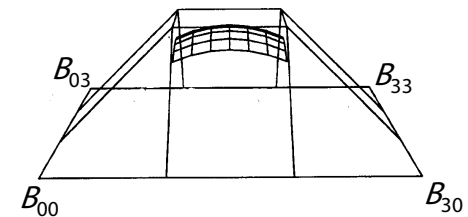
Which B-spline control points are always interpolated by the surface?

None

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Tensor product B-splines, cont.

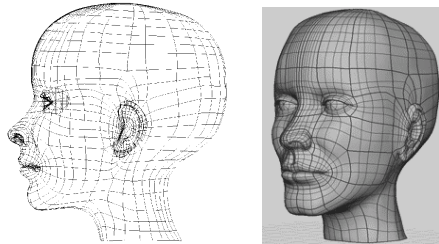
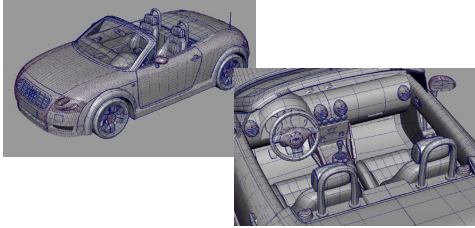
Another example:



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NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

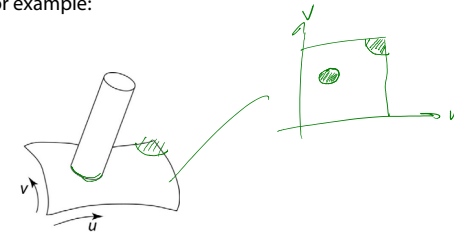


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Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:



We can do this by **trimming** the u - v domain.

- Define a closed curve in the u - v domain (a **trim curve**)
- Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

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Summary

What to take home:

- ♦ How to construct swept surfaces from a profile and trajectory curve:
 - with a fixed frame
 - with a Frenet frame
- ♦ How to construct tensor product Bézier surfaces
- ♦ How to construct tensor product B-spline surfaces

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