Lecture 3

Systems and (Convolutions)

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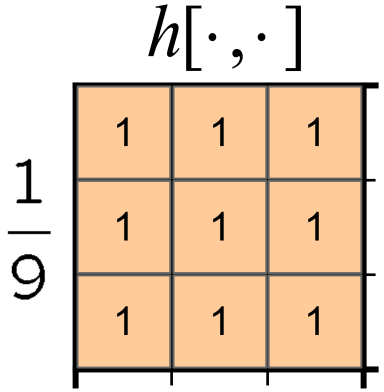
So far: 2D discrete system (filters)

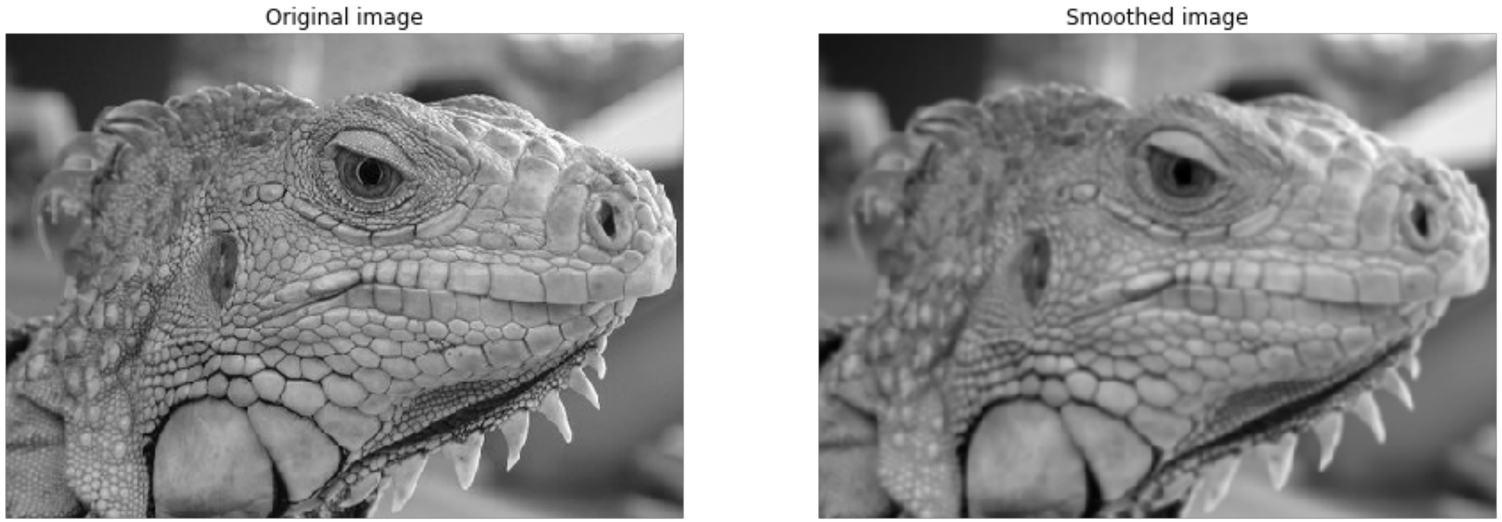
- **System:** a sequence of filter
- **S** is the **system operator**, defined as a mapping or assignment of possible input **function** f[n,m] to some possible output **function** g[n,m].

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

So far: Moving Average

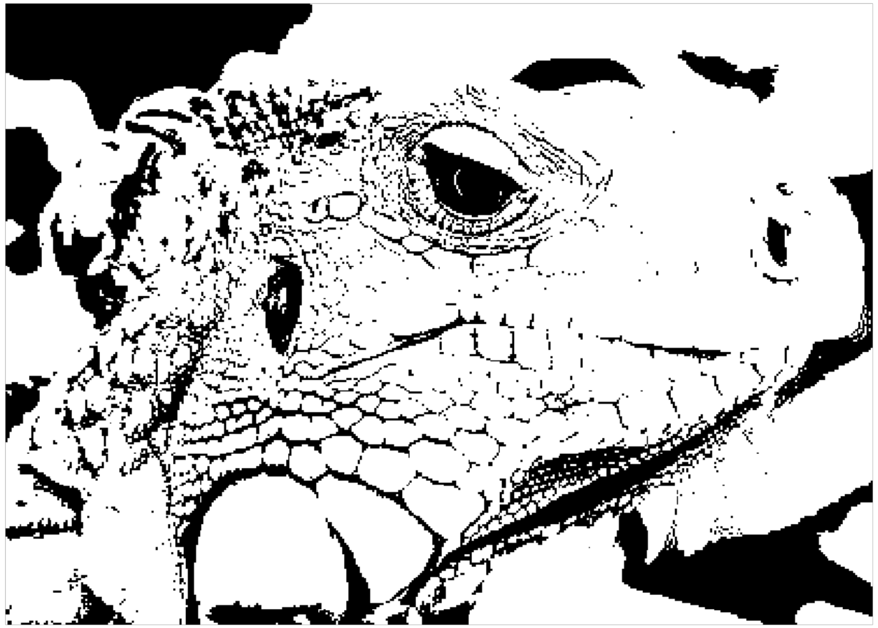


$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$



So far: Image Segmentation

- Use a simple pixel threshold: $$g[n,m] = \begin{cases} 255, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$



What we will learn today?

- Properties of filters
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response (Convolution)

Properties of systems

- Amplitude **linearity**:
  - Additivity – f(x+y) = f(x) + f(y)

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

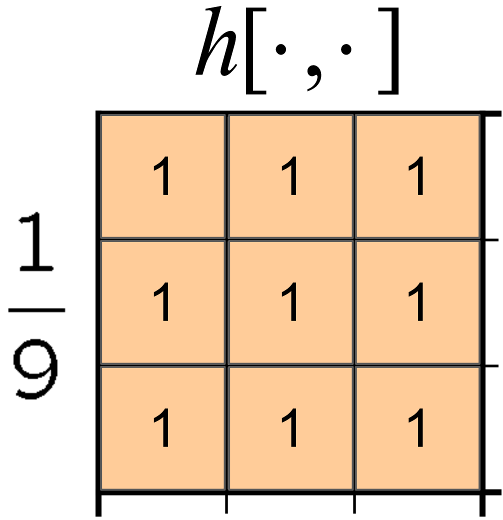
  - Homogeneity – f(αx) = α f(x)

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$

Is Moving Average Additive?

Goal: $\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$



$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Is Moving Average Additive?

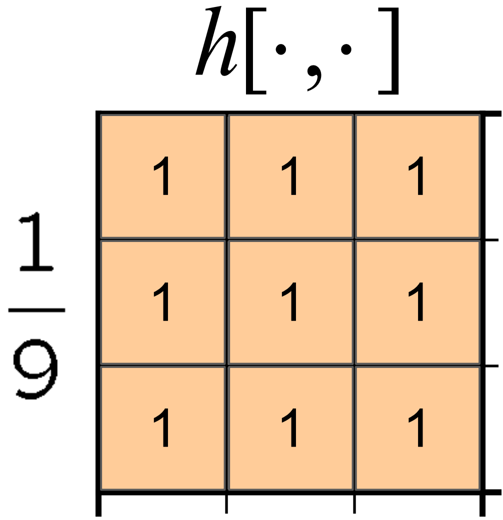
$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Goal: $\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f'[n,m]]$$

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$



Is Moving Average Additive?

Goal: $\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f'[n,m]]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k, m-l]$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Is Moving Average Additive?

Goal: $\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]$$

$$\begin{aligned}\mathcal{S}[f_i[n,m] + f_j[n,m]] &= \mathcal{S}[f'[n,m]] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k,m-l] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} [f_i[n-k,m-l] + f_j[n-k,m-l]]\end{aligned}$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Is Moving Average Additive?

Goal: $\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]$$

$$\begin{aligned}
\mathcal{S}[f_i[n,m] + f_j[n,m]] &= \mathcal{S}[f'[n,m]] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k,m-l] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} [f_i[n-k,m-l] + f_j[n-k,m-l]] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_i[n-k,m-l] + \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_j[n-k,m-l]]
\end{aligned}$$

Is Moving Average Additive?

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Goal: $\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f'[n,m]]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k, m-l]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} [f_i[n-k, m-l] + f_j[n-k, m-l]]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_i[n-k, m-l] + \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_j[n-k, m-l]]$$

$$= \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Is Moving Average Homogeneous?

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$

Exercise:
Showing moving average filter is homogeneous

$h[\cdot,\cdot]$

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Properties of systems

- Amplitude linearity:
  - Additivity

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

  - Homogeneity

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$

Exercise: prove homogeneity by your own

Properties of systems

- Amplitude linearity:
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$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

  - Homogeneity

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$

  - **From above, we get Superposition (Linear Combination)**

$$\mathcal{S}[\alpha f_i[n,m] + \beta f_j[n,m]] = \alpha \mathcal{S}[f_i[n,m]] + \beta \mathcal{S}[f_j[n,m]]$$

This is an important property. Make sure you know how to prove if any system has this property

Properties of systems

- Other properties:
  - Stability

$$\text{If } \forall n, m, \; |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

(It's related to Lipschitz condition in ML & Statistics)

Q. Is the moving average filter stable?

$$\text{If } \forall n,m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n,m,\ |f[n,m]| \leq k$$

$$\text{If } \forall n,m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n,m,\ |f[n,m]| \leq k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]|$$

$$\text{If } \forall n,m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n,m,\ |f[n,m]| \leq k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k,m-l]|$$

(Triangle Inequality)

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$|\mathcal{S} f[n,m]| = |\frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\leq \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} |f[n-k, m-l]| \qquad \text{(Triangle Inequality)}$$

$$\leq \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} k$$

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]|$$ (Triangle Inequality)

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k$$

$$\leq \frac{1}{9}(3)(3)k$$

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]| \quad \text{(Triangle Inequality)}$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k$$

$$\leq \frac{1}{9}(3)(3)k$$

$$\leq k$$

$$\leq ck, \text{ where } c = 1$$

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

  - Invertibility

$$\mathcal{S}^{-1}\mathcal{S}[f[n,m]] = f[n,m]$$

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

  - Invertibility

$$\mathcal{S}^{-1}\mathcal{S}[f[n,m]] = f[n,m]$$

Q. Is the 3x3 moving average filter invertible?

(Last time, we had a discussion of information loss of moving average)

A simple 1D moving average

- Consider a 1D moving avg problem of [1, 2, 3, 4] and the window size is 3

- The avg is [1, 2, 3, 7/3]
  - (0 + 1 + 2)/3 = 1
  - (1 + 2 + 3)/3 = 2
  - (2 + 3 + 4)/3 = 3
  - (3 + 4 + 0)/3 = 7/3

- Invertibility: can you infer back [1, 2, 3, 4] by [1, 2, 3, 7/3]?
  - if we know window size is 3 and zero paddings

A simple 1D moving average

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \frac{7}{3} \end{bmatrix}$$

- Boundary condition is important
- Exercise: what leads to invertible or non-invertible moving average?
- What is implication?
  - With different assumptions and conditions, we can make many operation invertible, which allows us to recover the image back (e.g. denoising)

Filters and Boundaries Miniquizz

https://tinyurl.com/cse455-3

Properties of systems

- Spatial properties
  - Shift invariance:

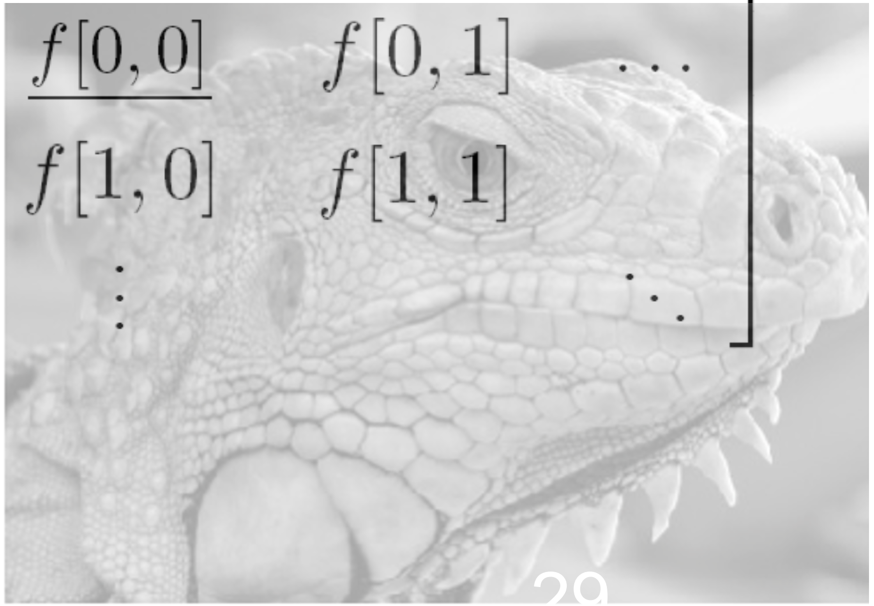
$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

What does shifting an image look like?

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \dots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \dots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

Original image



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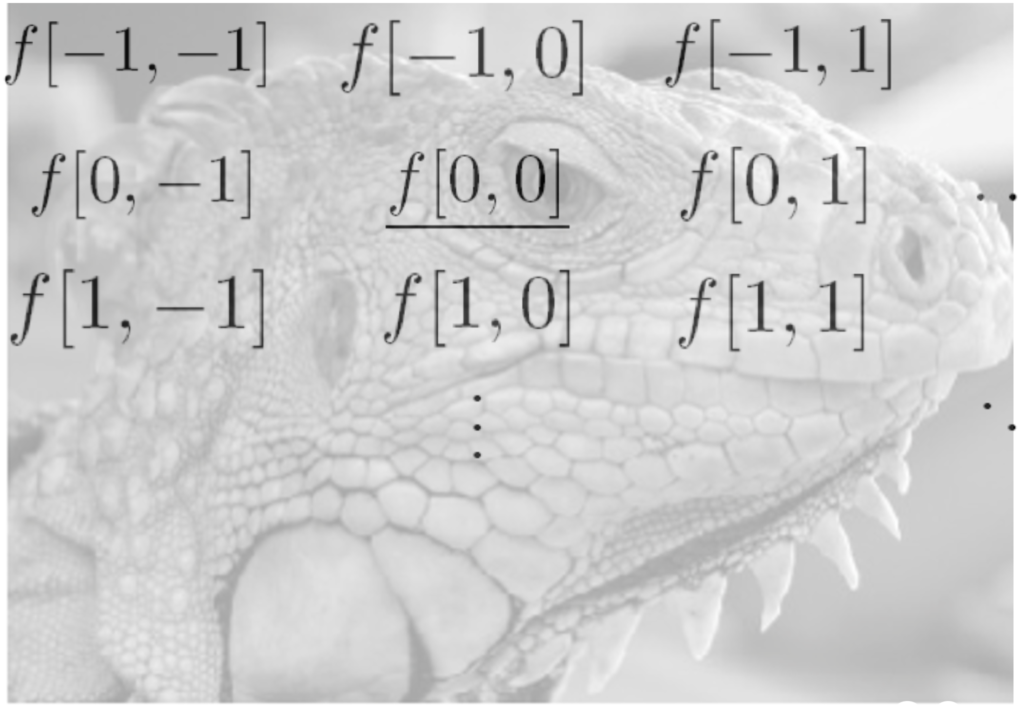

$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \dots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \dots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

Shifted image

$n_0 = 1$

$m_0 = 1$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

$\mathbf{f}[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\mathbf{g}[n,m]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

Is the moving average system is shift invariant?

$$g[n, m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n - k, m - l]$$

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$g[n-n_0, m-m_0] = g[n', m']$$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$g[n-n_0, m-m_0] = g[n', m']$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n'-k, m'-l]$$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

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Let $n' = n - n_0$ and $m' = m - m_0$

$$\begin{aligned} g[n-n_0, m-m_0] &= g[n', m'] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n'-k, m'-l] \\ &= \mathcal{S}[f[n', m']] \end{aligned}$$

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$\begin{aligned} g[n - n_0, m - m_0] &= g[n', m'] \\ &= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n'-k, m'-l] \\ &= \mathcal{S}[f[n', m']] \\ &= \mathcal{S}[f[n - n_0, m - m_0]] \end{aligned}$$

What we will learn today?

- Properties of filters (continued)
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response (Convolution)

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Linear filtering:
  - Form a new image whose pixels are a weighted sum of original pixel values
  - Use the same set of weights at each point
- **S** is a linear system (function) iff it *S satisfies*

$$S[\,\alpha f_i[n,m] + \beta f_j[k,l]\,] \quad = \quad \alpha\, S[\,f_i[n,m]] + \beta\, S[\,f_j[k,l]\,]$$

superposition property (linear combination)

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Q. Is the moving average a linear system? YES

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Q. Is the moving average a linear system?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

- Q. Is thresholding a linear system?

$$g[n,m] = \begin{cases} 1, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Q. Is the moving average a linear system?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

- Q. Is thresholding a linear system?
  - Let $f_1[0,0] = f_2[0,0] = 99$
  - So, $S[f_1[0,0]] = S[f_2[0,0]] = 0$
  - But $S[f_1[0,0] + f_2[0,0]] = 1$

$$g[n,m] = \begin{cases} 1, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$

Linear shift invariant (LSI) systems

- Satisfies two properties:

- Superposition (linear combination) property

$$S[\,\alpha f_i[n,m] + \beta f_j[k,l]\,] = \alpha\, S[\,f_i[n,m]] + \beta\, S[\,f_j[k,l]\,]$$

- Shift invariance:

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

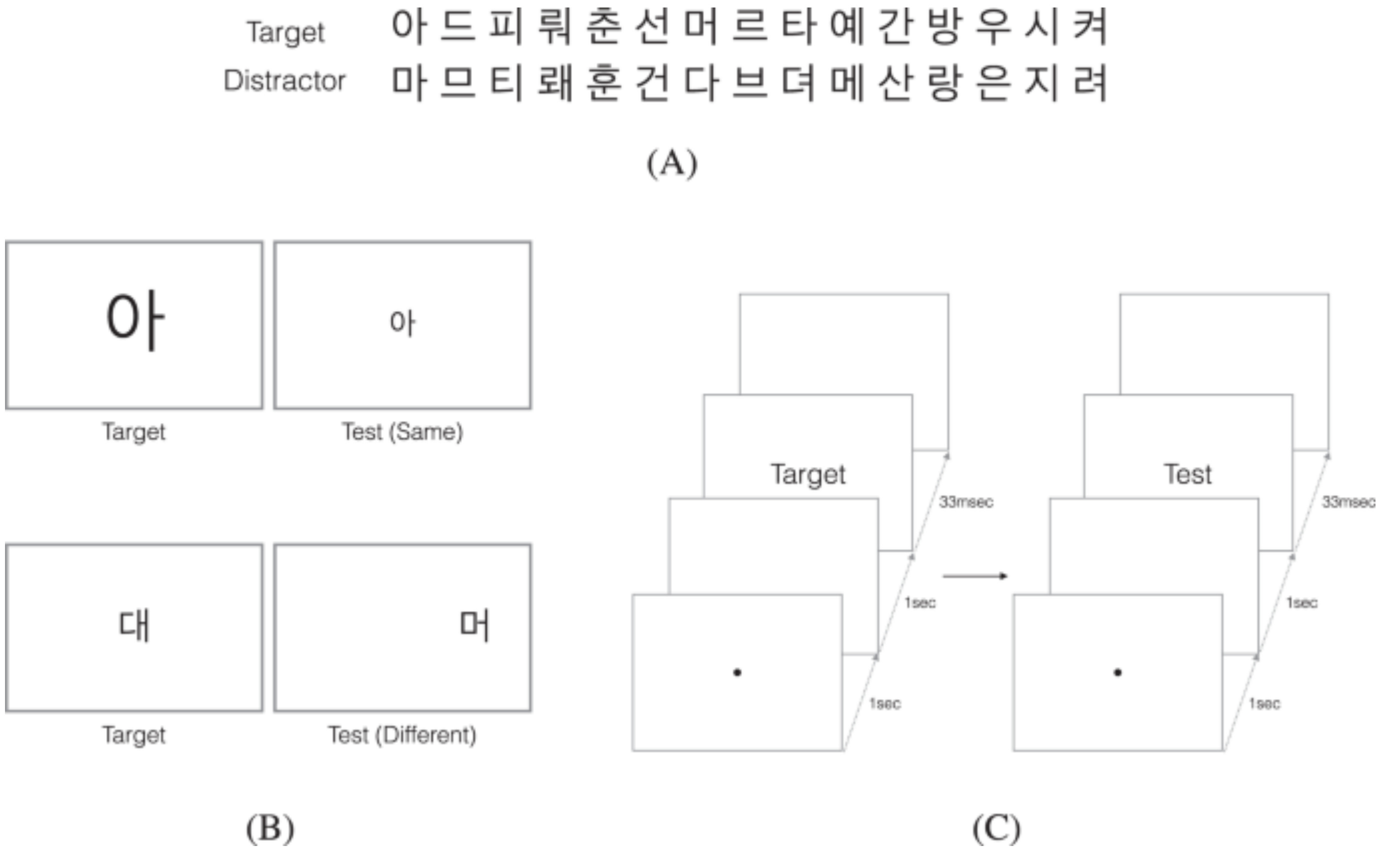
Moving average system is linear shift invariant (LSI)

- We are going to use this as an example to dive into interesting properties about linear shift-invariant systems.

- Why are linear shift invariant systems important?

Our visual system is (often) a shift invariant system, and linear is easy (for us)

Human vision are scale and translation invariant



Participants were shown some target Korean character once and were tested on whether they can identify the targets from other distractors

Han et al. Scale and translation-invariance for novel objects in human vision. Nature 2020 [link]

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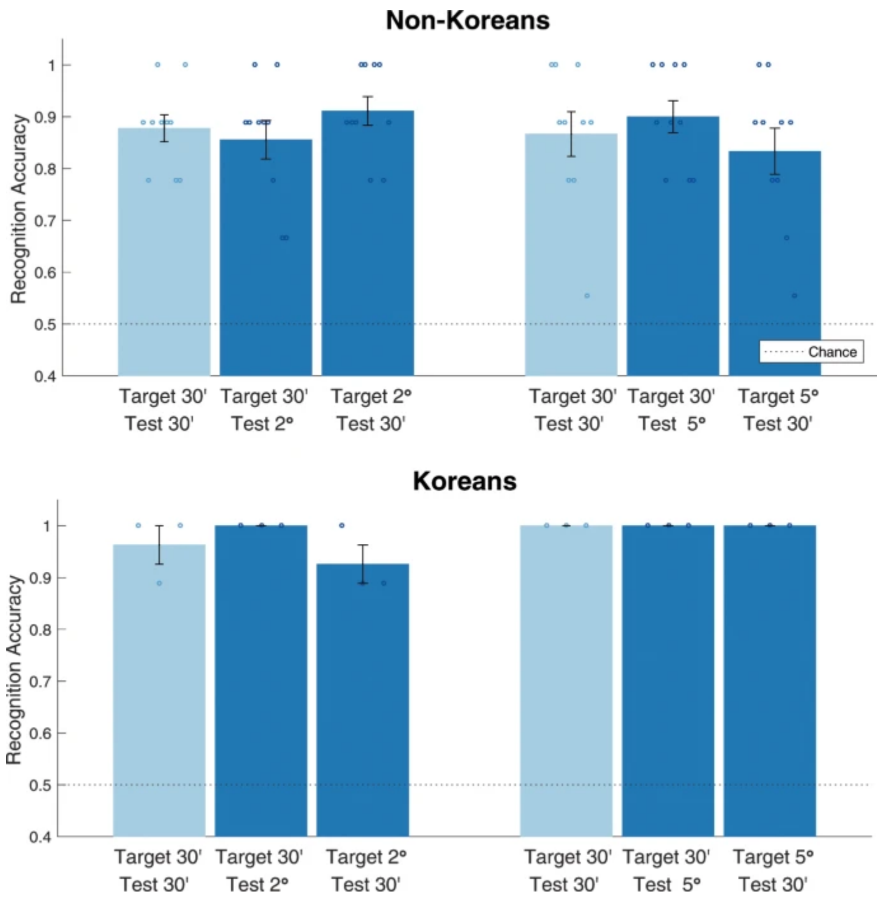
Human vision are scale and translation invariant



Very high recognition accuracies

Han et al. Scale and translation-invariance for novel objects in human vision. Nature 2020 [link]

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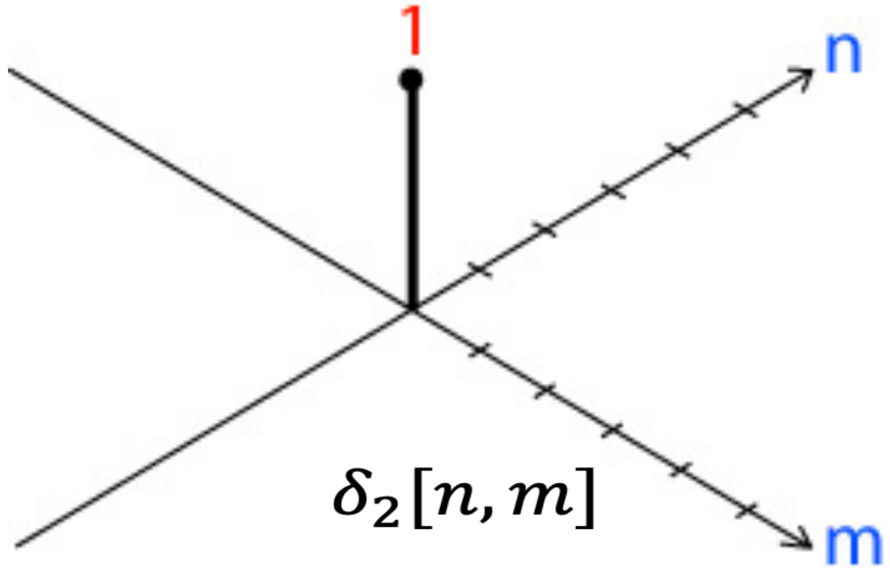
What we will learn today?

- Properties of filters (continued)
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response

2D impulse function

- Let’s look at a special function
- 1 at the origin [0,0].
- 0 everywhere else
  - (Similar to delta function)



2D impulse function as an image

- Let's look at a special function
- 1 at the origin [0,0].
- 0 everywhere else

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

What happens when we pass an impulse function through a LSI systems

- The moving average filter equation again: $g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$

$$\delta_2[n,m] \xrightarrow{S} h[n,m]$$

Pass in an impulse function **Record its response**

- By passing an impulse function into an LSI system, we get it's impulse response.
  - We will use h[n, m] to refer to the impulse response

What happens when we pass an impulse function through a LSI systems

Before we do this, let's remember how we used the moving average filter last lecture

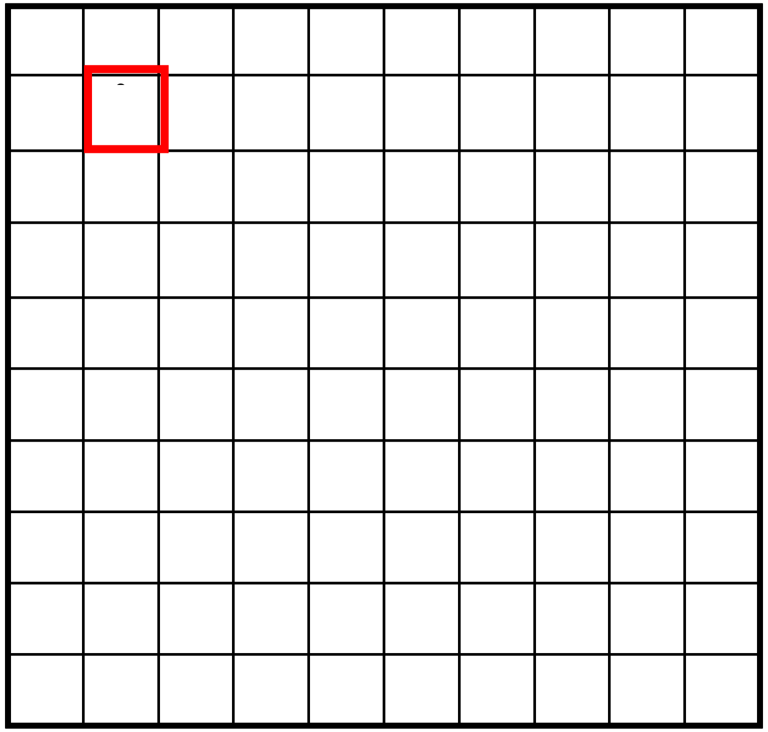
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

Remember the Moving Average filter from last lecture

$\mathrm{f}[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\mathrm{g}[n, m]$



Courtesy of S. Seitz

Remember the Moving Average filter from last lecture

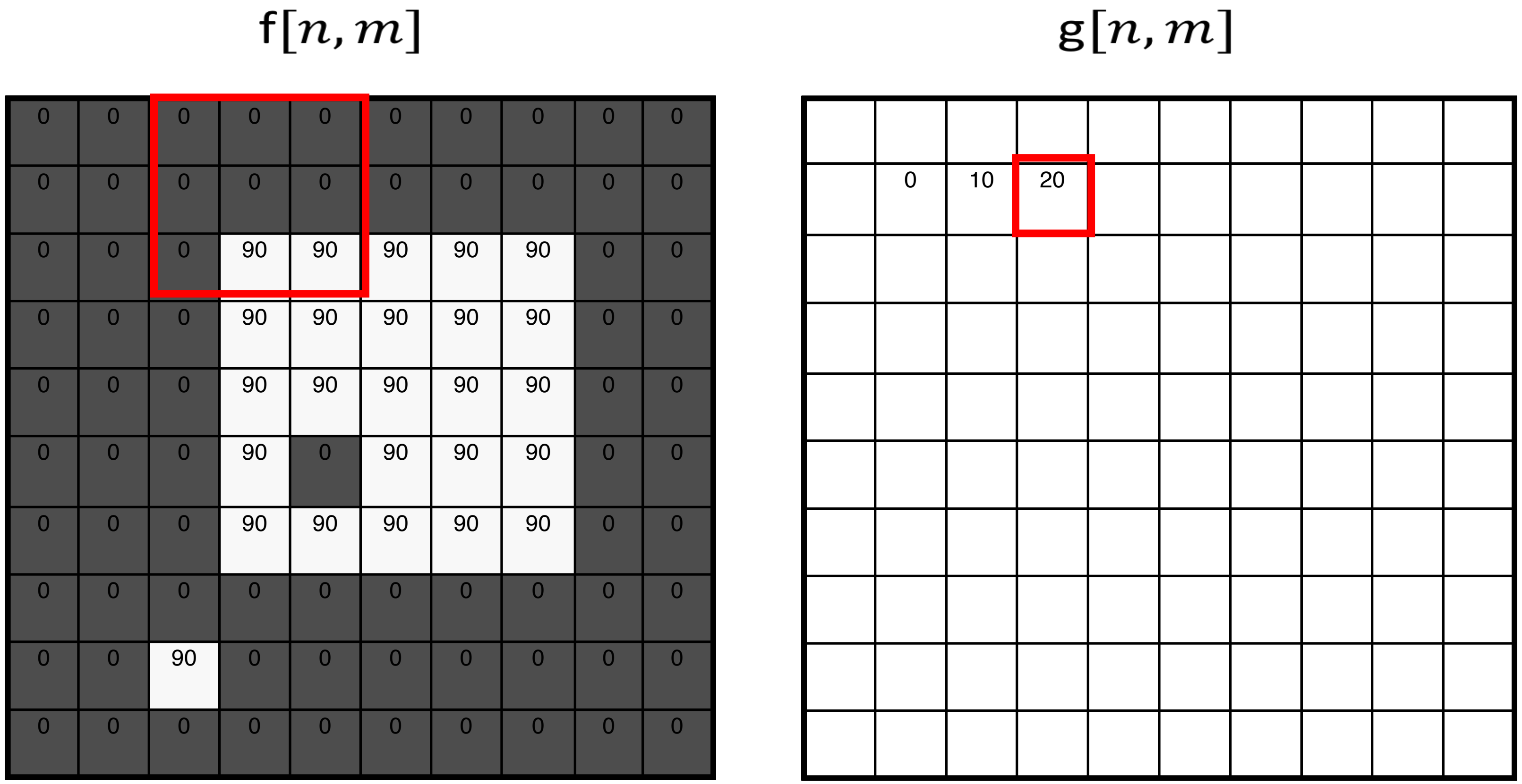
$\mathrm{f}[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\mathrm{g}[n, m]$

	0	10							

Remember the Moving Average filter from last lecture



Remember the Moving Average filter from last lecture

$\mathrm{f}[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\mathrm{g}[n, m]$

	0	10	20	30					

Remember the Moving Average filter from last lecture

$f[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n, m]$

	0	10	20	30	30				

Remember the Moving Average filter from last lecture

$\mathrm{f}[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\mathrm{g}[n, m]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

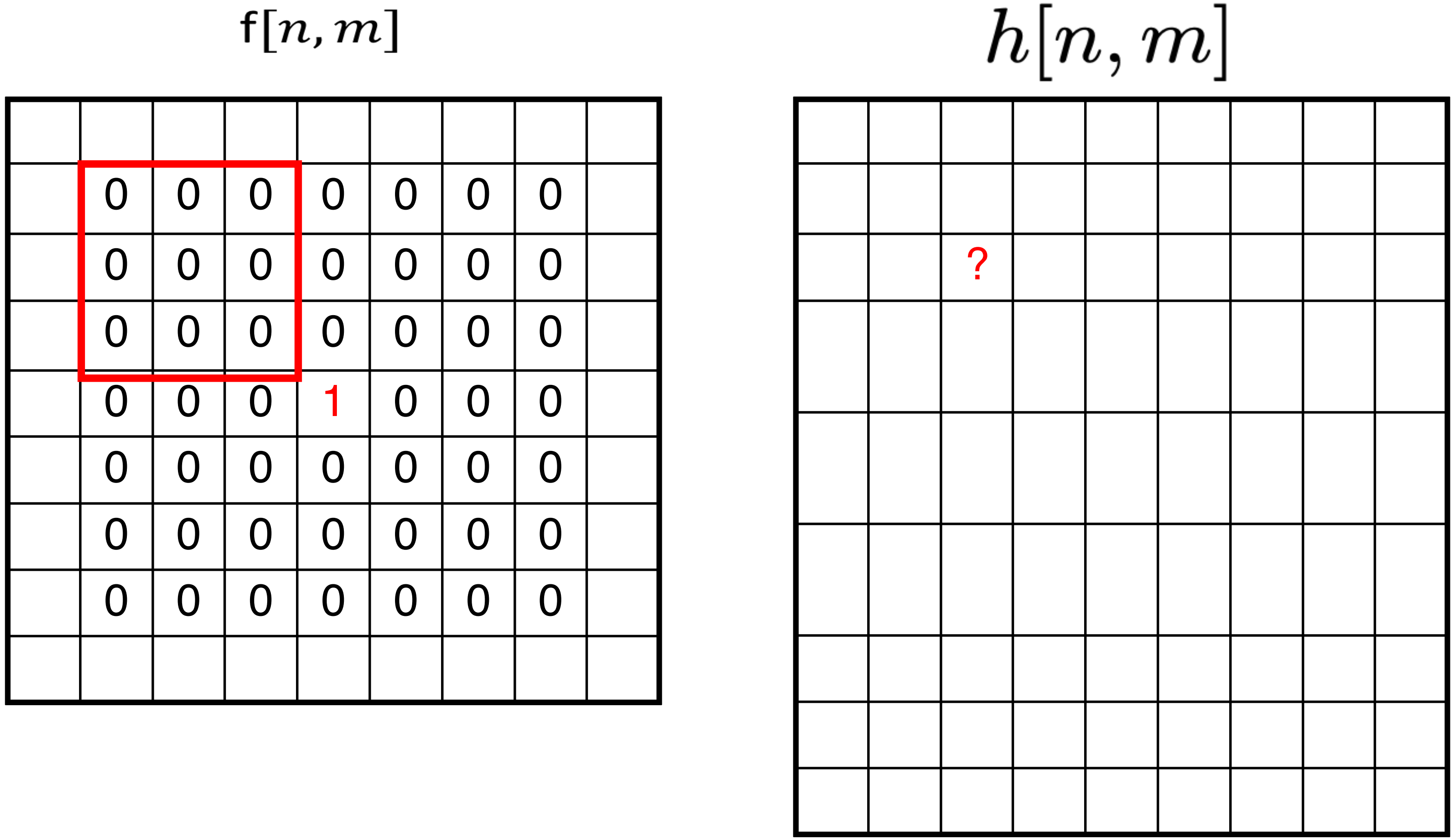
Now let's do the same thing with an impulse function

$\mathsf{f}[n,m]$

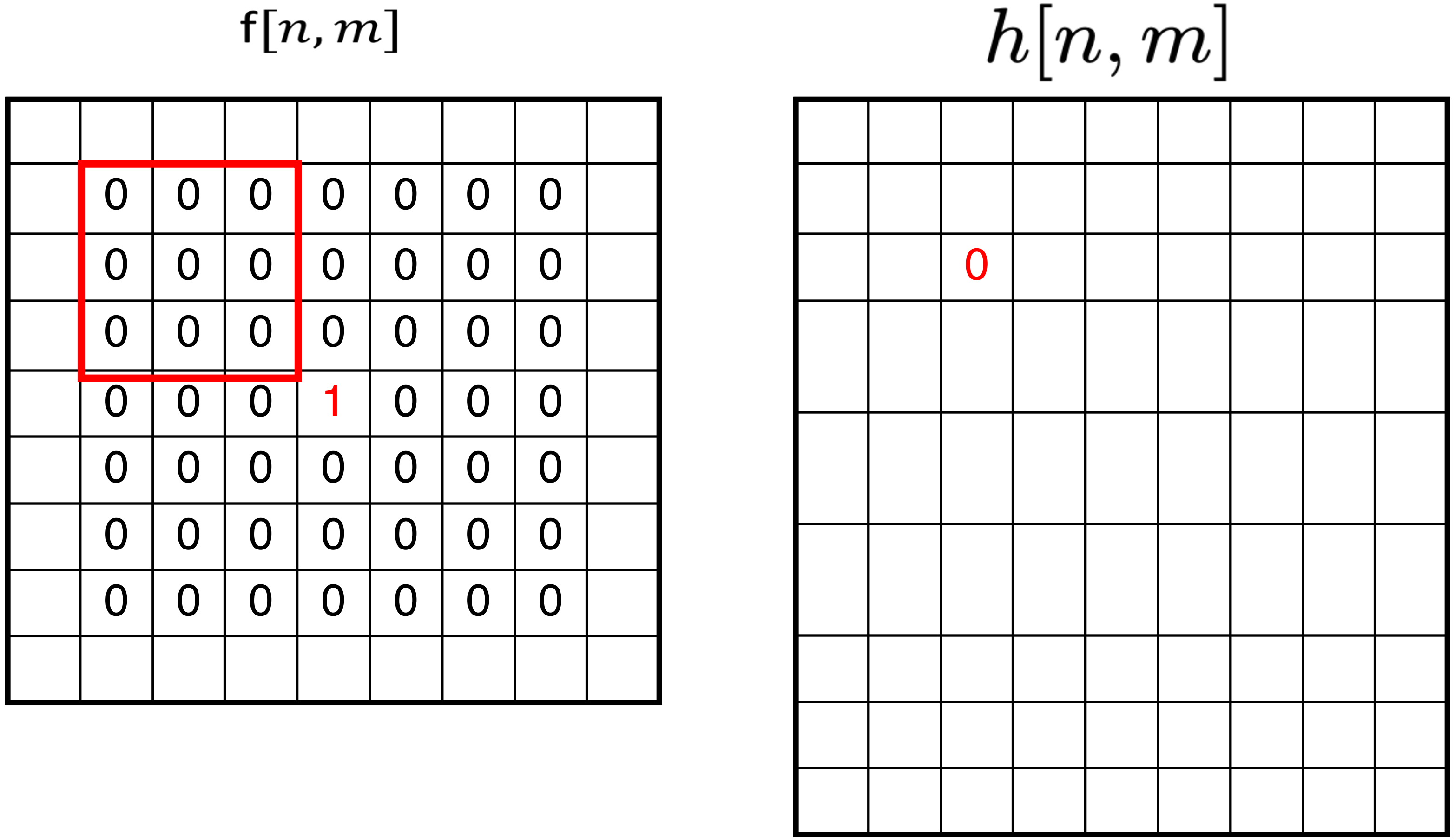
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

Now let's do the same thing with an impulse function



Now let’s do the same thing with an impulse function



Now let's do the same thing with an impulse function

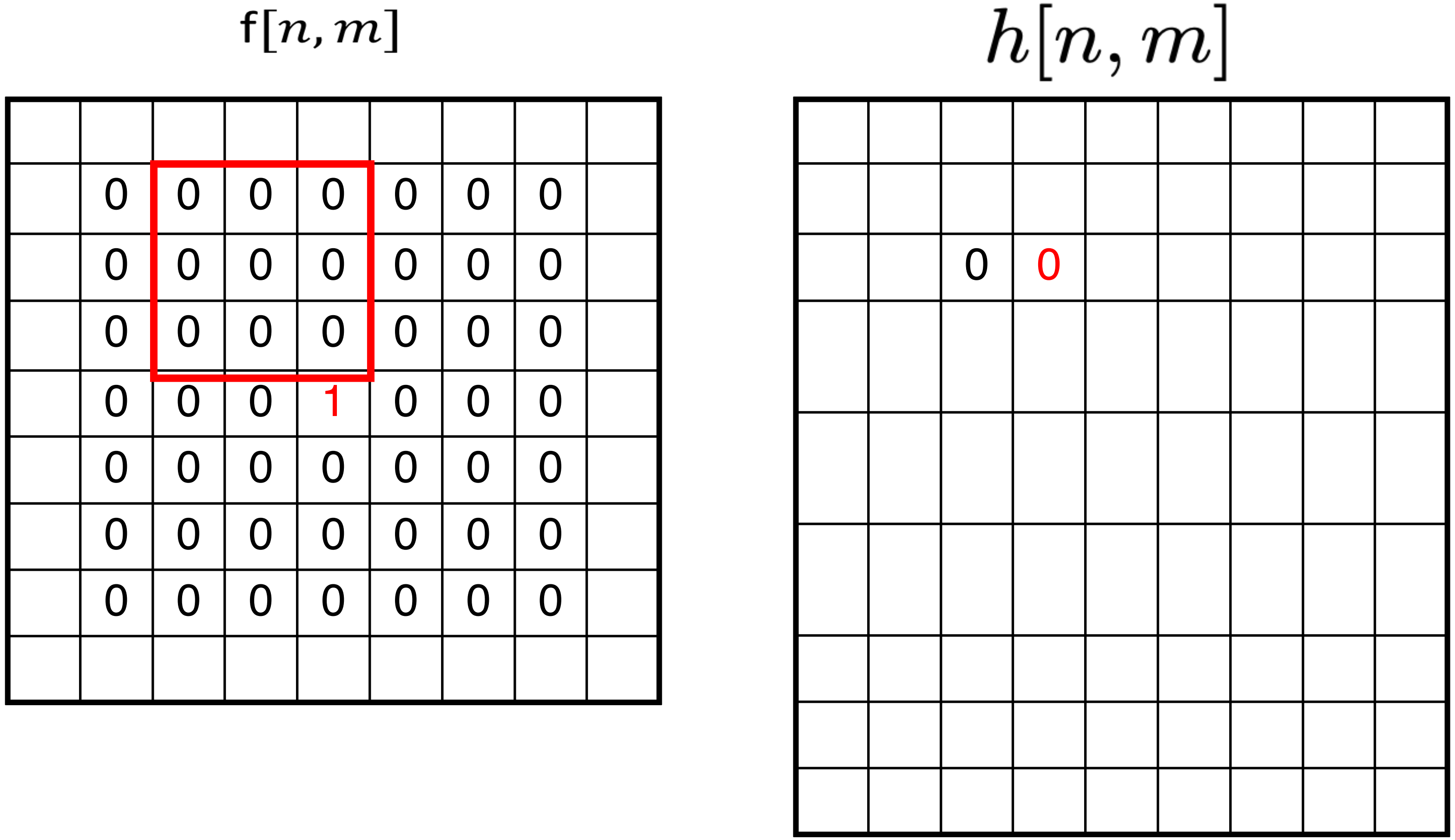
$\mathsf{f}[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

		0	?					

Now let's do the same thing with an impulse function



Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	?					

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9					

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	?				

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9				

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	?					

Now let's do the same thing with an impulse function

f$[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9					

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	?				

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	1/9				

Now let's do the same thing with an impulse function

$f[n, m]$

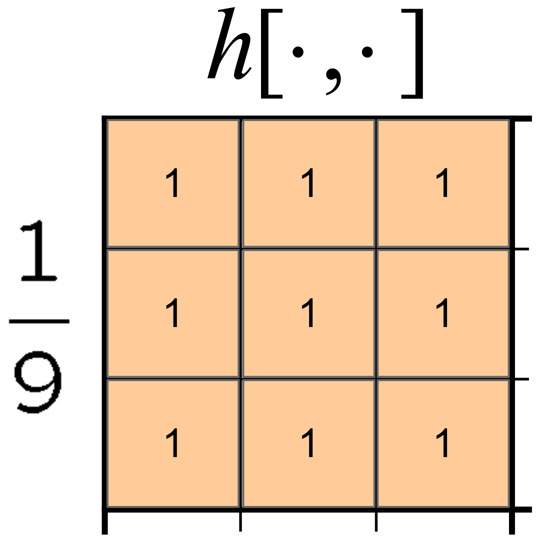
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

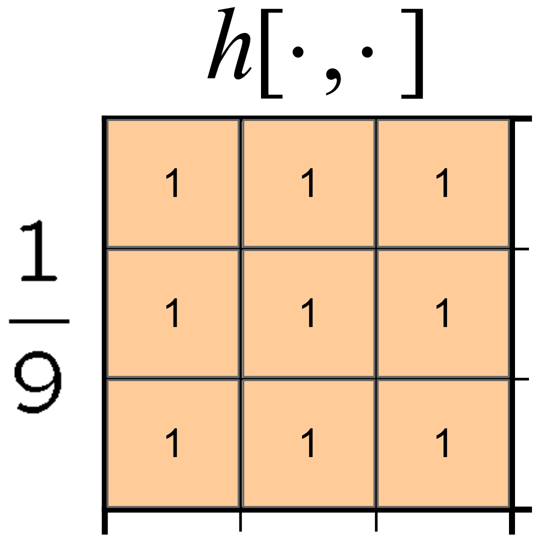
Impulse response of the 3 by 3 moving average filter

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & \underline{1/9} & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$



Notice that any response can be written as a summation of shifted impulse functions

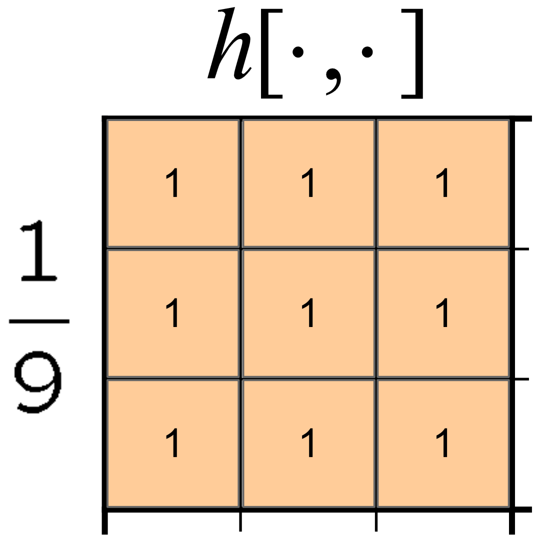
$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & \underline{1/9} & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$



$$h[0,0] = \frac{1}{9}\delta_2[0,0]$$

Notice that any response can be written as a summation of shifted impulse functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$



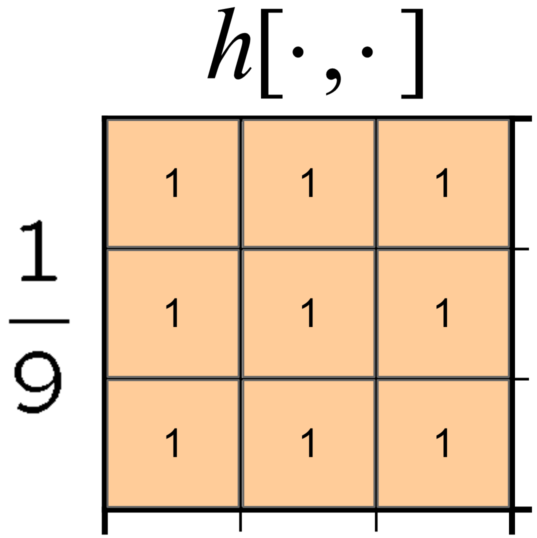
$$h[0,0] = \frac{1}{9}\delta_2[0,0]$$

$$h[0,1] = \frac{1}{9}\delta_2[0,0]$$

Notice that any response can be written as a summation of shifted impulse functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & \underline{1/9} & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$

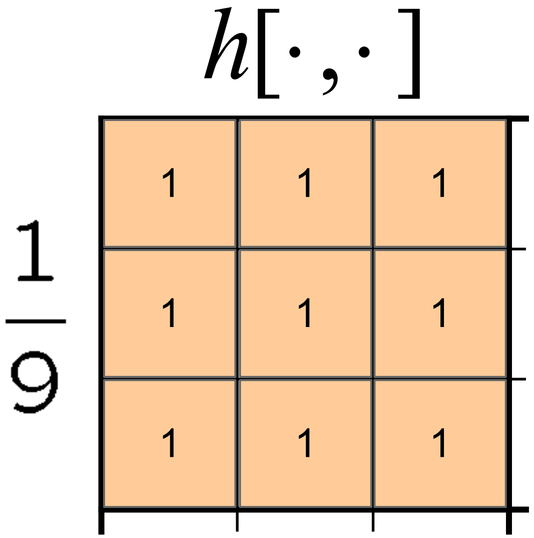


The general form for a moving average h[n,m]

Notice that any response can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & \underline{1/9} & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$

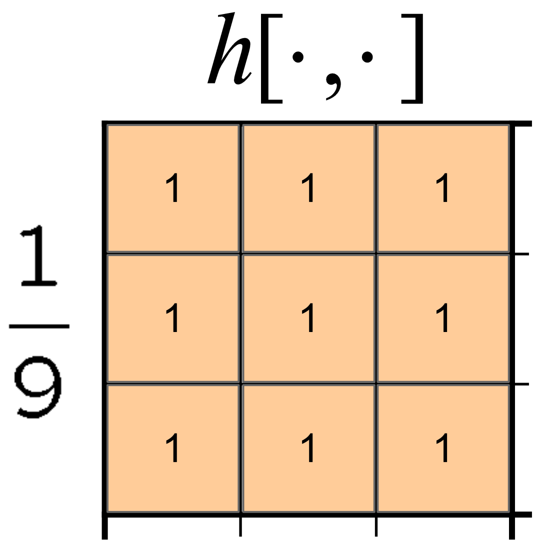


$h[0, 0] = 1/9 * (\delta_2[0+1, 0+1] + \delta_2[0+1, 0-0] + \delta_2[0+1, 0-1] +$
$\delta_2[0-0, 0+1] + \delta_2[0-0, 0-0] + \delta_2[0-0, 0-1] +$
$\delta_2[0-1, 0+1] + \delta_2[0-1, 0-0] + \delta_2[0-1, 0-1])$

Notice that any response can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$



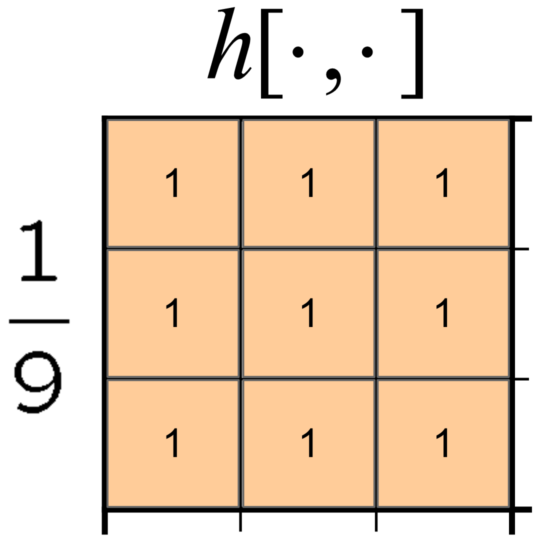
$h[0, 0] = 1/9 * (\delta_2[0+1, 0+1] + \delta_2[0+1, 0-0] + \delta_2[0+1, 0-1] +$
$\delta_2[0-0, 0+1] + \delta_2[0-0, 0-0] + \delta_2[0-0, 0-1] +$
$\delta_2[0-1, 0+1] + \delta_2[0-1, 0-0] + \delta_2[0-1, 0-1])$

$h[1, 1] = 1/9 * (\delta_2[1+1, 1+1] + \delta_2[1+1, 1-0] + \delta_2[1+1, 1-1] +$
$\delta_2[1-0, 1+1] + \delta_2[1-0, 1-0] + \delta_2[1-0, 1-1] +$
$\delta_2[1-1, 1+1] + \delta_2[1-1, 1-0] + \delta_2[1-1, 1-1])$

Notice that any response can be written as a summation of shifted impulse functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$



Exercise: What if we swap n-k for k-n. Does that also work?

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[k-n, l-m]$$

Yes because h is symmetric across the origin

Exercise: What if h was the filter on the right: h[:, -1] = 0

$$h[n, m]$$

(A) $$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$

(B) $$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[k-n, l-m]$$

$$h[\cdot,\cdot] = \frac{1}{9} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is A correct?
Is B correct?
Are both correct?
Are both wrong?

Exercise: What if h was the filter on the right: h[:, -1] = 0

$$h[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=0}^{1}\delta_2[n-k, m-l]$$

$h[\cdot\,,\cdot\,]$ = $\frac{1}{9}$ ×

0	1	1
0	1	1
0	1	1

h[-1, -1] = ~~δ_2[-1+1, -1+1] +~~ δ_2[-1+1, -1-0] + δ_2[-1+1, -1-1] +
~~δ_2[-1+1, -1+1] +~~ δ_2[-1-0, -1-0] + δ_2[-1-0, -1-1] +
~~δ_2[-1+1, -1+1] +~~ δ_2[-1-1, -1-0] + δ_2[-1-1, -1-1]

h[1, 1] = ~~δ_2[1+1, 1+1] +~~ δ_2[1+1, 1-0] + δ_2[1+1, 1-1] +
~~δ_2[1-0, 1+1] +~~ δ_2[1-0, 1-0] + δ_2[1-0, 1-1] +
~~δ_2[1-1, 1+1] +~~ δ_2[1-1, 1-0] + δ_2[1-1, 1-1]

Exercise: What if h was the filter on the right: h[:, -1] = 0

$$h[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=0}^{1}\delta_2[n-k, m-l]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{0}\delta_2[k-n, l-m]$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Because h is not symmetric, we need to invert the range if we invert m-l to l-m

Exercise: play with few numerical examples!

What we will learn today?

- Properties of filters (continued)
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response

Property of (LSI) systems

- An LSI system is completely specified by its impulse response.
  - For any input f, we can compute g using only the impulse response h.

$$f[n,m] \xrightarrow{S} g[n,m]$$

or we can use h to represent S

Property of (LSI) systems

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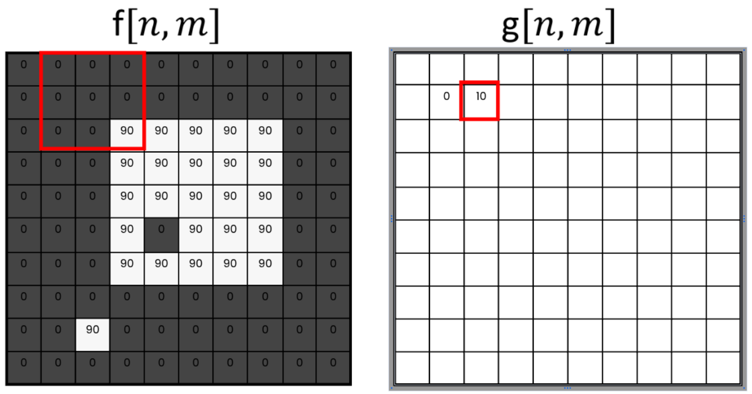

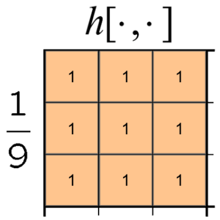

Property of (LSI) systems

- An LSI system is completely specified by its impulse response.
  - For any input f, we can compute g using only the impulse response h.

$$f[n,m] \xrightarrow{S} g[n,m]$$

  or we can use h to represent S

  - **Let's derive an expression for g in terms of h.**

Recall the 3 properties about LSI systems:

1. We know what happens when we send a delta function through an LSI system:

$$\delta_2[n,m] \longrightarrow \boxed{\text{System } \mathcal{S}} \longrightarrow h[n,m]$$

Recall the 3 properties about LSI systems:

1. We know what happens when we send a delta function through an LSI system:

$$\delta_2[n, m] \longrightarrow \boxed{\text{System } \mathcal{S}} \longrightarrow h[n, m]$$

1. We also know that LSI systems shift the output if the input is shifted:

$$\delta_2[n-k, m-l] \longrightarrow \boxed{\text{System } \mathcal{S}} \longrightarrow h[n-k, m-l]$$

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1. Finally, the superposition (linear combination) principle:

$$S\{\alpha f_1[n,m] + \beta f_2[n,m]\} = \alpha S\{f_1[n,m]\} + \beta S\{f_2[n,m]\}$$

Key idea: write down f as a sum of impulses

Let's say our input f is a 3x3 image:

f[0,0]	f[0,1]	f[1,1]
f[1,0]	f[1,1]	f[1,2]
f[2,0]	f[2,1]	f[2,2]

=

f[0,0]	0	0
0	0	0
0	0	0

+

0	f[0,1]	0
0	0	0
0	0	0

+ … +

0	0	0
0	0	0
0	0	f[2,2]

= f[0,0]×

1	0	0
0	0	0
0	0	0

+ f[0,1]×

0	1	0
0	0	0
0	0	0

+ … + f[2,2]×

0	0	0
0	0	0
0	0	1

$$= f[0,0] \cdot \delta_2[n,m] + f[0,1] \cdot \delta_2[n,m-1] + \ldots + f[2,2] \cdot \delta_2[n-2,m-2]$$

pixel value

shifted impulse function

Key idea: write down f as a sum of impulses

- More generally:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

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- More generally:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

- We can now use superposition to see what the output g is:

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \boxed{f[k,l]} \cdot \boxed{\delta_2[n-k, m-l]}$$

For given k, l,
this is a constant

This is a function
of n, m

Key idea: write down f as a sum of impulses

- Superposition

$$S\{\alpha f_1[n,m] + \beta f_2[n,m]\} = \alpha S\{f_1[n,m]\} + \beta S\{f_2[n,m]\}$$

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Key idea: write down f as a sum of impulses

- Superposition

$$S\{\alpha f_1[n,m] + \beta f_2[n,m]\} = \alpha S\{f_1[n,m]\} + \beta S\{f_2[n,m]\}$$

$$\mathcal{S}[\sum_i \alpha_i f_i[n,m]] = \sum_i \alpha_i \mathcal{S}[f_i[n,m]]$$

Exercise!

- We can now use superposition to see what the output g is:

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

For given k, l, this is a constant

This is a function of n, m

Key idea: write down f as a sum of impulses

- Superposition:

$$S\{\alpha f_1[n,m] + \beta f_2[n,m]\} = \alpha S\{f_1[n,m]\} + \beta S\{f_2[n,m]\}$$

$$\mathcal{S}[\sum_i \alpha_i f_i[n,m]] = \sum_i \alpha_i \mathcal{S}[f_i[n,m]]$$

- We can now use superposition to see what the output g is:

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

$$\xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot S\{\delta_2[n-k, m-l]\}$$

Key idea: write down f as a sum of impulses

- From previous slide:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

$$\xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot S\{\delta_2[n-k, m-l]\}$$

- Using shift invariance, we get a shifted impulse response:

$$S\{\delta_2[n-k, m-l]\} = h[n-k, m-l]$$

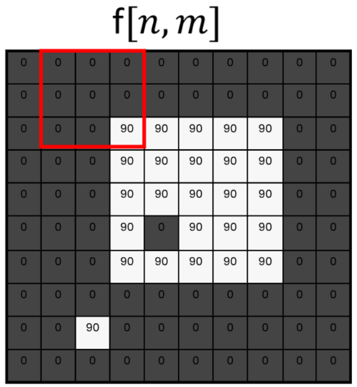
(From previous slide)

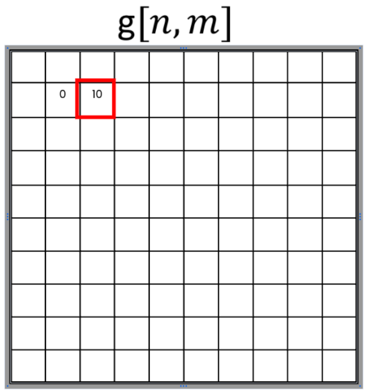
We can write g as a function of h

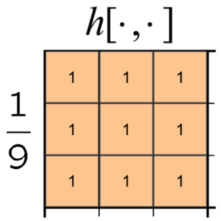
- We have:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

$$\xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot S$$







- Which means:

$$f[n,m] \xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

Linear Shift Invariant (LSI) systems

- An LSI system is completely specified by its impulse response.
  - For any input f, we can compute the output g in terms of the impulse response h.

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] \xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

Discrete Convolution

$$f[n,m] * h[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

Linear Shift Invariant (LSI) systems

- An LSI system is completely specified by its impulse response (we also call them as filters).

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$g[n,m] = f[n,m] * h[n,m]$$

$$f[n,m] * h[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

What we will learn today?

- Linear shift invariant systems
- Impulse functions
- LSI + impulse response
  - Why are they important?

Next time:

More Convolutions & Edges and Lines