Parametric surfaces

Reading

Required:

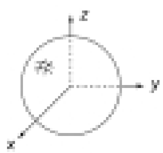
- Angel readings for “Parametric Curves” lecture, with emphasis on 10.1.2, 10.1.3, 10.1.5, 10.6.2, 10.7.3.

Optional

- Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling,* 1987.

Mathematical surface representations

- Explicit $z=f(x,y)$ (a.k.a., a “height field”)
  - what if the curve isn’t a function, like a sphere?



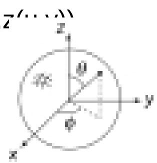
- Implicit $g(x,y,z) = 0$

- Parametric $S(u,v)=(x(u,v),y(u,v),z(u,v))$
  - For the sphere:

    $x(u,v) = r \cos 2\pi v \sin \pi u$

    $y(u,v) = r \sin 2\pi v \sin \pi u$

    $z(u,v) = r \cos \pi u$



As with curves, we’ll focus on parametric surfaces.

Surfaces of revolution

Idea: rotate a 2D **profile curve** around an axis.

What kinds of shapes can you model this way?

Constructing surfaces of revolution

Given: A curve $C(u)$ in the xy-plane:

$$C(u)=\begin{bmatrix} c_x(u) \\ c_y(u) \\ 0 \\ 1 \end{bmatrix}$$

Let $R_x(\theta)$ be a rotation about the x-axis.

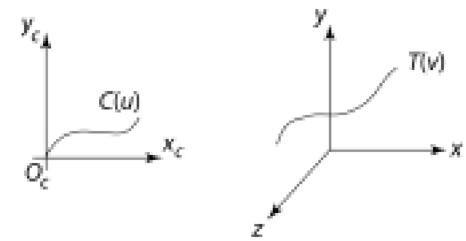
Find: A surface $S(u,v)$ which is $C(u)$ rotated about the x-axis.

Solution:

General sweep surfaces

The **surface of revolution** is a special case of a **swept surface**.

Idea: Trace out surface $S(u,v)$ by moving a **profile curve** $C(u)$ along a **trajectory curve** $T(v)$.



More specifically:

- Suppose that $C(u)$ lies in an (x_c,y_c) coordinate system with origin O_c.
- For every point along $T(v)$, lay $C(u)$ so that O_c coincides with $T(v)$.

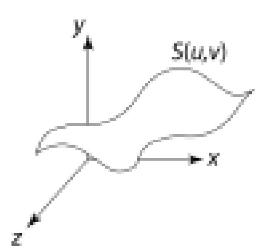
Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$?

Here are two options:

1. **Fixed** (or **static**): Just translate O_c along $T(v)$.

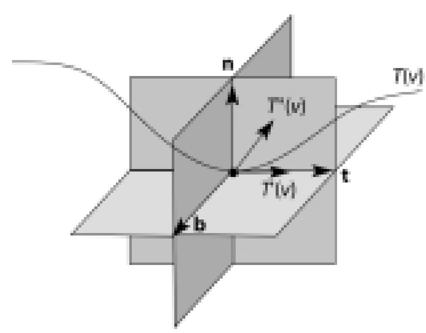


2. Moving. Use the **Frenet frame** of $T(v)$.

- Allows smoothly varying orientation.
- Permits surfaces of revolution, for example.

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly varying coordinate system.



To get a 3D coordinate system, we need 3 independent direction vectors.

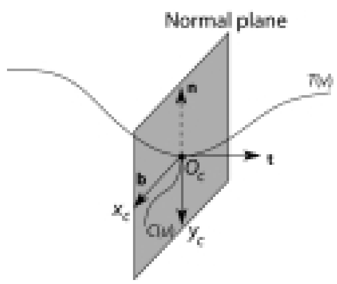
$$\mathbf{t}(v)=\text{normalize}[T'(v)]$$
$$\mathbf{b}(v)=\text{normalize}[T'(v)\times T''(v)]$$
$$\mathbf{n}(v)=\mathbf{b}(v)\times\mathbf{t}(v)$$

As we move along $T(v)$, the Frenet frame (t,b,n) varies smoothly.

Frenet swept surfaces

Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

- Put $C(u)$ in the **normal plane** .
- Place O_c on $T(v)$.
- Align x_c for $C(u)$ with **b**.
- Align y_c for $C(u)$ with **-n**.



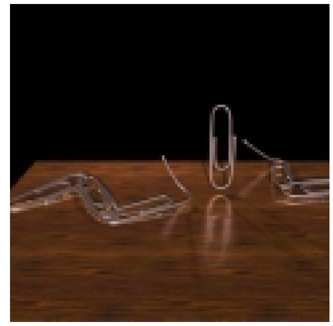
If $T(v)$ is a circle, you get a surface of revolution exactly!

What happens at inflection points, i.e., where curvature goes to zero?

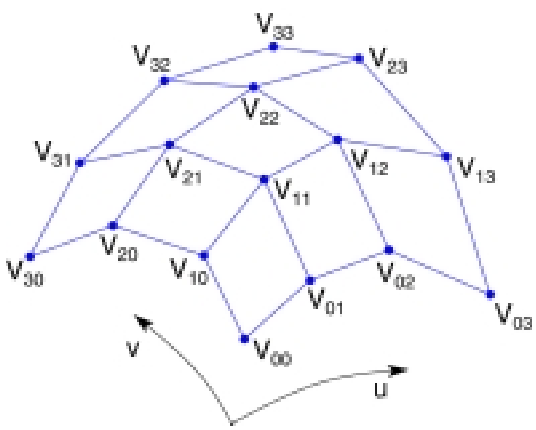
Variations

Several variations are possible:

- Scale $C(u)$ as it moves, possibly using length of $T(v)$ as a scale factor.
- Morph $C(u)$ into some other curve $\tilde{C}(u)$ as it moves along $T(v)$.
- …



Tensor product Bézier surfaces

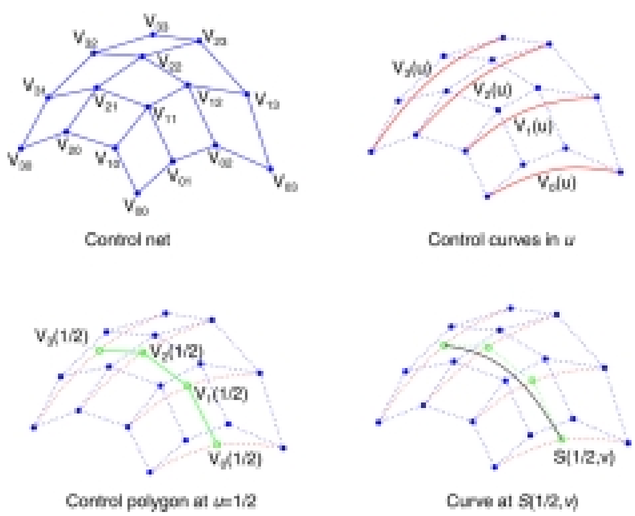


Given a grid of control points V_{ij}, forming a **control net**, contruct a surface $S(u,v)$ by:

- treating rows of V (the matrix consisting of the V_{ij}) as control points for curves $V_0(u),\ldots, V_n(u)$.
- treating $V_0(u),\ldots, V_n(u)$ as control points for a curve parameterized by v.

Tensor product Bézier surfaces, cont.

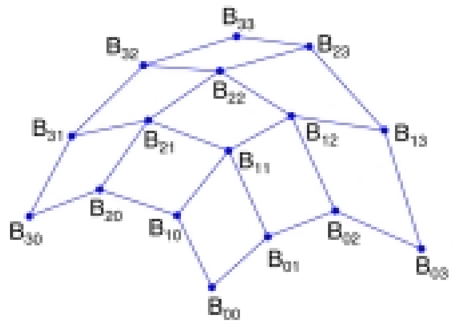
Let's walk through the steps:



Which control points are interpolated by the surface?

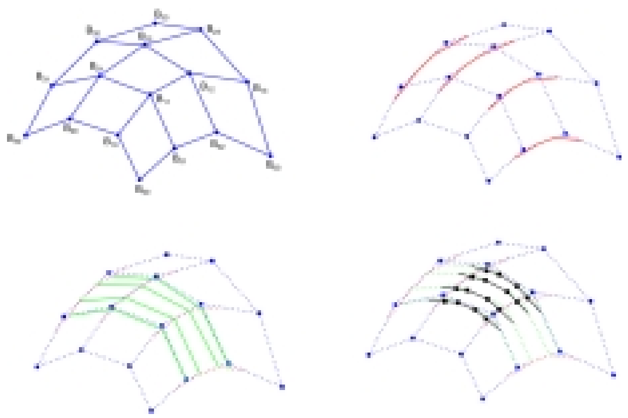
Tensor product B-spline surfaces

As with spline curves, we can piece together a sequence of Bézier surfaces to make a spline surface. If we enforce C^2 continuity and local control, we get B-spline curves:



- treat rows of *B* as control points to generate Bézier control points in *u*.
- treat Bézier control points in *u* as B-spline control points in *v*.
- treat B-spline control points in *v* to generate Bézier control points in *u*.

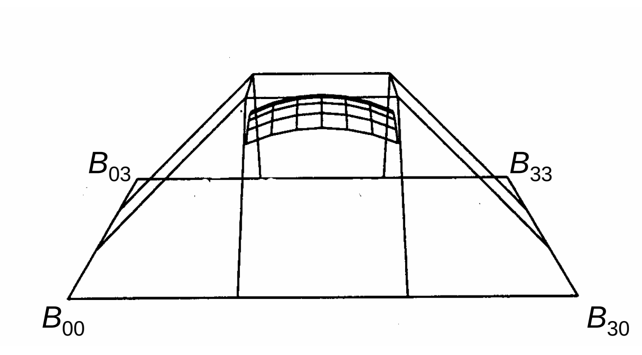
Tensor product B-spline surfaces, cont.



Which B-spline control points are interpolated by the surface?

Tensor product B-splines, cont.

Another example:



Summary

What to take home:

- How to construct swept surfaces from a profile and trajectory curve:
  - with a fixed frame
  - with a Frenet frame
- How to construct tensor product Bézier surfaces
- How to construct tensor product B-spline surfaces