Lecture 3

Systems and Convolutions

Administrative

A0 is due today.

- It is ungraded
- Meant to help you with python and numpy basics
- Learn how to do homeworks and submit them on gradescope.

A1 is out

- It is graded
- Due **Jan 24**

Administrative

Recitations (2 options)

- Friday mornings 9:30-10:20am @ MGH 231
- Friday afternoons 12:30-1:20pm @ CSE2 G01

This week:
We will go over Python & Numpy basics

So far: 2D discrete system (filters)

S is the **system operator**, defined as a mapping or assignment of possible inputs f[n,m] to some possible outputs g[n,m].

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

So far: Moving Average

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

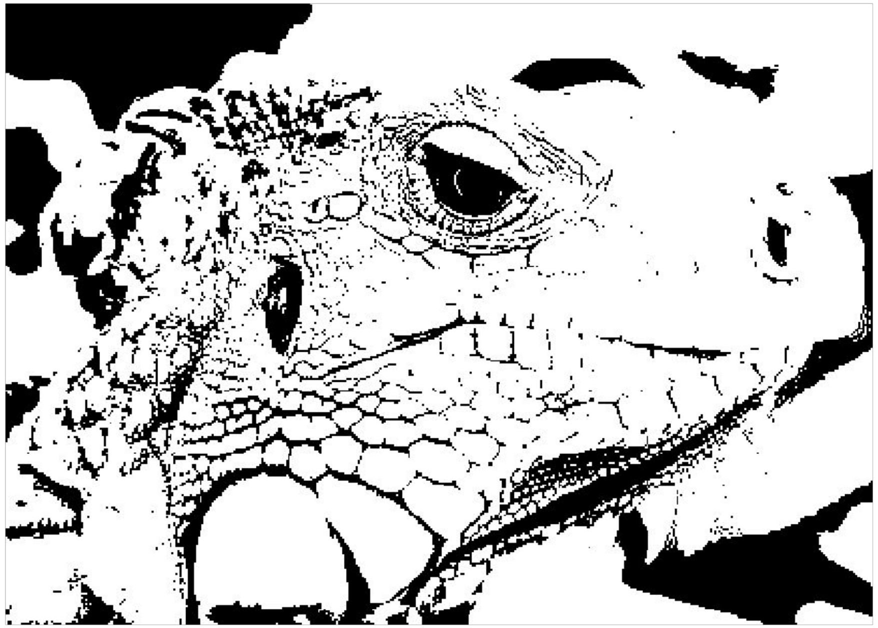
$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Original image

Smoothed image

So far: Image Segmentation

- Use a simple pixel threshold: $$g[n,m] = \begin{cases} 255, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$



So far: Properties of systems

- **Amplitude properties:**
  - Additivity

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

  - Homogeneity

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$

What we will learn today?

- Properties of filters (continued)
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response

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Properties of systems

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  - Homogeneity

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$

  - Superposition

$$\mathcal{S}[\alpha f_i[n,m] + \beta f_j[n,m]] = \alpha \mathcal{S}[f_i[n,m]] + \beta \mathcal{S}[f_j[n,m]]$$

This is an important property. Make sure you know how to prove if any system has this property

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m, \ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

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$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Q. Is the moving average filter stable?

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

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Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]|$$

If $\forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck$ for some constant c and k

Proof of stability

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$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]|$$

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$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k,m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k$$

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$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k,m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k$$

$$\leq \frac{1}{9}(3)(3)k$$

If $\forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck$ for some constant c and k

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$\begin{aligned}
|\mathcal{S}f[n,m]| &= |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k \\
&\leq \frac{1}{9}(3)(3)k \\
&\leq k
\end{aligned}$$

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

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$$\begin{aligned}
|\mathcal{S}f[n,m]| &= |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k \\
&\leq \frac{1}{9}(3)(3)k \\
&\leq k \\
&\leq ck, \text{ where } c = 1
\end{aligned}$$

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m, \ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

  - Invertibility

$$\mathcal{S}^{-1}\mathcal{S}[f[n,m]] = f[n,m]$$

Properties of systems

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  - Stability

$$\text{If } \forall n, m, \ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

  - Invertibility

$$\mathcal{S}^{-1}\mathcal{S}[f[n,m]] = f[n,m]$$

Q. Is the 3x3 moving average filter invertible?

Properties of systems

- Spatial properties
  - Causality

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

Is the moving average filter causal?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

⬅ HINT

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

Properties of systems

- Spatial properties
  - Causality

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

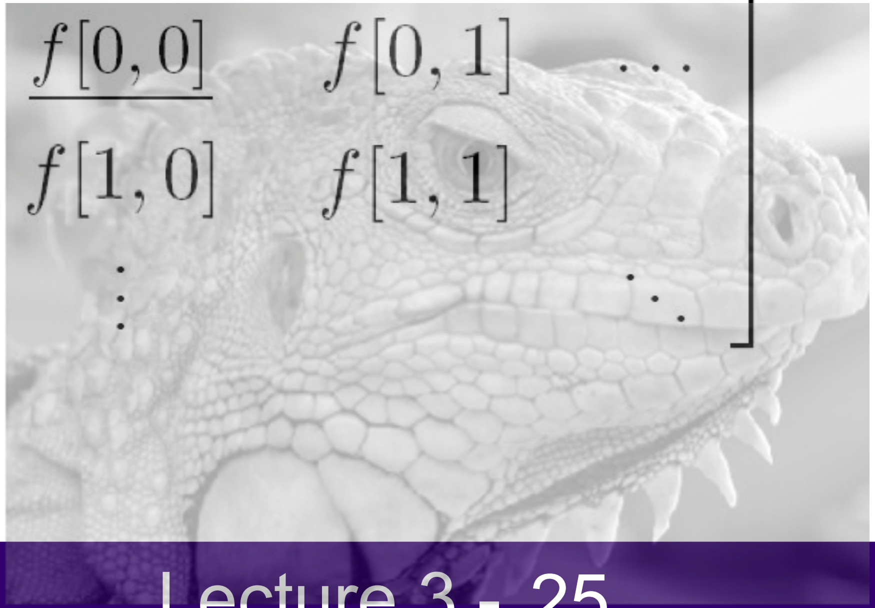
  - Shift invariance:

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

What does shifting an image look like?

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

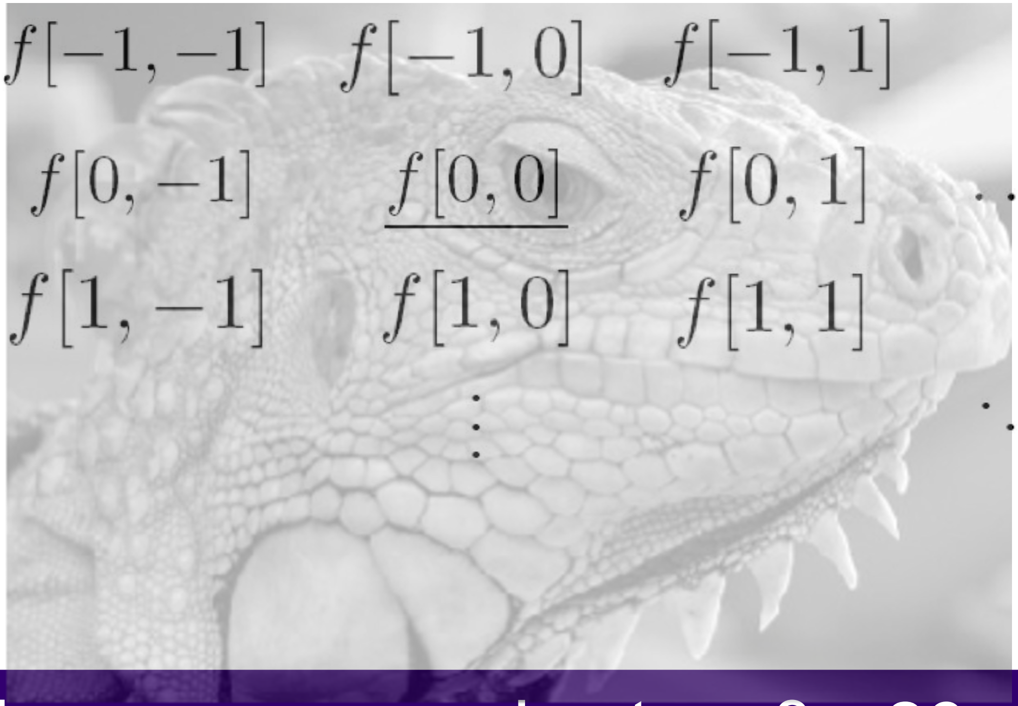
$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \ldots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \ldots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$



Original image

What does shifting an image look like?

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$



$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \dots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \dots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

Shifted image

$n_0 = 1$

$m_0 = 1$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

Is the moving average system is shift invariant?

$$g[n, m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n - k, m - l]$$

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

Is the moving average system is shift invariant?

$$g[n, m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n - k, m - l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$g[n-n_0, m-m_0] = g[n', m']$$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

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$$g[n-n_0, m-m_0] = g[n', m']$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n'-k, m'-l]$$

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Let $n' = n - n_0$ and $m' = m - m_0$

$$\begin{aligned} g[n - n_0, m - m_0] &= g[n', m'] \\ &= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n' - k, m' - l] \\ &= \mathcal{S}[f[n', m']] \end{aligned}$$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

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$$\begin{aligned} g[n-n_0, m-m_0] &= g[n', m'] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n'-k, m'-l] \\ &= \mathcal{S}[f[n', m']] \\ &= \mathcal{S}[f[n-n_0, m-m_0]] \end{aligned}$$

What we will learn today?

- Properties of filters (continued)
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response
- Convolutions and Cross-Correlation

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Linear filtering:
  - Form a new image whose pixels are a weighted sum of original pixel values
  - Use the same set of weights at each point
- **S** is a linear system (function) iff it *S satisfies*

$$S[\,\alpha f_i[n,m] + \beta f_j[k,l]\,] \quad = \quad \alpha\, S[\,f_i[n,m]] + \beta\, S[\,f_j[k,l]\,]$$

superposition property

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Q. Is the moving average a linear system?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Q. Is the moving average a linear system?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

- Q. Is thresholding a linear system?

$$g[n,m] = \begin{cases} 1, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Q. Is the moving average a linear system?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

- Q. Is thresholding a linear system?
  - Let f1[0,0] = f2[n,m] = 0.4
  - Let T = 0.5
  - So, S[f1[0,0]] = S[f2[0,0]] = 0
  - But S[f1[0,0] + f2[0,0]] = 1

$$g[n,m] = \begin{cases} 1, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$

Linear shift invariant (LSI) systems

- Satisfies two properties:

- Superposition property

$$S[\alpha f_i[n,m] + \beta f_j[k,l]] = \alpha S[f_i[n,m]] + \beta S[f_j[k,l]]$$

- Shift invariance:

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

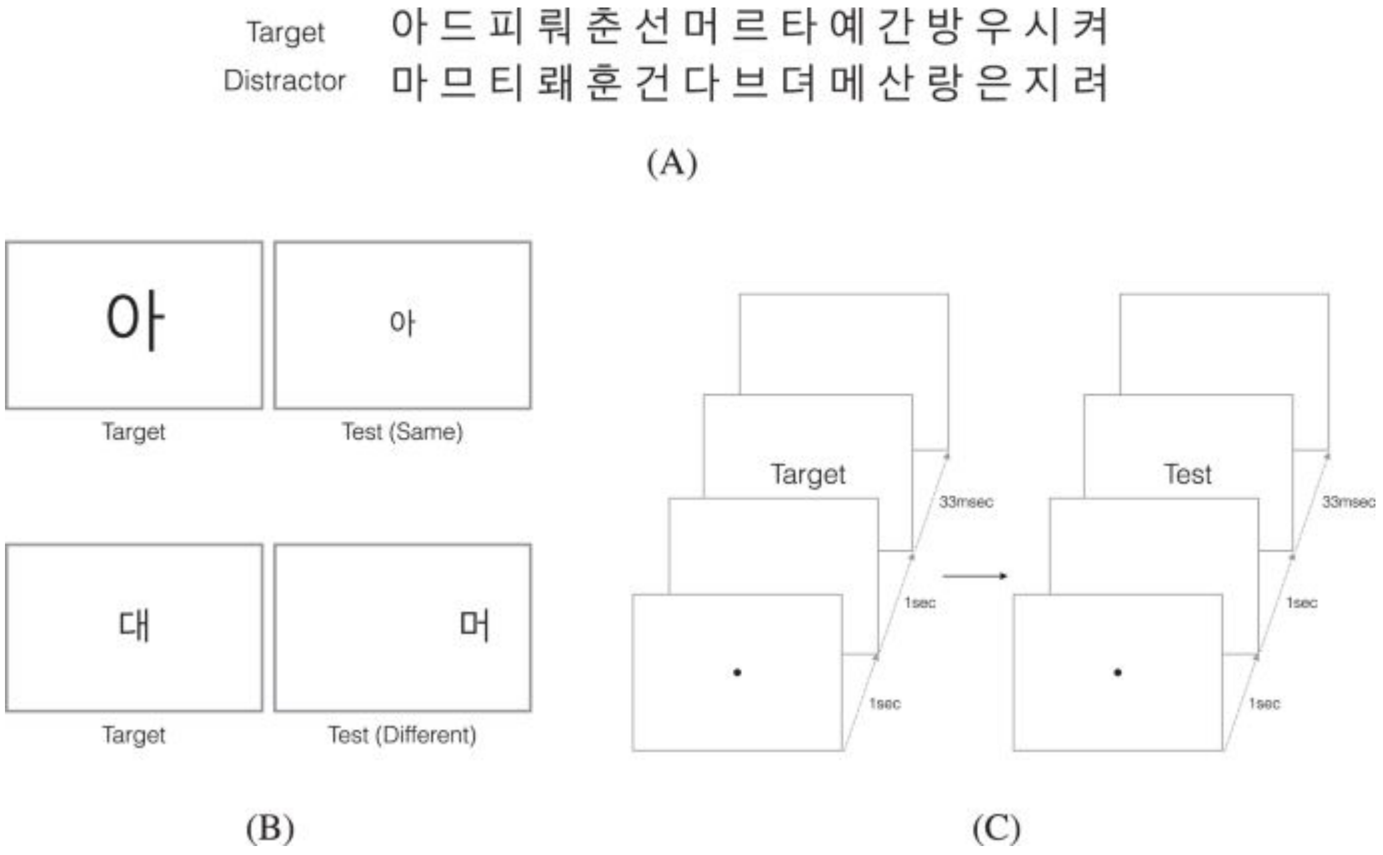
Moving average system is linear shift invariant (LSI)

- We are going to use this as an example to dive into interesting properties about linear shift-invariant systems.

- Why are linear shift invariant systems important?

Our visual system is a
linear shift invariant system

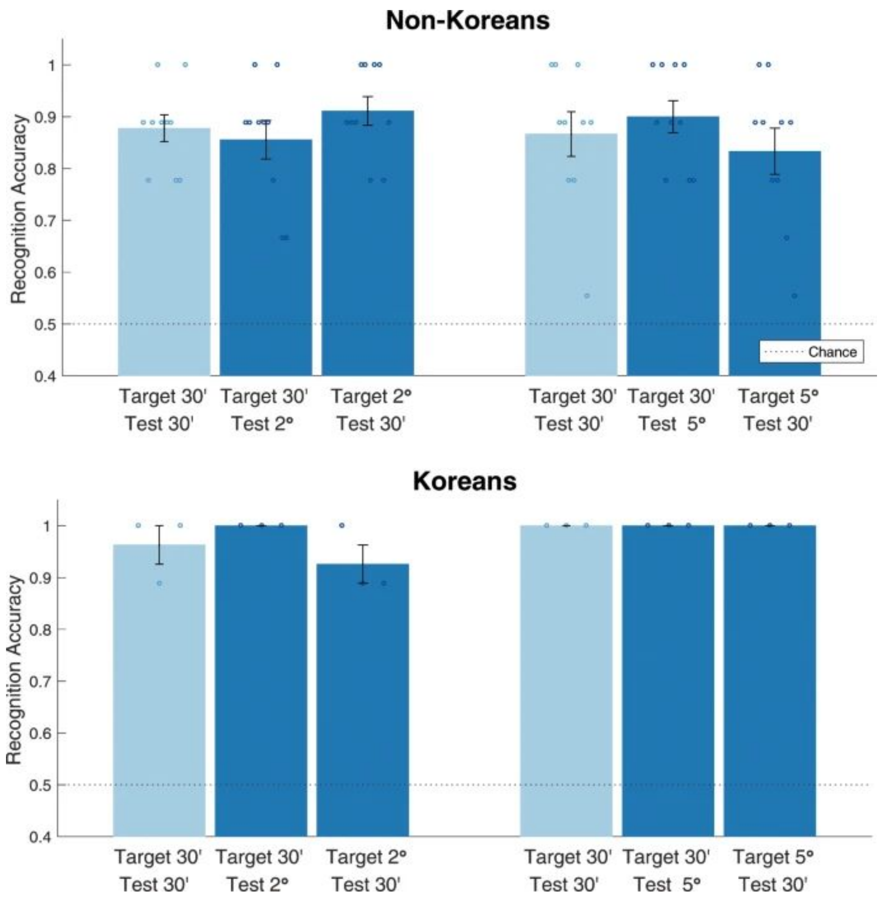
Human vision are scale and translation invariant



Participants were shown some target Korean character once and were tested on whether they can identify the targets from other distractors

Han et al. Scale and translation-invariance for novel objects in human vision. Nature 2020 [link]

Human vision are scale and translation invariant



Very high recognition accuracies

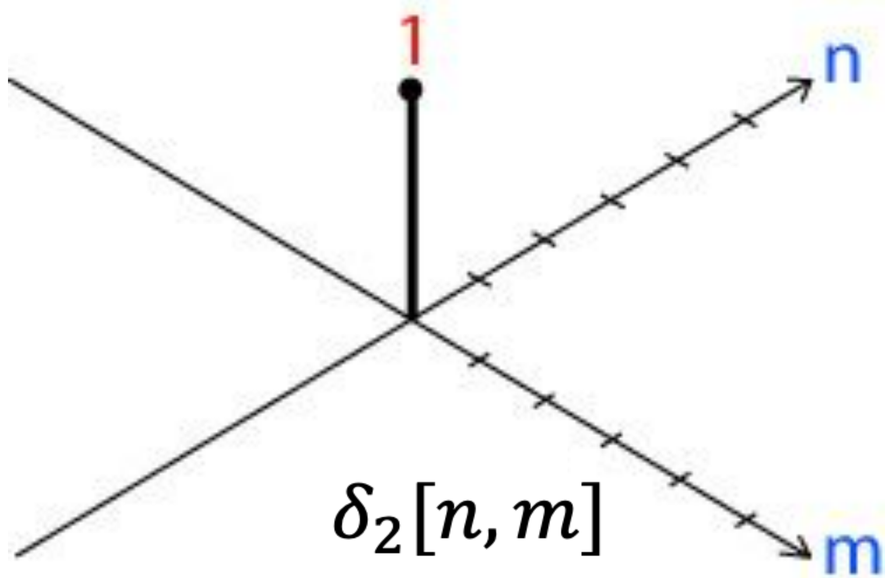
Han et al. Scale and translation-invariance for novel objects in human vision. Nature 2020 [link]

What we will learn today?

- Properties of filters (continued)
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response

2D impulse function

- Let's look at a special function
- 1 at the origin [0,0].
- 0 everywhere else



2D impulse function as an image

- Let’s look at a special function
- 1 at the origin [0,0].
- 0 everywhere else

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

What happens when we pass an impulse function through a LSI systems

- The moving average filter equation again: $g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$

$$\delta_2[n,m] \xrightarrow{S} h[n,m]$$

Pass in an impulse function **Record its response**

- By passing an impulse function into an LSI system, we get it's impulse response.
  - We will use h[n, m] to refer to the impulse response

What happens when we pass an impulse function through a LSI systems

Before we do this, let's remember how we used the moving average filter last lecture

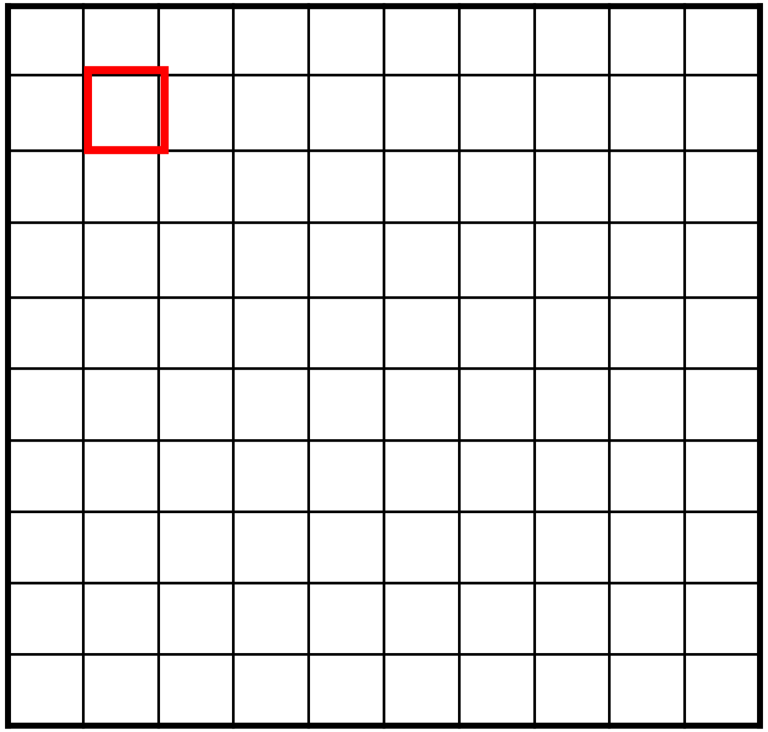
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

Remember the Moving Average filter from last lecture

$\mathrm{f}[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\mathrm{g}[n, m]$



Courtesy of S. Seitz

Remember the Moving Average filter from last lecture

$f[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n, m]$

	0	10							

Remember the Moving Average filter from last lecture

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$

	0	10	20						

Remember the Moving Average filter from last lecture

$f[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n, m]$

	0	10	20	30					

Remember the Moving Average filter from last lecture

$f[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n, m]$

	0	10	20	30	30				

Remember the Moving Average filter from last lecture

$f[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n, m]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

		?						

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

		0						

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

		0	?					

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

		0	0					

Now let's do the same thing with an impulse function

$\mathrm{f}[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	?						

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0						

Now let's do the same thing with an impulse function

$\mathrm{f}[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	?					

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9					

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	?				

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9				

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	?					

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9					

Now let's do the same thing with an impulse function

$f[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	?				

Now let's do the same thing with an impulse function

$f[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n,m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	1/9				

Now let's do the same thing with an impulse function

$\mathsf{f}[n, m]$

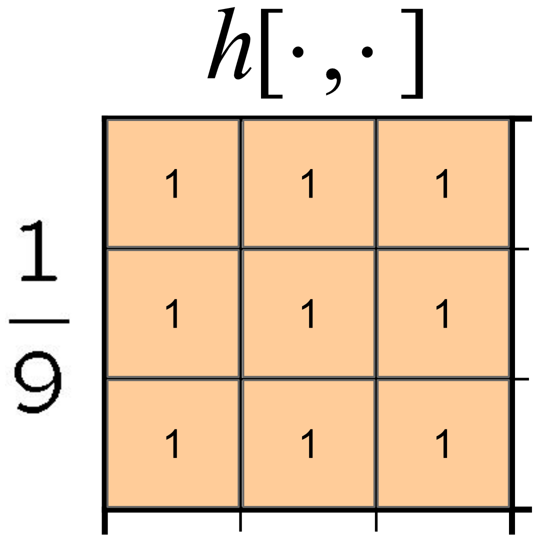
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	1	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

$h[n, m]$

	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	1/9	1/9	1/9	0	0	
	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	

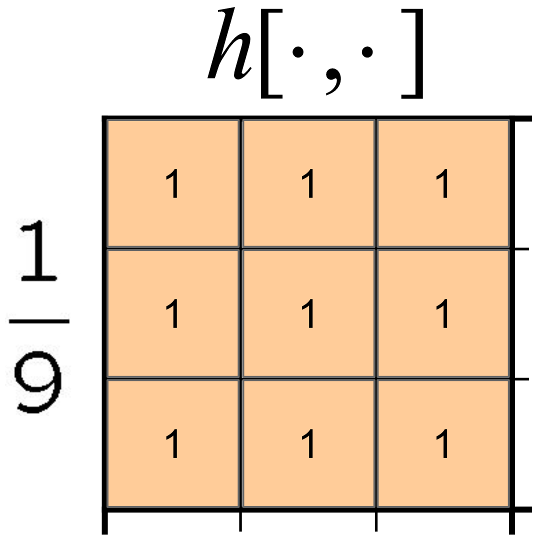
Impulse response of the 3 by 3 moving average filter

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & \underline{1/9} & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$



Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & \underline{1/9} & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$



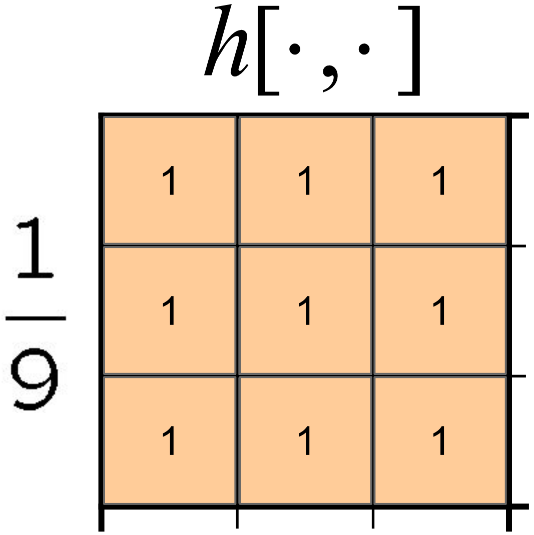
$$h[0,0] = \frac{1}{9}\delta_2[0,0]$$

Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

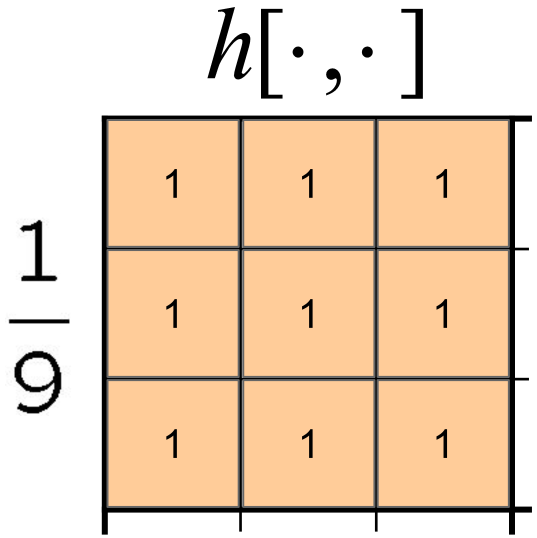
$$h[0,0] = \frac{1}{9}\delta_2[0,0]$$

$$h[0,1] = \frac{1}{9}\delta_2[0,0]$$



Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$



$$h[0,0] = \frac{1}{9}\delta_2[0,0]$$

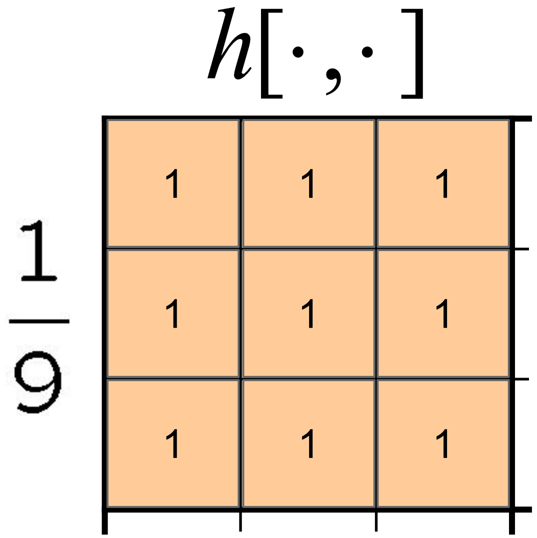
$$h[0,1] = \frac{1}{9}\delta_2[0,0]$$

Q. For what values of **n** and **m** is h[,] **not** zero?

Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$

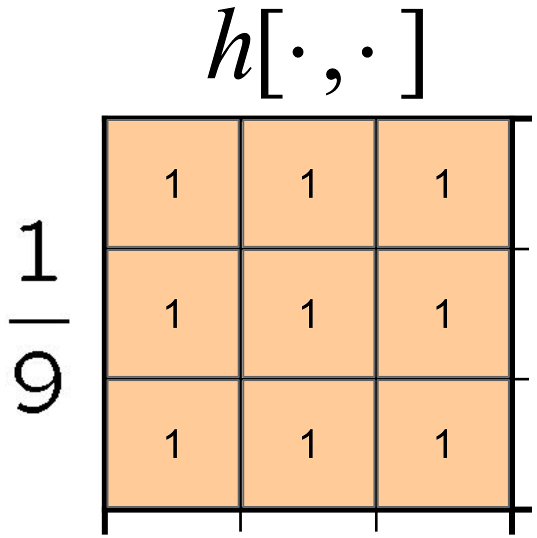


The general form for a moving average h[n,m]

Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$

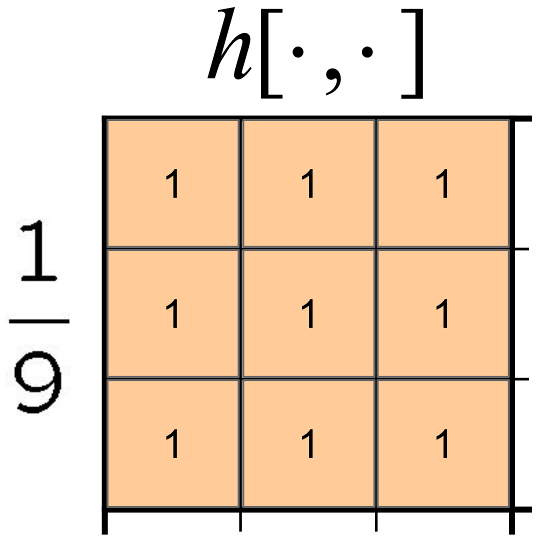


Q. Why is this the general form?

Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$



Q. Why is this the general form?

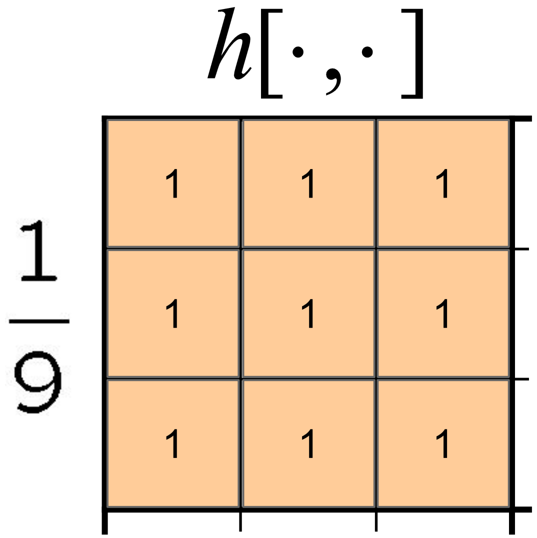
As long as n-1, n, or n+1 is 0, the value is 1/9

Same for m

Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$



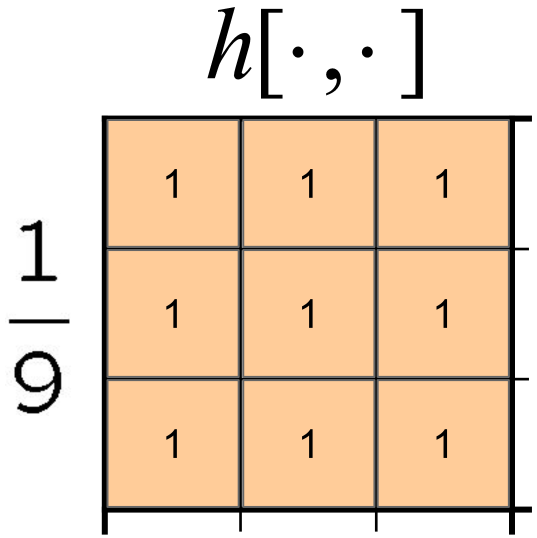
Q. What if we swap n-k for k-n. Does that also work?

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[k-n, l-m]$$

Notice that any filter can be written as a summation of shifted delta functions

$$h[n,m] = \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$$



Q. What if we swap n-k for k-n. Does that also work?

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[k-n, l-m]$$

Yes because h is symmetric across the origin

Q. What if h was the filter on the right:

h[:, -1] = 0

$h[n, m]$

(A) $= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[n-k, m-l]$

(B) $= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} \delta_2[k-n, l-m]$

$h[\cdot, \cdot]$

$\frac{1}{9}$

0	1	1
0	1	1
0	1	1

Is A correct?
Is B correct?
Are both correct?
Are both wrong?

Q. What if h was the filter on the right:

h[:, -1] = 0

$$h[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=0}^{1}\delta_2[n-k, m-l]$$

$h[\cdot\,,\cdot\,]$

$\frac{1}{9}$

0	1	1
0	1	1
0	1	1

Q. What if h was the filter on the right:

h[:, -1] = 0

$$h[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=0}^{1}\delta_2[n-k, m-l]$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{0}\delta_2[k-n, l-m]$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Because h is not symmetric, we need to invert the range if we invert m-l to l-m

What we will learn today?

- Properties of filters (continued)
- Linear shift invariant systems
- Impulse functions
- LSI + impulse response

Property of (LSI) systems

- An LSI system is completely specified by its impulse response.
  - For any input f, we can compute g using only the impulse response h.

$$f[n,m] \xrightarrow{S} g[n,m]$$

Property of (LSI) systems

- An LSI system is completely specified by its impulse response.
  - For any input f, we can compute g using only the impulse response h.

$$f[n,m] \xrightarrow{S} g[n,m]$$

  - **Let's derive an expression for g in terms of h.**

Recall the 3 properties about LSI systems:

1. We know what happens when we send a delta function through an LSI system:

$$\delta_2[n,m] \longrightarrow \boxed{\text{System } \mathcal{S}} \longrightarrow h[n,m]$$

Recall the 3 properties about LSI systems:

1. We know what happens when we send a delta function through an LSI system:

$$\delta_2[n,m] \longrightarrow \boxed{\text{System } \mathcal{S}} \longrightarrow h[n,m]$$

2. We also know that LSI systems shift the output if the input is shifted:

$$\delta_2[n-k,m-l] \longrightarrow \boxed{\text{System } \mathcal{S}} \longrightarrow h[n-k,m-l]$$

Recall the 3 properties about LSI systems:

1. We know what happens when we send a delta function through an LSI system:

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2. We also know that LSI systems shift the output if the input is shifted:

$$\delta_2[n-k,m-l] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow h[n-k,m-l]$$

3. Finally, the superposition principle:

$$S\{\alpha f_1[n,m] + \beta f_2[n,m]\} = \alpha S\{f_1[n,m]\} + \beta S\{f_2[n,m]\}$$

Key idea: write down f as a sum of impulses

Let's say our input f is a 3x3 image:

f[0,0]	f[0,1]	f[1,1]
f[1,0]	f[1,1]	f[1,2]
f[2,0]	f[2,1]	f[2,2]

=

f[0,0]	0	0
0	0	0
0	0	0

+

0	f[0,1]	0
0	0	0
0	0	0

+ . . . +

0	0	0
0	0	0
0	0	f[2,2]

Key idea: write down f as a sum of impulses

Let's say our input f is a 3x3 image:

$$
\begin{bmatrix} f[0,0] & f[0,1] & f[1,1] \\ f[1,0] & f[1,1] & f[1,2] \\ f[2,0] & f[2,1] & f[2,2] \end{bmatrix}
=
\begin{bmatrix} f[0,0] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
+
\begin{bmatrix} 0 & f[0,1] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
+ \ldots +
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f[2,2] \end{bmatrix}
$$

$$
= f[0,0] \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
+ f[0,1] \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
+ \ldots + f[2,2] \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Key idea: write down f as a sum of impulses

Let's say our input f is a 3x3 image:

f[0,0]	f[0,1]	f[1,1]
f[1,0]	f[1,1]	f[1,2]
f[2,0]	f[2,1]	f[2,2]

=

f[0,0]	0	0
0	0	0
0	0	0

+

0	f[0,1]	0
0	0	0
0	0	0

+ … +

0	0	0
0	0	0
0	0	f[2,2]

= f[0,0] ×

1	0	0
0	0	0
0	0	0

+ f[0,1] ×

0	1	0
0	0	0
0	0	0

+ … + f[2,2] ×

0	0	0
0	0	0
0	0	1

$$= f[0,0] \cdot \delta_2[n,m] + f[0,1] \cdot \delta_2[n,m-1] + \ldots + f[2,2] \cdot \delta_2[n-2,m-2]$$

Key idea: write down f as a sum of impulses

- More generally:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

Key idea: write down f as a sum of impulses

- More generally:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

- We can now use superposition to see what the output g is:

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

Key idea: write down f as a sum of impulses

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- We can now use superposition to see what the output g is:

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \boxed{f[k,l]} \cdot \delta_2[n-k, m-l]$$

For given k, l,
this is a constant

Key idea: write down f as a sum of impulses

- More generally:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

- We can now use superposition to see what the output g is:

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \boxed{f[k,l]} \cdot \boxed{\delta_2[n-k, m-l]}$$

For given k, l, this is a constant

This is a function of n, m

Key idea: write down f as a sum of impulses

- Superposition

$$S\{\alpha f_1[n,m] + \beta f_2[n,m]\} = \alpha S\{f_1[n,m]\} + \beta S\{f_2[n,m]\}$$

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$$\mathcal{S}[\sum_i \alpha_i f_i[n,m]] = \sum_i \alpha_i \mathcal{S}[f_i[n,m]]$$

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$$\xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot S\{\delta_2[n-k, m-l]\}$$

Key idea: write down f as a sum of impulses

- Superposition:

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Key idea: write down f as a sum of impulses

- From previous slide:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

$$\xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot S\{\delta_2[n-k, m-l]\}$$

- Using shift invariance, we get a shifted impulse response:

$$S\{\delta_2[n-k, m-l]\} = h[n-k, m-l]$$

We can write g as a function of h

- We have:

$$f[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot \delta_2[n-k, m-l]$$

$$\xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot S\{\delta_2[n-k, m-l]\}$$

- Which means:

$$f[n,m] \xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

Linear Shift Invariant (LSI) systems

- An LSI system is completely specified by its impulse response.
  - For any input f, we can compute the output g in terms of the impulse response h.

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$f[n,m] \xrightarrow{S} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

Discrete Convolution

$$f[n,m] * h[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

Linear Shift Invariant (LSI) systems

- An LSI system is completely specified by its impulse response.

$$f[n,m] \xrightarrow{S} g[n,m]$$

$$g[n,m] = f[n,m] * h[n,m]$$

$$f[n,m] * h[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l] \cdot h[n-k, m-l]$$

What we will learn today?

- Linear shift invariant systems
- Impulse functions
- LSI + impulse response

Next time:

Edges and lines