Dot Product

Reference

Section A.3, Dot Products and Distances, *Computer Graphics, Principles and Practice,* Foley, van Dam

Dot Product

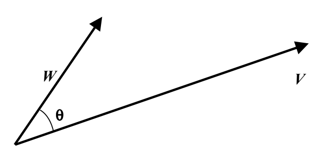
The dot product or inner product of two vectors is a very useful operation in computer graphics and is applied in numerous ways

These notes are a short review of what the dot product is and some examples of how it gets used

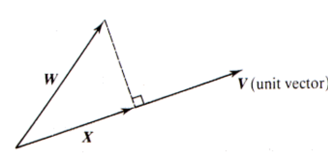
Definition

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$



$$\begin{aligned} v \cdot w &= v_1 w_1 + v_2 w_2 + v_3 w_3 \\ &= \|v\| \|w\| \cos(\theta) \end{aligned}$$

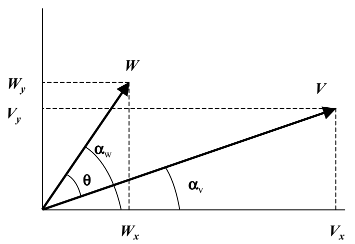


if v is a unit vector, then

$$\begin{aligned} v \cdot w &= v_1 w_1 + v_2 w_2 + v_3 w_3 \\ &= \|w\| \cos(\theta) \end{aligned}$$

and so $v \cdot w$ is the length of the projection of w onto v

Illustration of *V·W*



$$V_x = \|V\|\cos(\alpha_v)$$
$$V_y = \|V\|\sin(\alpha_v)$$
$$W_x = \|W\|\cos(\alpha_w)$$
$$W_y = \|W\|\sin(\alpha_w)$$

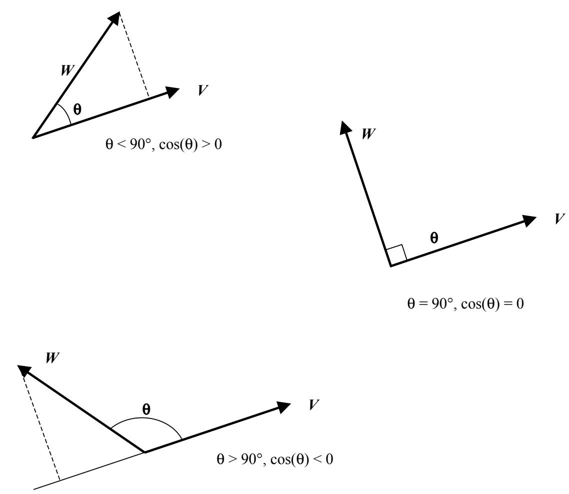
$$
\begin{aligned}
V \cdot W &= V_x W_x + V_y W_y \\
&= \|V\|\cos(\alpha_v)\|W\|\cos(\alpha_w) + \|V\|\sin(\alpha_v)\|W\|\sin(\alpha_w) \\
&= \|V\|\|W\|[\cos(\alpha_v)\cos(\alpha_w) + \sin(\alpha_v)\sin(\alpha_w)] \\
&= \|V\|\|W\|\cos(\alpha_w - \alpha_v) \\
&= \|V\|\|W\|\cos(\theta)
\end{aligned}
$$

The cosine is a useful function ...

if both v and w are unit vectors, then

$$
\begin{aligned}
v \cdot w &= v_1 w_1 + v_2 w_2 + v_3 w_3 \\
&= \|v\|\|w\|\cos(\theta) \\
&= \cos(\theta)
\end{aligned}
$$

and so $v \cdot w$ is just the cosine of the angle between the vectors



Unit vectors

The dot product of v with itself is

$$
\begin{aligned}
v \cdot v &= v_1 v_1 + v_2 v_2 + v_3 v_3 \\
&= \|v\|\|v\|\cos(0) \\
&= \|v\|^2
\end{aligned}
$$

and so $v \cdot v$ is the square of its length

and if v is a unit vector then $v \cdot v$ is 1

the columns of a rotation matrix are perpendicular unit vectors

$\mathrm{a} \cdot \mathrm{a} = \cos\theta\cos\theta + \sin\theta\sin\theta = 1$

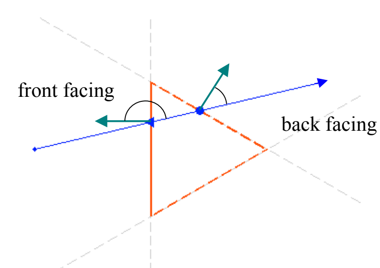
$\mathrm{a} \cdot \mathrm{b} = \cos\theta(\text{-}\sin\theta) + \sin\theta\cos\theta = 0$

and so the transpose of a rotation matrix is its inverse

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

note that the transpose is rotation through -θ, since -sin(- θ)=sin(θ), which also shows that the transpose is the inverse of the original rotation matrix.

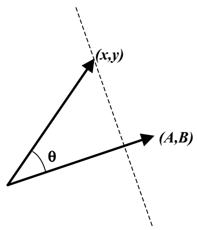
Front facing polygon?



surface normal dot product with ray direction

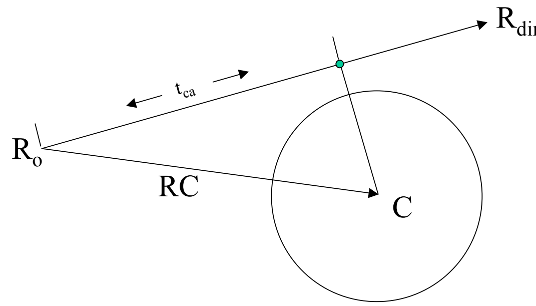
Equation of a line

$Ax+By+C=0$



All vectors (x,y) for which (A,B) · (x,y) = -C

Where on ray is closest approach to C?



$RC = C - R_o$

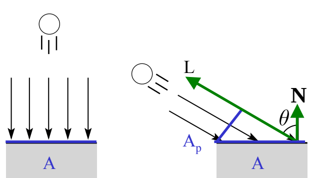
R_o is ray origin

R_{dir} is ray direction vector (unit vector)

so

$t_{ca} = RC \cdot R_{dir}$

Diffuse reflection of light



$$\frac{A_p}{A} = \cos\theta$$

$$N \cdot L = \|L\| \cos\theta$$

$$= \|L\| \frac{A_p}{A}$$

so $N \cdot L$ gives a scaled value for diffuse reflected light intensity