

## 6. Affine transformations

1

## Reading

Required:

- ♦ Foley, et al, Chapter 5.1-5.5.

Further reading:

- ♦ David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2<sup>nd</sup> Ed., McGraw-Hill, New York, 1990, Chapter 2.

2

## Geometric transformations

Geometric transformations will map points in one space to points in another:  $(x',y',z') = f(x,y,z)$ .

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

We'll start in 2D...

3

## Representation

We can represent a **point**,  $\mathbf{p} = (x,y)$ , in the plane

- ♦ as a column vector  $\begin{bmatrix} x \\ y \end{bmatrix}$
- ♦ as a row vector  $[x \ y]$

4

## Representation, cont.

We can represent a **2-D transformation**  $M$  by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If  $\mathbf{p}$  is a column vector,  $M$  goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If  $\mathbf{p}$  is a row vector,  $M^T$  goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

5

## Two-dimensional transformations

Here's all you get with a  $2 \times 2$  transformation matrix  $M$ :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$

$$y' = cx + dy$$

We will develop some intimacy with the elements  $a, b, c, d, \dots$

6

## Identity

Suppose we choose  $a=d=1, b=c=0$ :

- Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

7

## Scaling

Suppose we set  $b=c=0$ , but let  $a$  and  $d$  take on any *positive* value:

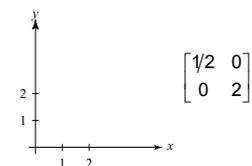
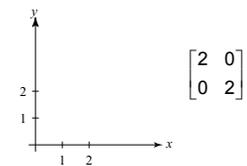
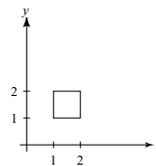
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- Provides **differential scaling** in  $x$  and  $y$ :

$$x' = ax$$

$$y' = dy$$



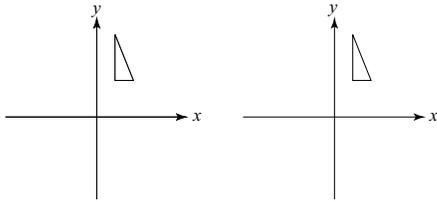
8

Suppose we keep  $b=c=0$ , but let either  $a$  or  $d$  go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



9

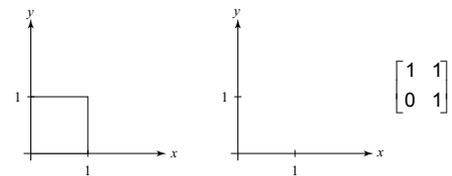
Now let's leave  $a=d=1$  and experiment  $b$  . . .

The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$\begin{aligned} x' &= x + by \\ y' &= y \end{aligned}$$



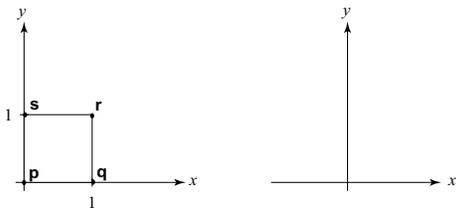
10

### Effect on unit square

Let's see how a general  $2 \times 2$  transformation  $M$  affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q & r & s \end{bmatrix} = \begin{bmatrix} p' & q' & r' & s' \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$



11

### Effect on unit square, cont.

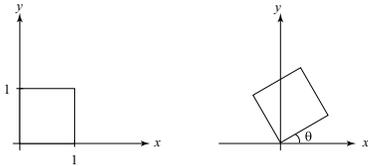
Observe:

- Origin invariant under  $M$
- $M$  can be determined just by knowing how the corners  $(1,0)$  and  $(0,1)$  are mapped
- $a$  and  $d$  give  $x$ - and  $y$ -scaling
- $b$  and  $c$  give  $x$ - and  $y$ -shearing

12

## Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



•  $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$

•  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$

Thus,

$$M = R(\theta) = \begin{bmatrix} & \\ & \end{bmatrix}$$

13

## Limitations of the 2 x 2 matrix

A 2 x 2 matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

**Q:** What important operation does that leave out?

14

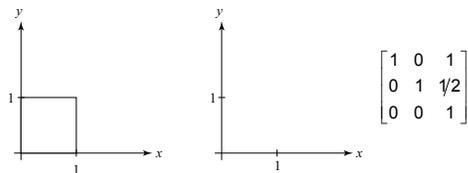
## Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(t) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



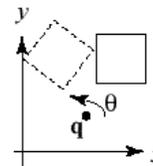
... gives translation!

15

## Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation,  $\theta$ , about any point  $\mathbf{q} = [q_x \ q_y]^T$  with a matrix.



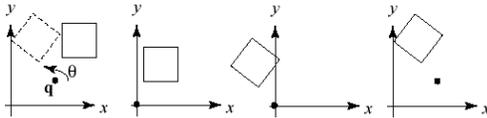
**Q:** how would you find the matrix for rotating about  $\mathbf{q}$  by  $\theta$ ?

16

## Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation,  $\theta$ , about any point  $\mathbf{q} = [q_x \ q_y]^T$  with a matrix:



1. Translate  $\mathbf{q}$  to origin
2. Rotate
3. Translate back

Note: Transformation order is important!!

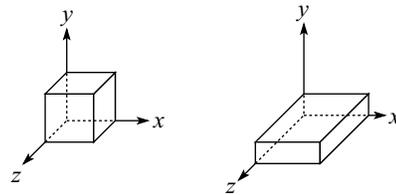
17

## Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, scaling:

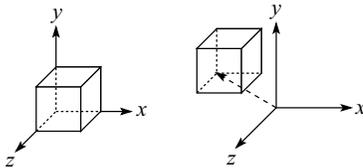
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



18

## Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



19

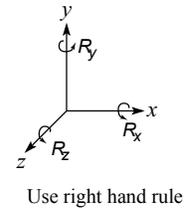
## Rotation in 3D

Rotation now has more possibilities in 3D:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

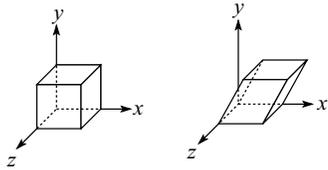


20

## Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



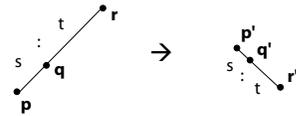
21

## Properties of affine transformations

All of the transformations we've looked at so far are examples of "affine transformations."

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios along a line are always preserved)



$$\text{ratio} = \frac{\|pq\|}{\|qr\|} = \frac{s}{t} = \frac{\|p'q'\|}{\|q'r'\|}$$

22

## Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.

23