NURBS & Parametric Surfaces

CSE 457

Rational polynomial curves

Remarkably, parametric polynomial curves **cannot** represent something as simple as a circle!

BUT, ratios of polynomials can. We can write these in terms of homogeneous coordinates, which we then normalize:

$$Q(u) = \begin{bmatrix} \sum_{k=0}^{n} a_k u^k \\ \sum_{k=0}^{n} b_k u^k \\ \sum_{k=0}^{n} c_k u^k \end{bmatrix} \xrightarrow[\div \sum_{k=0}^{n} c_k u^k]{\text{Normalize}} \tilde{Q}(u) = \begin{bmatrix} \sum_{k=0}^{n} a_k u^k \Big/ \sum_{k=0}^{n} c_k u^k \\ \sum_{k=0}^{n} b_k u^k \Big/ \sum_{k=0}^{n} c_k u^k \\ 1 \end{bmatrix}$$

The equations above describe a **rational Bézier** curve.

It can be represented in terms of control points, but now we add the homogenous dimension. So for a 2D curve, we have control points with *three* components (lofted up into 3D), where the homogenous component can be something other than 1.

Rational polynomial curves (cont'd)

What do we get for the following curve?

$$Q(u) = \begin{bmatrix} 2u \\ 1-u^2 \\ 1+u^2 \end{bmatrix}$$

Q: How does Illustrator represent a circle?

NURBS

In general, we can spline together rational Bézier curves, to get things like **rational B-splines**.

Another thing we can do is vary the range of *u* so that it is not always [0..1] in each Bézier segment of a spline. E.g, it could be [0..1] in one segment and then [0..2] in the next.

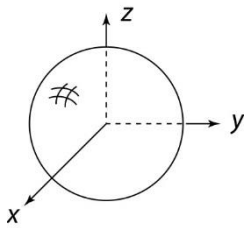
The *u*-range affects placement of control points. The result is a **non-uniform** spline.

A very common type of spline is a **Non-Uniform Rational B-Spline** or **NURBS**.

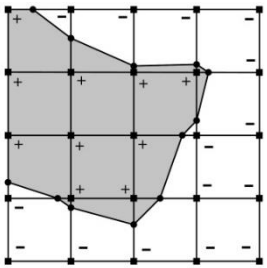
(The "B" in B-spline technically stands for "Basis.")

Mathematical surface representations

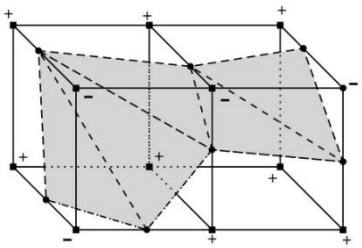
- Explicit $z=f(x,y)$ (a.k.a., a "height field")
  - what if the curve isn't a function, like a sphere?



- Implicit $g(x,y,z) = 0$



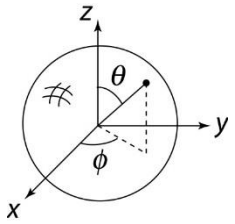
Isocontour from "marching squares"



Isocontour from "marching cubes"

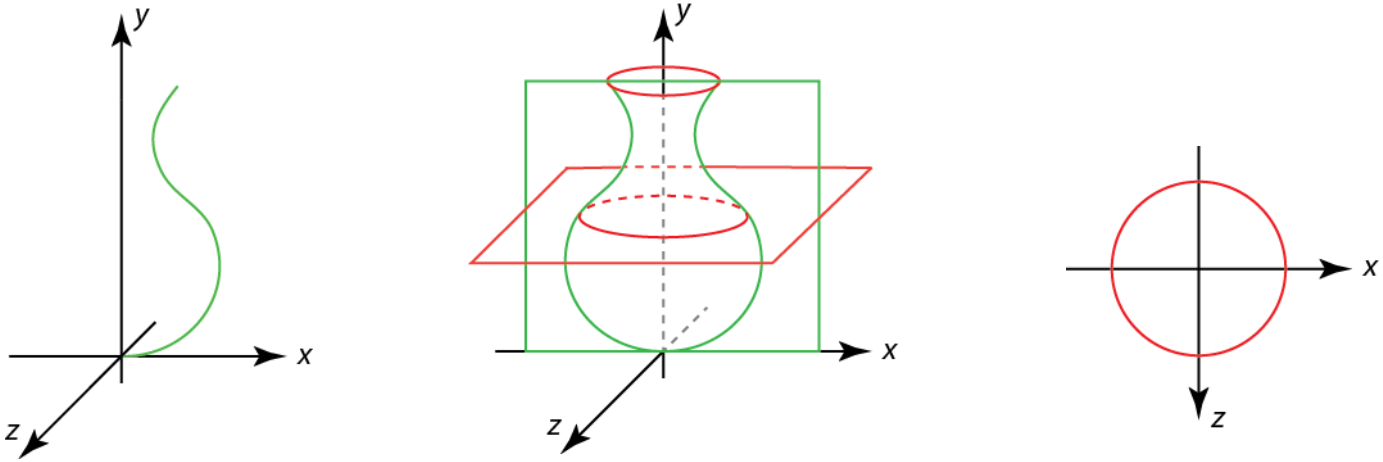
- Parametric $S(u,v)=(x(u,v),y(u,v),z(u,v))$
  - For the sphere:

$$x(u,v) = r \cos 2\pi v \sin \pi u$$
$$y(u,v) = r \sin 2\pi v \sin \pi u$$
$$z(u,v) = r \cos \pi u$$



As with curves, we'll focus on parametric surfaces.

Surfaces of revolution



Given: A curve $C(u)$ in the xy-plane:

$$C(u) = \begin{bmatrix} c_x(u) \\ c_y(u) \\ 0 \\ 1 \end{bmatrix}$$

Let $R_y(\theta)$ be a rotation about the y-axis.

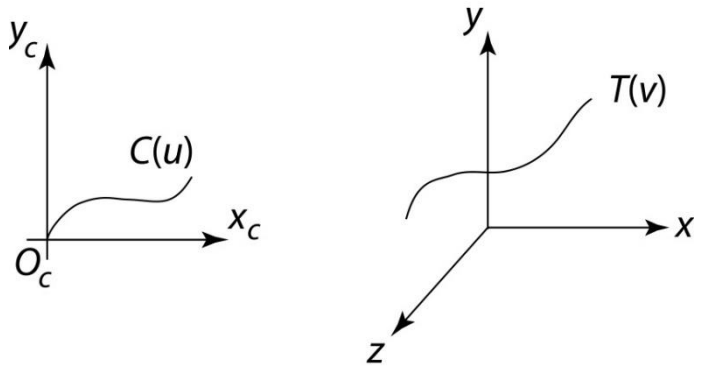
Find: A surface $S(u,v)$ which is $C(u)$ rotated about the y-axis, where $u, v \in [0, 1]$.

Solution:

General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u,v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



More specifically:

- Suppose that $C(u)$ lies in an (x_c,y_c) coordinate system with origin O_c.
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

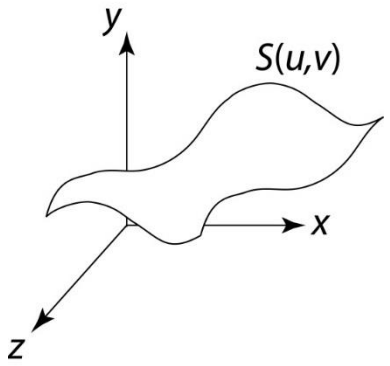
Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

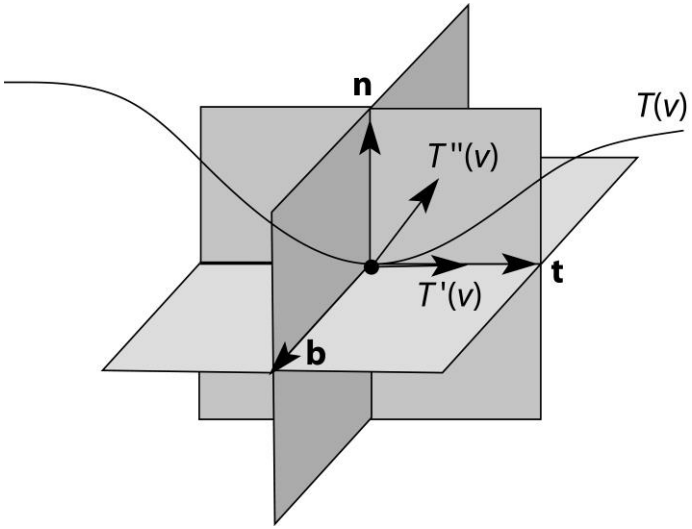
1. **Fixed** (or **static**): Just translate O_c along $T(v)$.



2. Moving. Use the **Frenet frame** of $T(v)$.
   - Allows smoothly varying orientation.
   - Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

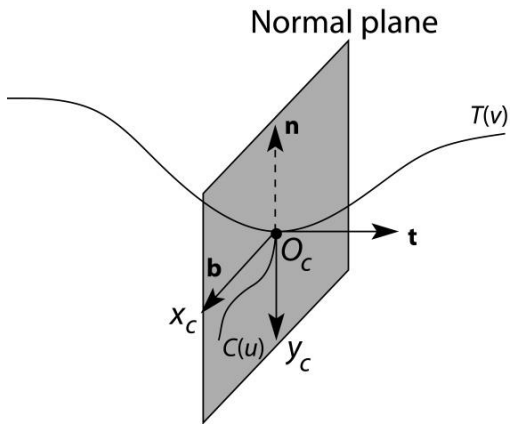
$$\text{Tangent:} \quad \mathbf{t}(v) = \text{normalize}[T'(v)]$$
$$\text{Binormal:} \quad \mathbf{b}(v) = \text{normalize}[T'(v) \times T''(v)]$$
$$\text{Normal:} \quad \mathbf{n}(v) = \mathbf{b}(v) \times \mathbf{t}(v)$$

As we move along $T(v)$, the Frenet frame (t,b,n) varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

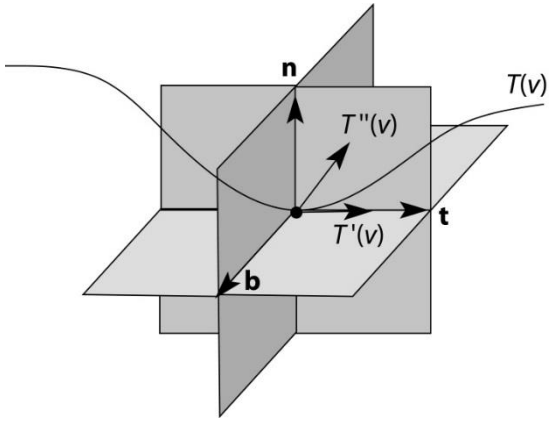
- Put $C(u)$ in the **normal plane** .
- Place O_c on $T(v)$.
- Align x_c for $C(u)$ with **b**.
- Align y_c for $C(u)$ with -**n**.



If $T(v)$ is a circle, you get a surface of revolution exactly!

Degenerate frames

Let's look back at where we computed the coordinate frames from curve derivatives:

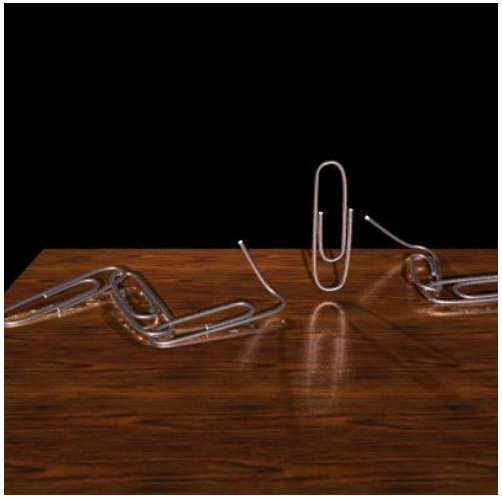


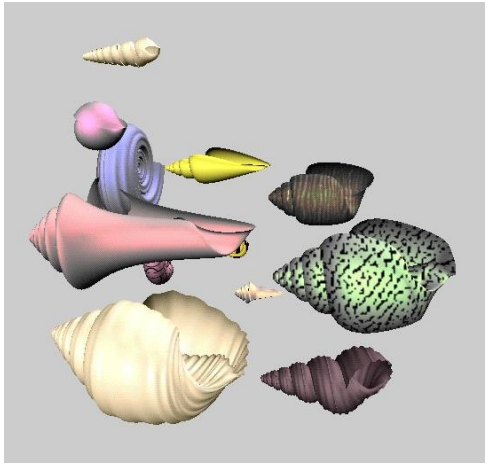
Where might these frames be ambiguous or undetermined?

Variations

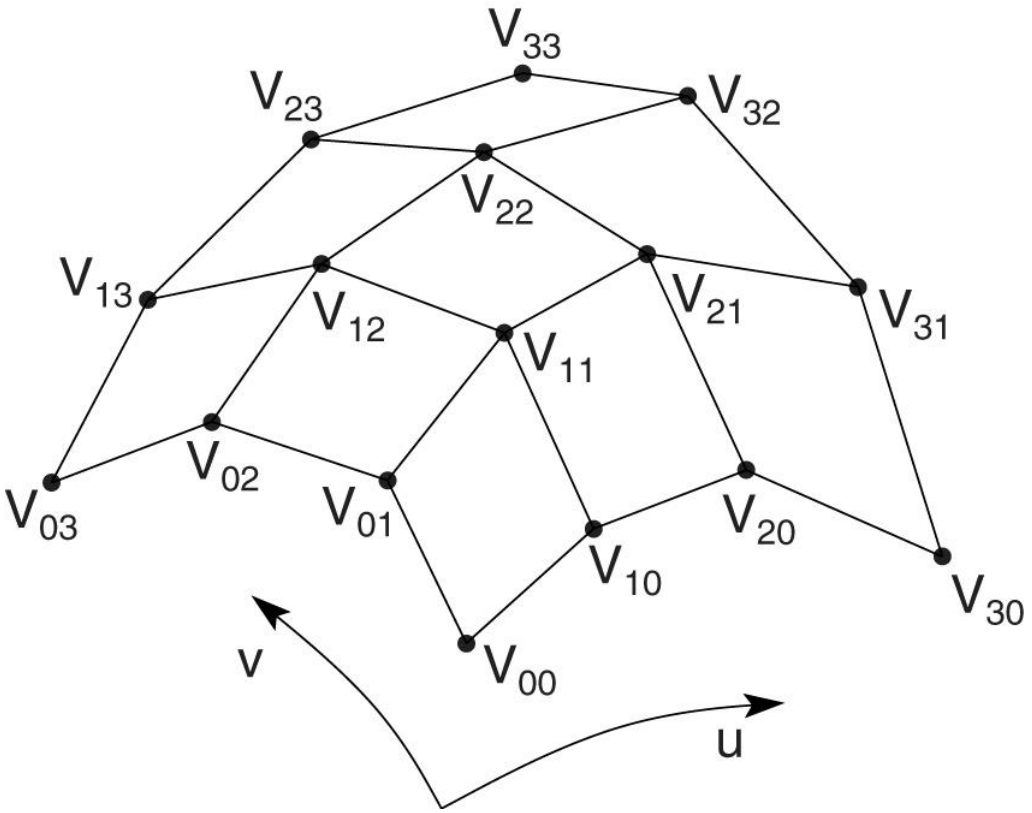
Several variations are possible:

- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- …





Tensor product Bézier surfaces

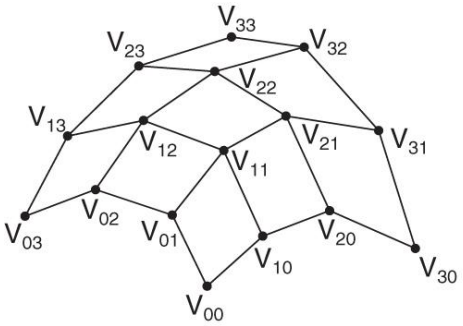


Given a grid of control points V_{ij}, forming a **control net**, construct a surface $S(u,v)$ by:

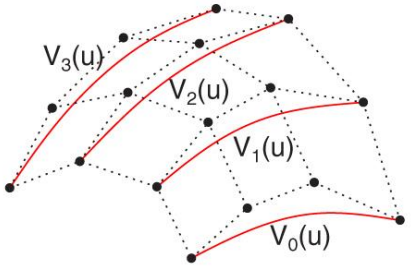
- treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0(u),\ldots, V_n(u)$.
- treating $V_0(u),\ldots, V_n(u)$ as control points for a curve parameterized by v.

Tensor product Bézier surfaces, cont.

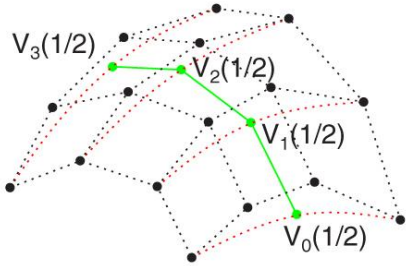
Let's walk through the steps:



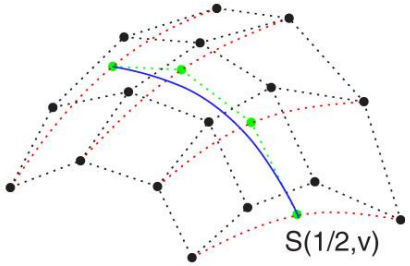
Control net



Control curves in *u*



Control polygon at u=1/2



Curve at $S(1/2, v)$

Which control points are interpolated by the surface?

Polynomial form of Bézier surfaces

Recall that cubic Bézier *curves* can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^{n} V_i b_i(u)$$

A tensor product Bézier surface can be written as:

$$S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n} V_{ij} b_i(u) b_j(v)$$

In the previous slide, we constructed curves along u, and then along v. This corresponds to re-grouping the terms like so:

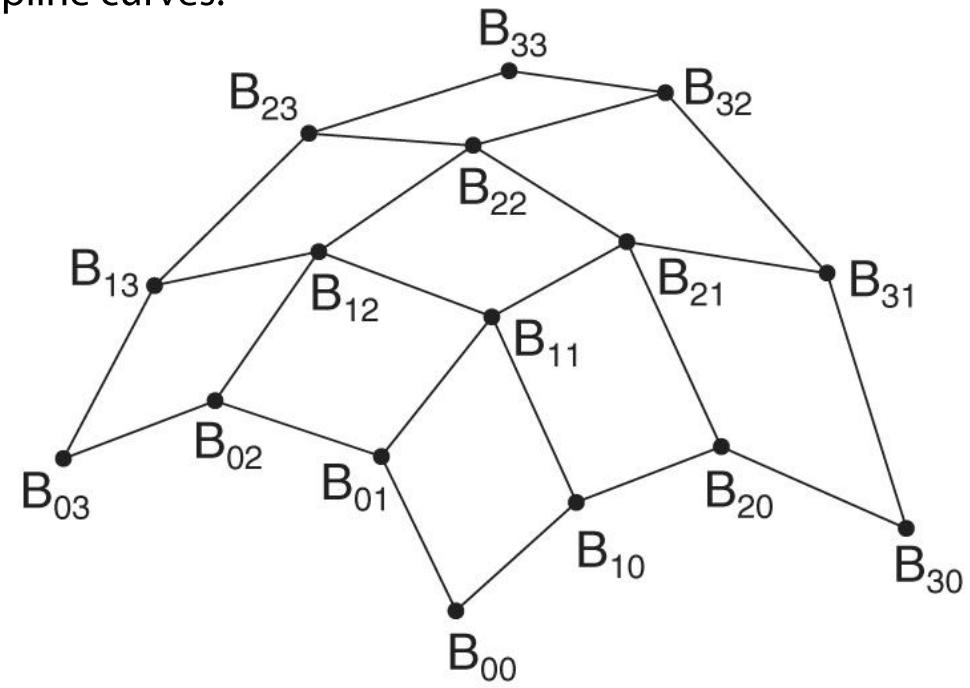
$$S(u,v) = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} V_{ij} b_i(u) \right) b_j(v)$$

But, we could have constructed them along v, then u:

$$S(u,v) = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} V_{ij} b_j(v) \right) b_i(u)$$

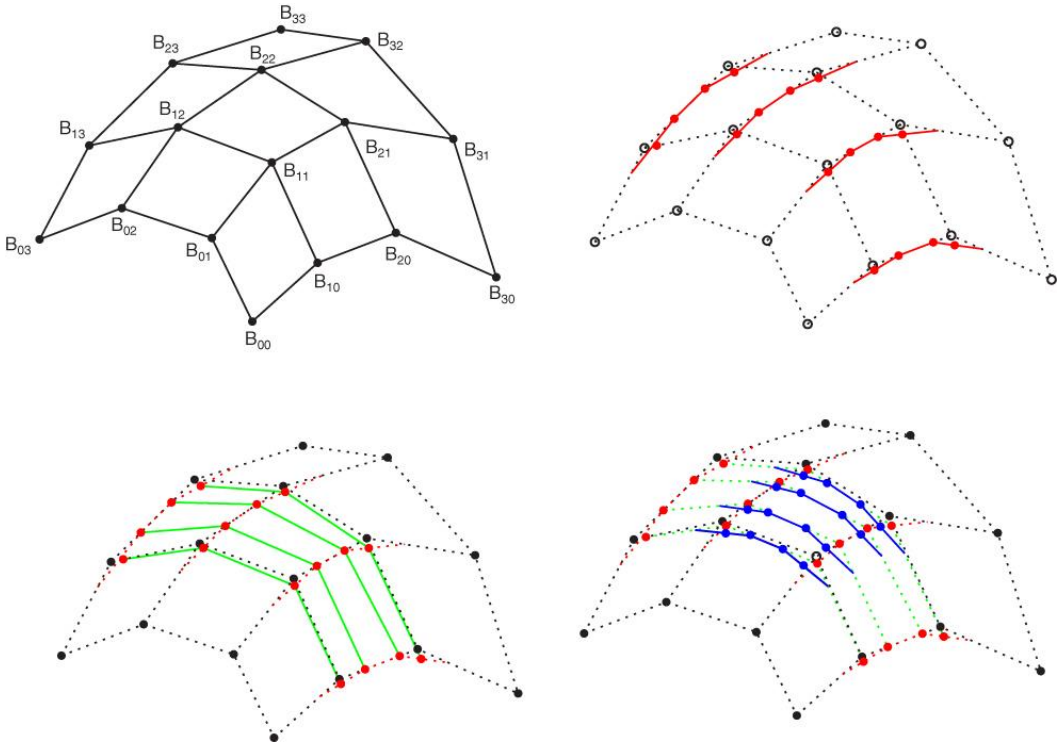
Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- treat rows of *B* as control points to generate Bézier control points in *u*.
- treat Bézier control points in *u* as B-spline control points in *v*.
- treat B-spline control points in *v* to generate Bézier control points in *u*.

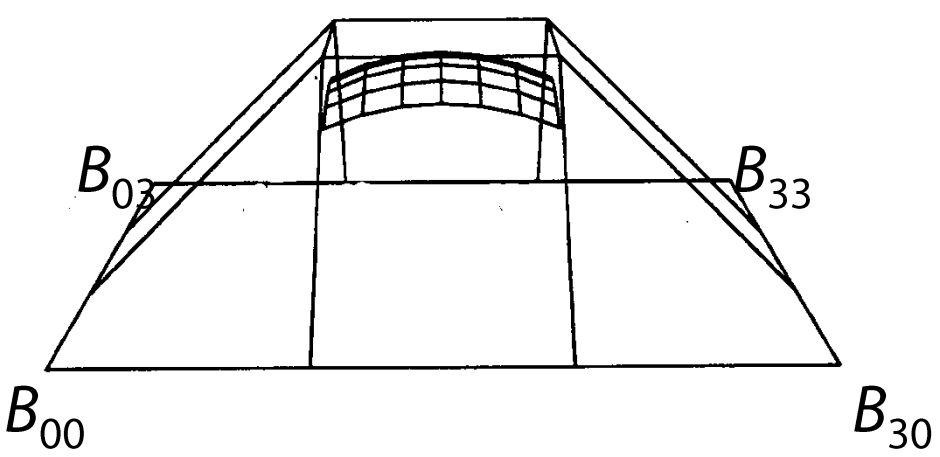
Tensor product B-spline surfaces, cont.



Which B-spline control points are interpolated by the surface?

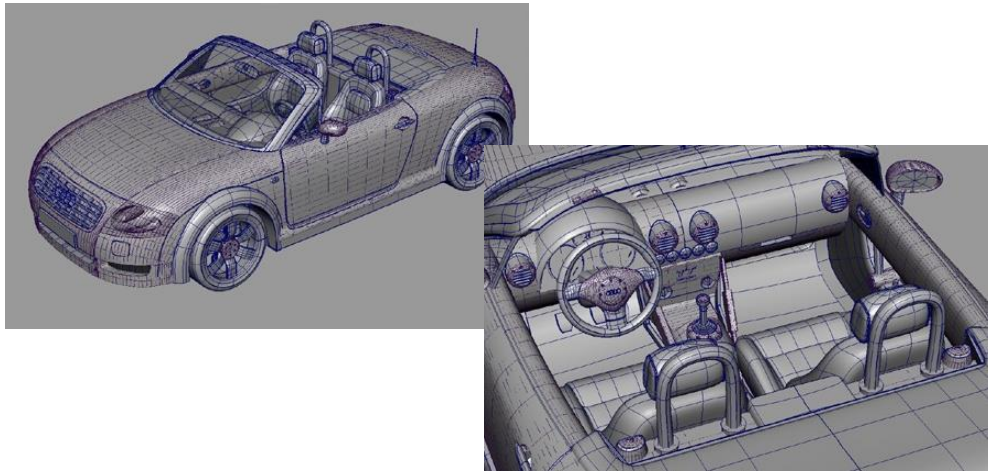
Tensor product B-splines, cont.

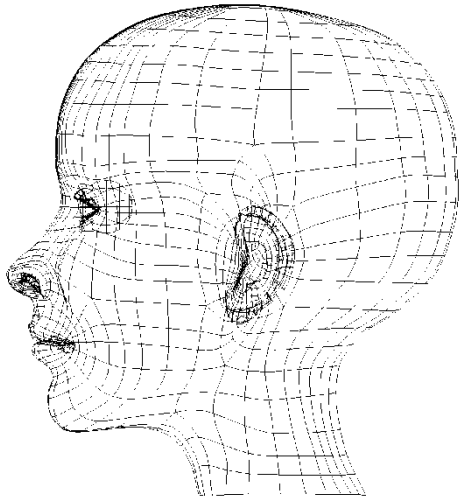
Another example:

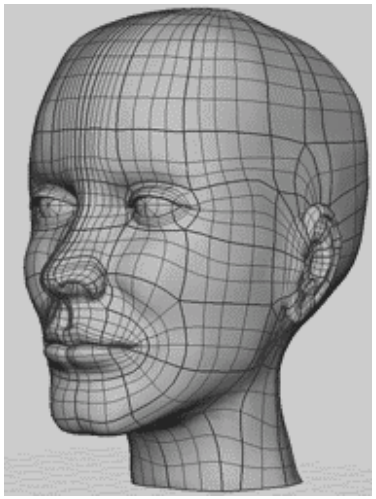


NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.



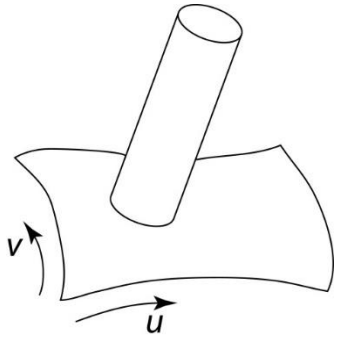




Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:



We can do this by **trimming** the *u*-*v* domain.

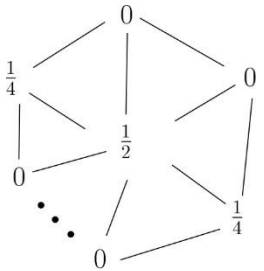
- Define a closed curve in the *u*-*v* domain (a **trim curve**)
- Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

Adding creases without trim curves

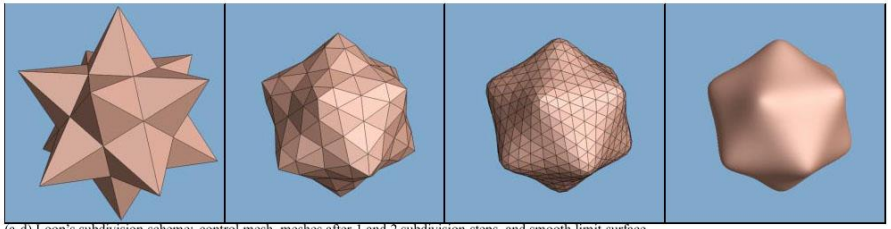
For NURBS surfaces, adding sharp features like creases required the use of trim curves.

For subdivision surfaces, we can just modify the subdivision masks. E.g., we can mark some edges and vertices as "creases" and modify the subdivision mask for them (and their children):

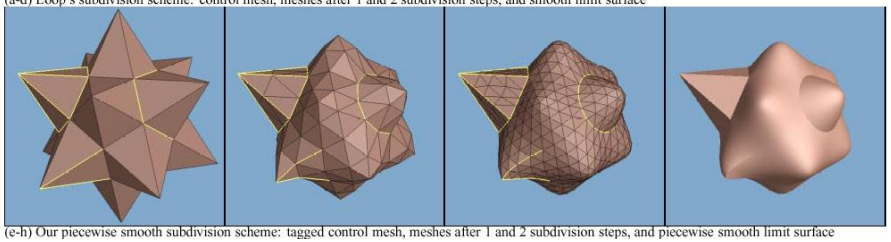


This gives rise to G^0 continuous surfaces (i.e., having positional but not tangent plane continuity).

[Hoppe, SIGGRAPH 1994]



(a-d) Loop's subdivision scheme: control mesh, meshes after 1 and 2 subdivision steps, and smooth limit surface



(e-h) Our piecewise smooth subdivision scheme: tagged control mesh, meshes after 1 and 2 subdivision steps, and piecewise smooth limit surface

Summary

What to take home:

- How to construct swept surfaces from a profile and trajectory curve:
  - with a fixed frame
  - with a Frenet frame
- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces