# Affine transformations

**Brian Curless**
**CSE 457**
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## Reading

Required:

- Angel 3.1, 3.7-3.11

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

## Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

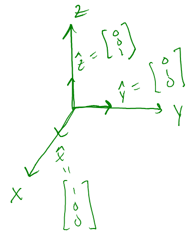
## Vector representation

We can represent a **point**, $\mathbf{p} = (x, y)$, in the plane or $\mathbf{p} = (x, y, z)$ in 3D space

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

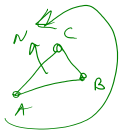
- ~~as row vectors $\begin{bmatrix} x & y \end{bmatrix}$ $\begin{bmatrix} x & y & z \end{bmatrix}$~~

## Canonical axes



$\hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

right-handed coordinate systems

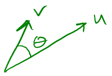


$\hat{x} \times \hat{y} = \hat{z}$

$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$

## Vector length and dot products

$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ $u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$



$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ $v \cdot v = \|v\|^2$

$u \cdot v \equiv u_x v_x + u_y v_y + u_z v_z$

$u \cdot v = v \cdot u$ ✓

$u \cdot v = u^T v = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$

$v \cdot u = v^T u$

$u \cdot v = \|u\| \|v\| \cos\theta$

$u \cdot v = 0 \iff u \perp v$ perpendicular orthogonal

$(\|u\| \neq 0, \|v\| \neq 0)$

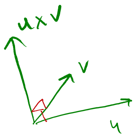
$\|u\| = \|v\| = 1 \Rightarrow u \cdot v = \cos\theta$

$\hat{w} \leftarrow \frac{w}{\|w\|}$ vector normalization

$\|\hat{w}\| = 1$

"direction"

## Vector cross products

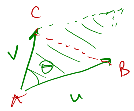


$u \times v \overset{det}{\equiv} \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = (u_y v_z - u_z v_y)\hat{x} + (u_z v_x - u_x v_z)\hat{y} + (u_x v_y - u_y v_x)\hat{z}$

$(u \times v) \cdot u = 0$

$(u \times v) \cdot v = 0$

$u \times v = -v \times u$



$\|u \times v\| = \|u\| \|v\| \sin\theta = \text{Area}(\square_{u,v}) = 2\,\text{Area}(\Delta_{u,v})$ $\Delta_{ABC}$

$N_{ABC} \sim u \times v$

$\|\hat{u} \times \hat{v}\| = 1 \iff 90^\circ$ $u \perp v$

$\|u \times v\| = 0 \Rightarrow u \cancel{\propto} \alpha v$

## Representation, cont.

We can represent a **2-D transformation** *M* by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$(AB)^T = B^T A^T$

$u \cdot v = u^T v = v^T u$

If **p** is a column vector, *M* goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$(AB)^{-1} = B^{-1} A^{-1}$

If **p** is a row vector, $M^T$ goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ax + by & cx + dy \end{bmatrix}$$

We will use **column vectors**.

## Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix *M*:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d*…

## Identity

Suppose we choose $a=d=1, b=c=0$:

- Gives the **identity** matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad x' = x \quad y' = y$$

- Doesn't move the points at all

## Scaling

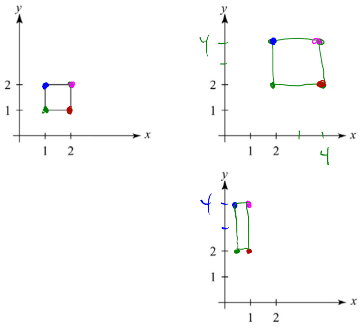
Suppose we set $b=c=0$, but let *a* and *d* take on any *positive* value:

- Gives a **scaling** matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Provides **differential (non-uniform) scaling** in *x* and *y*:

$$x' = ax$$
$$y' = dy$$



$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ← $a=d$ uniform scale

$x' = 2x$
$y' = 2y$

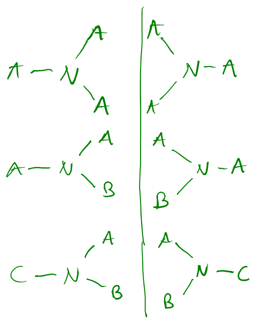
$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$ non-uniform scale

$a \neq d$

$x' = 1/2x$
$y' = 2y$

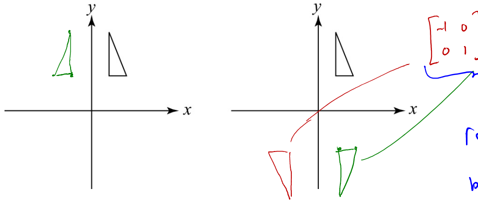
## Reflection (mirror)

Suppose we keep $b=c=0$, but let either *a* or *d* go negative.



Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{rotation by } 180^\circ} \begin{bmatrix} x \\ y \end{bmatrix}$$

Chiral center
(organic chem.)

## Shear

Now let's leave *a*=*d*=*1* and experiment with *b*. . .

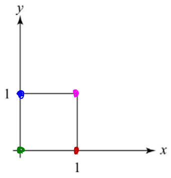
The matrix

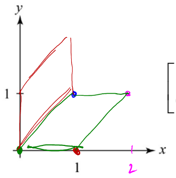
$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$





$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

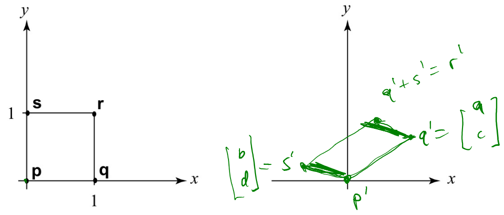
$$x' = x + y$$
$$y' = y$$

## Effect on unit square

Let's see how a general 2 x 2 transformation *M* affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [\mathbf{p} \quad \mathbf{q} \quad \mathbf{r} \quad \mathbf{s}] = [\mathbf{p'} \quad \mathbf{q'} \quad \mathbf{r'} \quad \mathbf{s'}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$



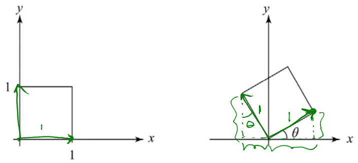
## Effect on unit square, cont.

Observe:

- Origin invariant under *M*
- *M* can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- *a* and *d* give *x*- and *y*-scaling
- *b* and *c* give *x*- and *y*-shearing

## Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

## Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

**Q**: What important operation does that leave out?

translation

## Homogeneous coordinates

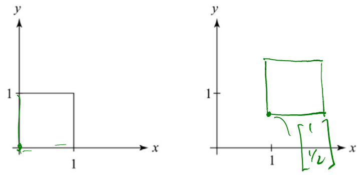
Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "*w*" component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

. . . gives **translation**!

## Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + t_x \\ cx + dy + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \\ 1 \end{bmatrix}$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added *w*-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

## Rotation about arbitrary points

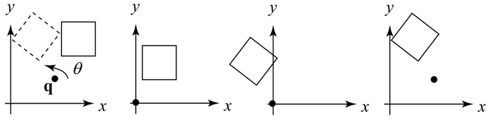
$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, $\theta$, about any point $\mathbf{q} = [q_x \ q_y]^T$ with a matrix.

$T(t_x, t_y)$

$T(\mathbf{t})$



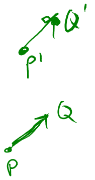
$M \neq \overrightarrow{T(-q)}$ $R(\theta)$ $T(q)$

$M = T(q)R(\theta)T(-q)$

1. Translate **q** to origin
2. Rotate
3. Translate back

<u>Note</u>: Transformation order is important!!

## Points and vectors



Vectors have an additional coordinate of $w = 0$. Thus, a change of origin has no effect on vectors.

**Q**: What happens if we multiply a vector by an affine matrix?

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} av_x + bv_y \\ cv_x + dv_y \\ 0 \end{bmatrix}$$

Q-P

$$\begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix} - \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} = \begin{bmatrix} Q_x - P_x \\ Q_y - P_y \\ 0 \end{bmatrix}$$

These representations reflect some of the rules of affine operations on points and vectors:

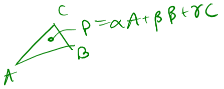
| | | |
|---|---|---|
| vector + vector | → | vector |
| scalar · vector | → | vector |
| point - point | → | vector |
| point + vector | → | point |
| point + point | → | chaos |

$\alpha P + \beta Q$

$$\alpha \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} + \beta \begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha P_x + \beta Q_x \\ \alpha P_y + \beta Q_y \\ \alpha + \beta \end{bmatrix}$$

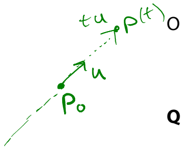
$\alpha + \beta = 1 \Rightarrow$ point

$\alpha + \beta = 0 \Rightarrow$ vectr



$p = \alpha A + \beta B + \gamma C$

One useful combination of affine operations is:



$$\mathbf{p}(t) = \mathbf{p}_o + t\mathbf{u}$$

**Q**: What does this describe?

$t \in (-\infty, \infty) \Rightarrow$ line

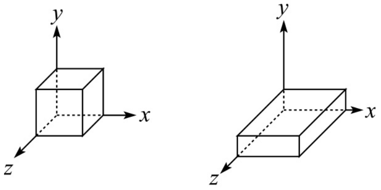
$t \in [0, \infty) \Rightarrow$ ray (half-line)

## Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, <u>scaling</u>:

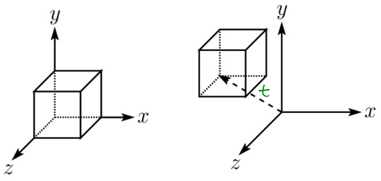
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



## Translation in 3D

$t$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



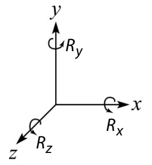
## Rotation in 3D (cont'd)

These are the rotations about the canonical axes:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

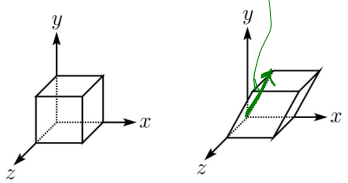
A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

"quaternion" → specifies rotation about arbitrary direction

## Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

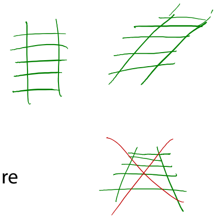


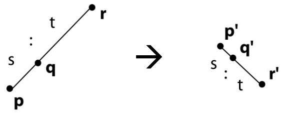
We call this a shear with respect to the x-z plane.

## Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)





$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

## Affine transformations in OpenGL

OpenGL maintains a "modelview" matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` **M** ← **I**
  - set **M** to identity
- glTranslatef($t_x$, $t_y$, $t_z$) **M** ← **MT**
  - translate by ($t_x$, $t_y$, $t_z$)
- `glRotatef(θ, x, y, z)` **M** ← **MR**
  - rotate by angle θ about axis (x, y, z)
- glScalef($s_x$, $s_y$, $s_z$) **M** ← **MS**
  - scale by ($s_x$, $s_y$, $s_z$)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

## Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.