Parametric surfaces

Daniel Leventhal
Adapted from Brian Curless
CSE 457
Autumn 2011

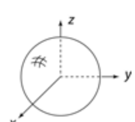
Reading

Optional

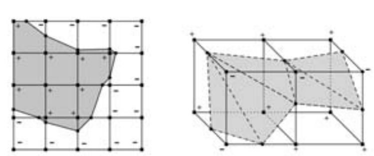
- Angel readings for "Parametric Curves" lecture, with emphasis on 12.1.2, 12.1.3, 12.1.5, 12.6.2, 12.7.3, 12.9.4.
- Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling,* 1987.

Mathematical surface representations

- Explicit z= (x,y) (a.k.a., a "height field")
  - what if the curve isn't a function, like a sphere?



- Implicit (,) = 0



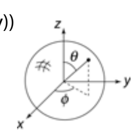
Isocontour from "marching squares" Isocontour from "marching cubes"

- Parametric $S(u,v$ x u,v y u,v),z(u,v))
  - For the sphere:

    $x(u,v) = r$ $2\pi v$ sin πu

    y(u,v) = sin 2π sin π

    z(u,v) = cos π



As with curves, we'll focus on parametric surfaces.

Surfaces of revolution

Idea: rotate a 2D **profile curve**

What kinds of shapes can you model this way?

Constructing surfaces of revolution

Given: -plane:

$$C(u) = \begin{bmatrix} c_x(u) \\ c_y(u) \\ 0 \\ 1 \end{bmatrix}$$

Let R_y θ) be a rotation about the y-axis.

Find:
-axis.

Solution:

Isoparameter curves and tangents

We can follow curves where v is constant, and u varies or vice versa. These are called **isoparameter curves** (where one parameter is held constant):

If we sample at equal spacing in and , we can create a quadrilateral mesh (or a triangle mesh).

We can compute tangents to the surface at any point by looking at (infintesimally) nearby points.

Holding one parameter constant, we can find nearby points by varying the other parameter. Thus, we can get two tangents:

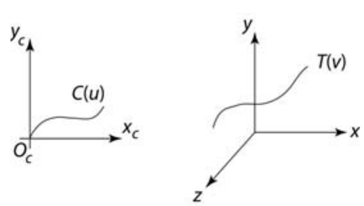
$$\mathbf{t}_u = \frac{\partial S(u,v)}{\partial u} \qquad \mathbf{t}_v = \frac{\partial S(u,v)}{\partial v}$$

How would we compute the normal?

General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**

profile
$C(u)$ along a **trajectory curve** T v



More specifically:

- Suppose that ( ) lies in an ($_c$ y_c system with origin O_c
- For every point along ( ), lay ( ) so that coincides with $T(v)$.

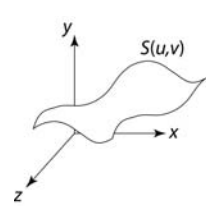
Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

1. **Fixed** **static**): Just translate along ( ).

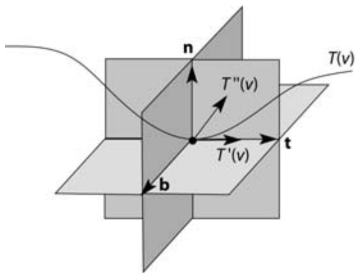


2. Moving. Use the of ( ).

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve ( ), we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

Tangent: $\mathbf{t}(v) = \text{normalize}[T'(v)]$

Binormal: $\mathbf{b}(v) = \text{normalize}[T'(v) \times T''(v)]$

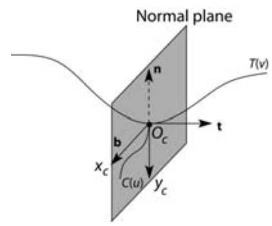
Normal: $\mathbf{n}(v) = \mathbf{b}(v) \times \mathbf{t}(v)$

As we move along *T*(*v*), the Frenet frame (*t*,*b n* smoothly.

Frenet swept surfaces

Orient the profile curve *C*(*u*) using the Frenet frame of the trajectory ( ):

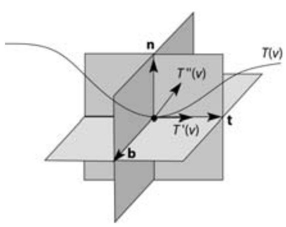
- Put ( ) in the **normal plane** .
- Place O_c on *T*(*v*).
- Align for ( ) with **b**
- Align for ( ) with -**n**.



If *T*(*v*) is a circle, you get a surface of revolution exactly!

Degenerate frames

Let's look back at where we computed the coordinate frames from curve derivatives:

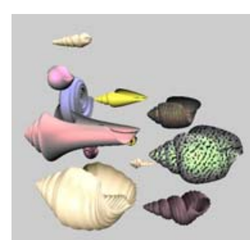


Where might these frames be ambiguous or undetermined?

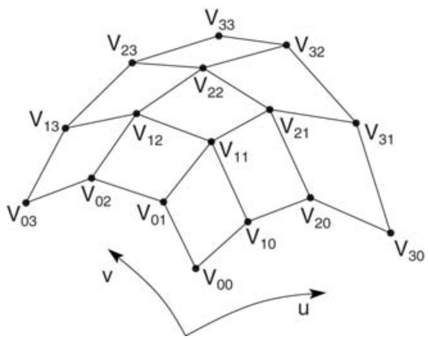
Variations

Several variations are possible:

- Scale *C*(*u*) as it moves, possibly using length of *T*(*v*) as a scale factor.
- Morph ( ) into some other curve $\tilde{C}(u)$ as it moves along *T*(*v*).
- …



Tensor product Bézier surfaces

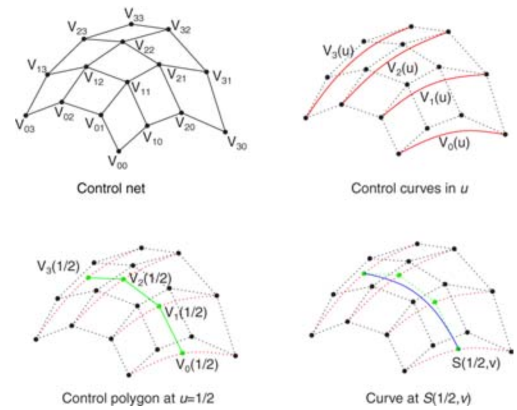


Given a grid of control points V_{ij}
construct a surface $S(u,v)$ by:

- treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0\ u$ $V_n\ u$
- treating $V_0(u),\ldots, V_n(u)$ as control points for a curve

Tensor product Bézier surfaces, cont.

Let's walk through the steps:



Which control points are interpolated by the surface?

Polynomial form of Bézier surfaces

Recall that cubic Bézier *curves* can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^{n} V_i b_i(u)$$

A tensor product Bézier surface can be written as:

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n} V_{ij} b_i(u) b_j(v)$$

In the previous slide, we constructed curves along u, and then along v. This corresponds to re-grouping the terms like so:

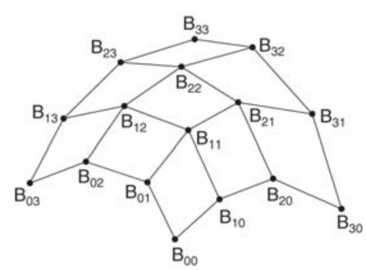
$$S(u,v) = \sum_{j=0}^{n}\left(\sum_{i=0}^{n} V_{ij} b_i(u)\right) b_j(v)$$

But, we could have constructed them along v, then u:

$$S(u,v) = \sum_{i=0}^{n}\left(\sum_{j=0}^{n} V_{ij} b_j(v)\right) b_i(u)$$

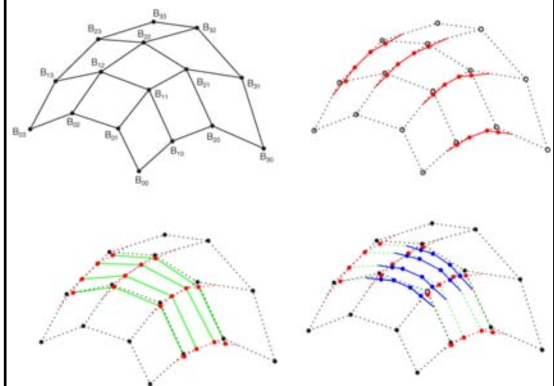
Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- treat rows of B
Bézier control points in u
- treat Bézier control points in u as B-spline control points in v
- treat B-spline control points in v to generate Bézier control points in u.

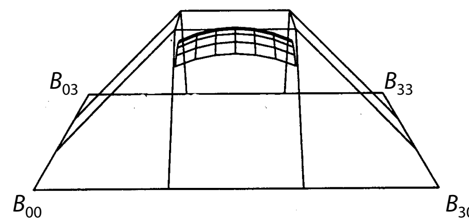
Tensor product B-spline surfaces, cont.



Which B-spline control points are interpolated by the surface?

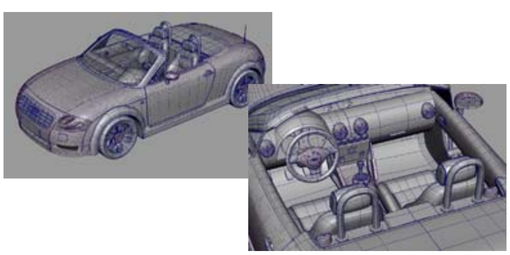
Tensor product B-splines, cont.

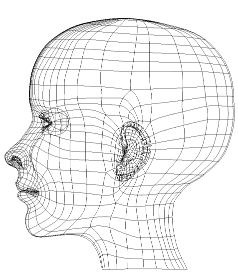
Another example:

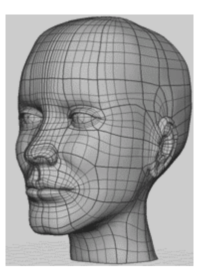


NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.



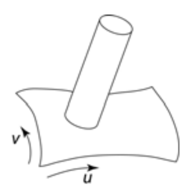




Trimmed NURBS surfaces

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:



We can do this by **trimming** -v

- **trim curve**
- Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

Summary

What to take home:

- How to construct a surface of revolution
- How to construct swept surfaces from a profile and trajectory curve:
  - with a fixed frame
  - with a Frenet frame
- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces