

Parametric surfaces

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Reading

Required:

- ♦ Angel readings for “Parametric Curves” lecture, with emphasis on 11.1.2, 11.1.3, 11.1.5, 11.6.2, 11.7.3, 11.9.4.

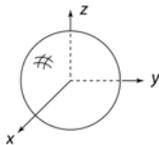
Optional

- ♦ Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987.

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Mathematical surface representations

- ♦ Explicit $z=f(x,y)$ (a.k.a., a “height field”)
 - what if the curve isn’t a function?



- ♦ Implicit $g(x,y,z) = 0$

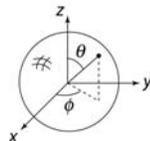
- ♦ Parametric $S(u,v)=(x(u,v),y(u,v),z(u,v))$

- For the sphere:

$$x(u,v) = r \cos 2\pi v \sin \pi u$$

$$y(u,v) = r \sin 2\pi v \sin \pi u$$

$$z(u,v) = r \cos \pi u$$



As with curves, we’ll focus on parametric surfaces.

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Surfaces of revolution

Idea: rotate a 2D **profile curve** around an axis.

What kinds of shapes can you model this way?

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Constructing surfaces of revolution

Given: A curve $C(u)$ in the xy -plane:

$$C(u) = \begin{bmatrix} c_x(u) \\ c_y(u) \\ 0 \\ 1 \end{bmatrix}$$

Let $R_x(\theta)$ be a rotation about the x -axis.

Find: A surface $S(u,v)$ which is $C(u)$ rotated about the x -axis.

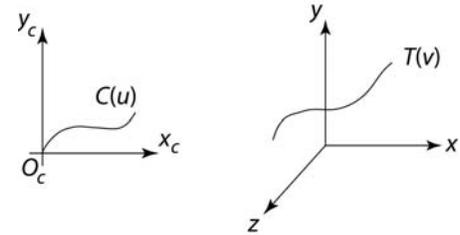
Solution:

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General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u,v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



More specifically:

- Suppose that $C(u)$ lies in an (x_c, y_c) coordinate system with origin O_c .
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

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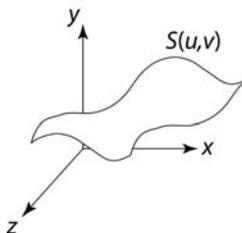
Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

1. **Fixed** (or **static**): Just translate O_c along $T(v)$.



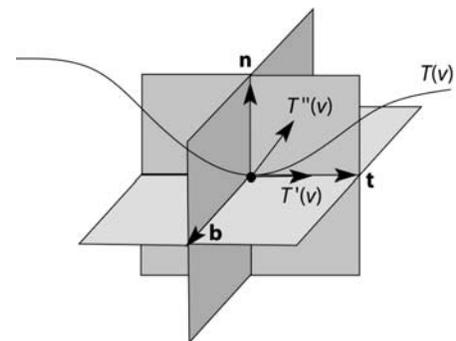
2. **Moving**. Use the **Frenet frame** of $T(v)$.

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.

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Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

$$\begin{aligned} \mathbf{t}(v) &= \text{normalize}[T'(v)] \\ \mathbf{b}(v) &= \text{normalize}[T'(v) \times T''(v)] \\ \mathbf{n}(v) &= \mathbf{b}(v) \times \mathbf{t}(v) \end{aligned}$$

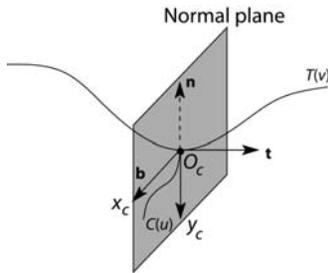
As we move along $T(v)$, the Frenet frame (t, b, n) varies smoothly.

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Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

- ♦ Put $C(u)$ in the **normal plane** .
- ♦ Place O_c on $T(v)$.
- ♦ Align x_c for $C(u)$ with \mathbf{b} .
- ♦ Align y_c for $C(u)$ with $-\mathbf{n}$.



If $T(v)$ is a circle, you get a surface of revolution exactly!

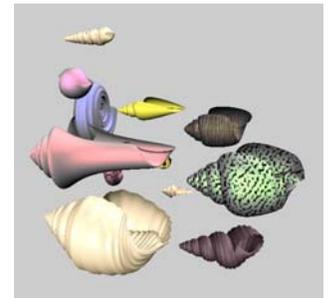
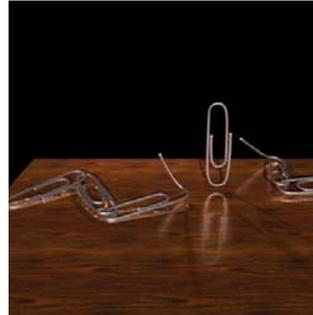
Where might these frames be ambiguous or undetermined?

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Variations

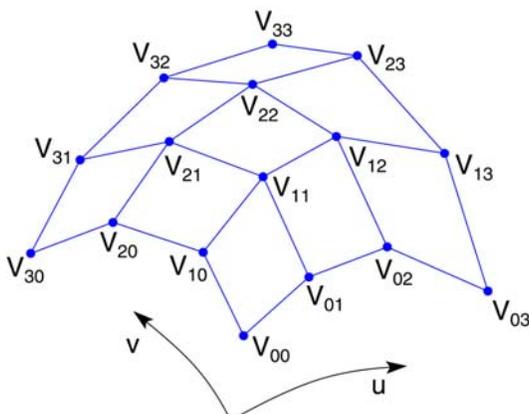
Several variations are possible:

- ♦ Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- ♦ Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- ♦ ...



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Tensor product Bézier surfaces



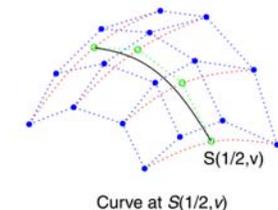
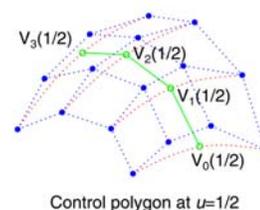
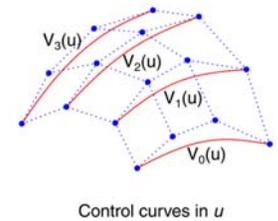
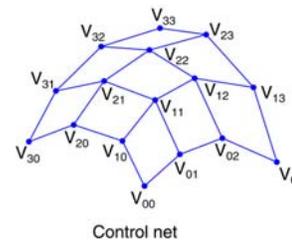
Given a grid of control points V_{ij} forming a **control net**, construct a surface $S(u,v)$ by:

- ♦ treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0(u), \dots, V_n(u)$.
- ♦ treating $V_0(u), \dots, V_n(u)$ as control points for a curve parameterized by v .

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Tensor product Bézier surfaces, cont.

Let's walk through the steps:



Which control points are interpolated by the surface?

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Matrix form of Bézier curves and surfaces

Recall that Bézier curves can be written in terms of the Bernstein polynomials:

$$Q(u) = \sum_{i=0}^n V_i b_i(u)$$

They can also be written in a matrix form:

$$Q^T(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_0^T \\ V_1^T \\ V_2^T \\ V_3^T \end{bmatrix}$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{M}_{\text{Bezier}} \mathbf{V}_{\text{curve}}$$

Tensor product surfaces can be written out similarly:

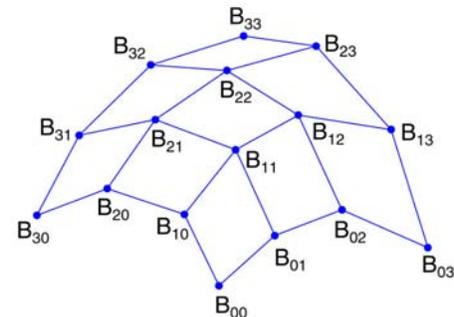
$$S(u, v) = \sum_{i=0}^n \sum_{j=0}^n V_{ij} b_i(u) b_j(v)$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{M}_{\text{Bezier}} \mathbf{V}_{\text{surface}} \mathbf{M}_{\text{Bezier}}^T \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

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Tensor product B-spline surfaces

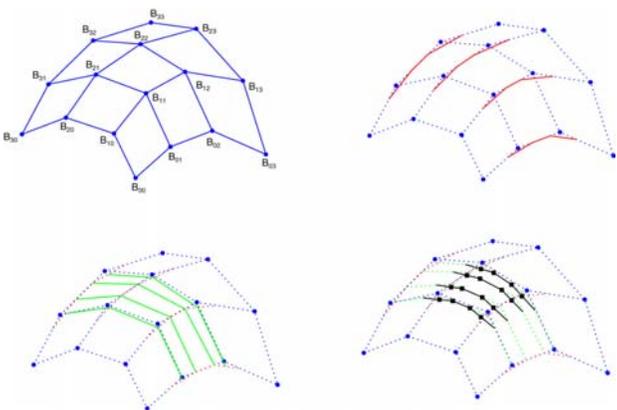
As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- ♦ treat rows of B as control points to generate Bézier control points in u .
- ♦ treat Bézier control points in u as B-spline control points in v .
- ♦ treat B-spline control points in v to generate Bézier control points in u .

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Tensor product B-spline surfaces, cont.



Which B-spline control points are interpolated by the surface?

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Matrix form of B-spline surfaces

For curves, we can write a matrix that generates Bézier control points from B-spline control points:

$$\begin{bmatrix} V_0^T \\ V_1^T \\ V_2^T \\ V_3^T \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} B_0^T \\ B_1^T \\ B_2^T \\ B_3^T \end{bmatrix}$$

$$\mathbf{V}_{\text{curve}} = \mathbf{M}_{\text{B-spline}} \mathbf{B}_{\text{curve}}$$

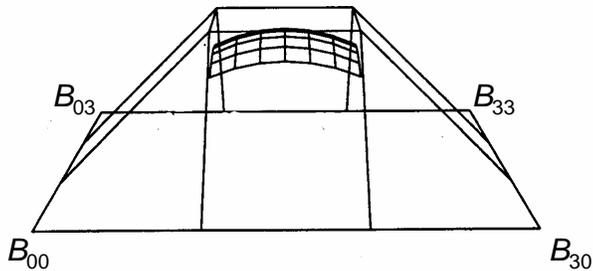
We can arrive at a similar form for tensor product B-spline surfaces:

$$\mathbf{V}_{\text{surface}} = \mathbf{M}_{\text{B-spline}} \mathbf{B}_{\text{surface}} \mathbf{M}_{\text{B-spline}}^T$$

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Tensor product B-splines, cont.

Another example:



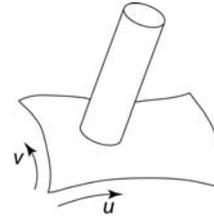
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Trimmed NURBS surfaces

Uniform B-spline surfaces are a special case of NURBS surfaces.

Sometimes, we want to have control over which parts of a NURBS surface get drawn.

For example:



We can do this by **trimming** the u - v domain.

- ◆ Define a closed curve in the u - v domain (a **trim curve**)
- ◆ Do not draw the surface points inside of this curve.

It's really hard to maintain continuity in these regions, especially while animating.

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Summary

What to take home:

- ◆ How to construct swept surfaces from a profile and trajectory curve:
 - with a fixed frame
 - with a Frenet frame
- ◆ How to construct tensor product Bézier surfaces
- ◆ How to construct tensor product B-spline surfaces

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