# Lecture 16

Recognition, kNN and PCA

# Administrative

A4 should be out soon
- Due May 30th

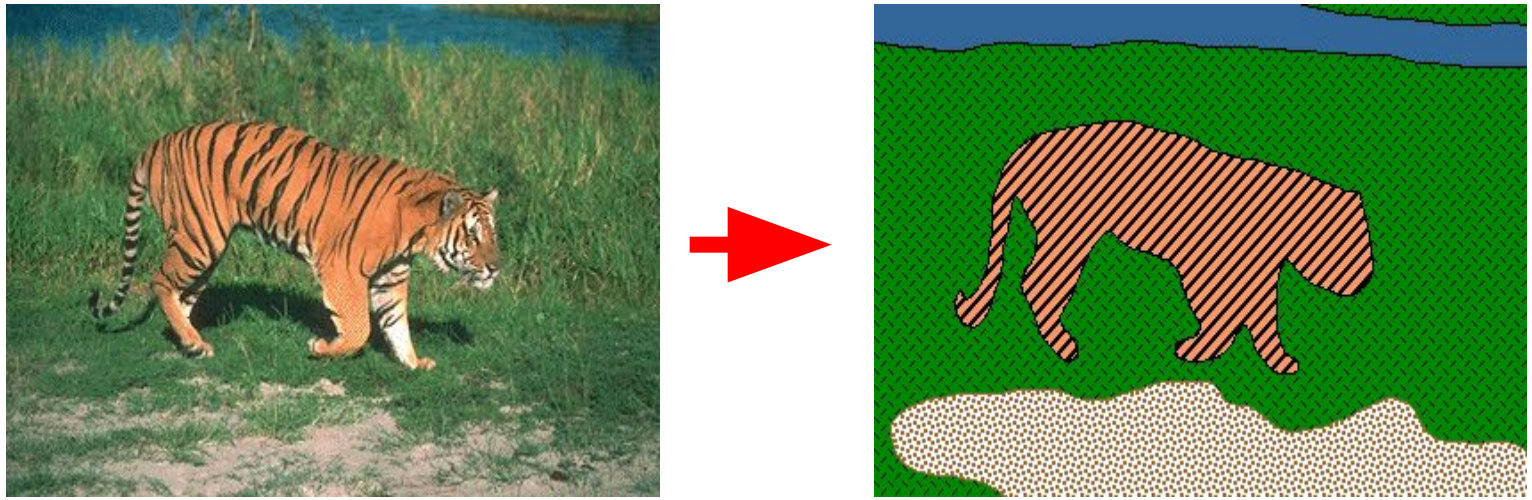
Memorial day:
- No class on May 26th.
- We will cover some dimensionality reduction today (PCA)

Recitation this week:
- Will cover LDA for dimensionality reduction

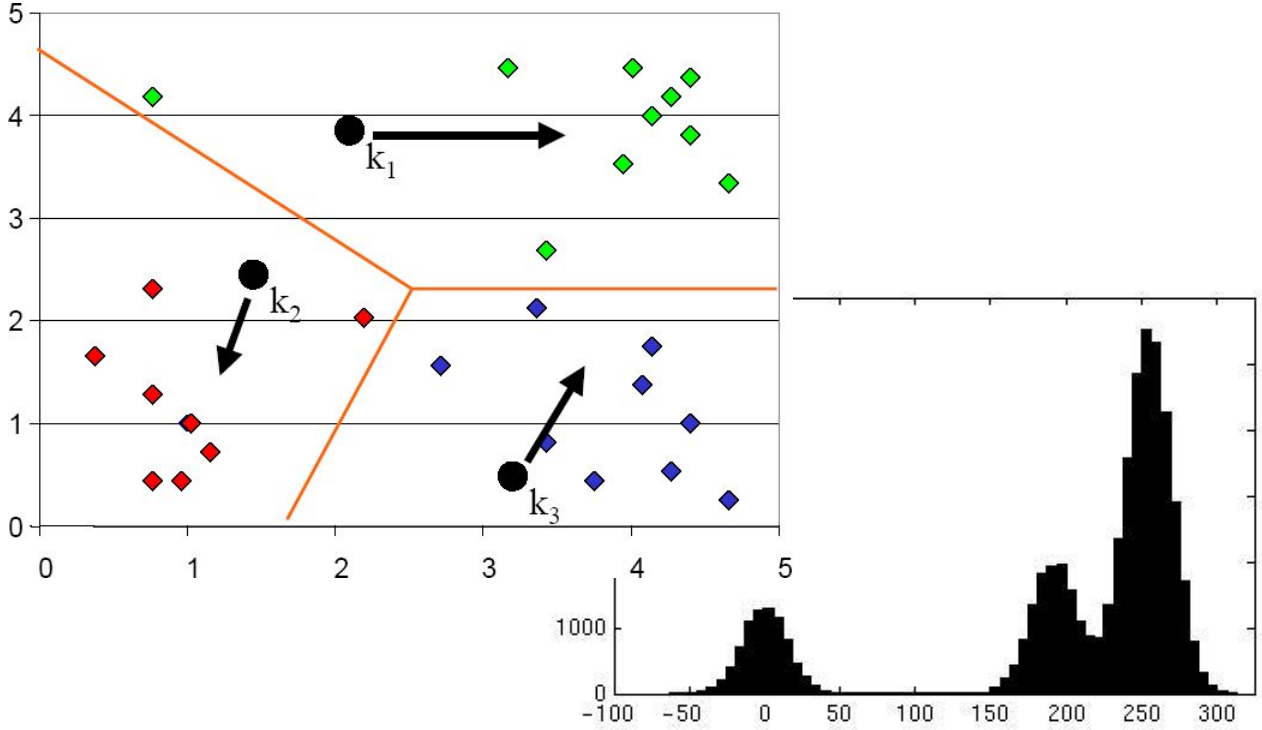
Final exam:
- June 9th, Make up on June 6th
- T/F, MCQ, Short answers

# So far: Segmentation

cluster meaningful groups of pixels

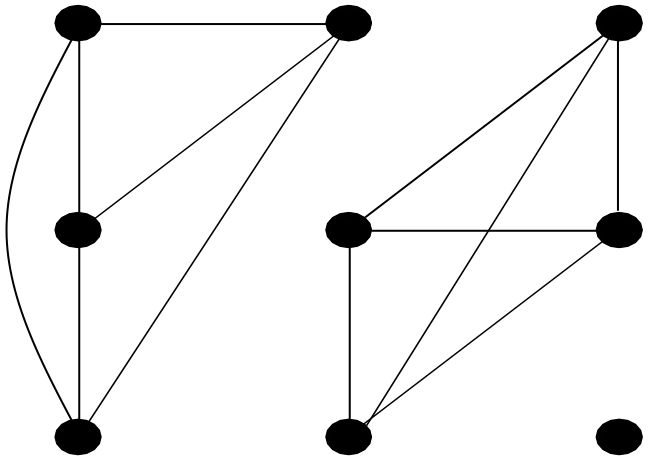


# So far: Segmentation




K-Means and
Mean-shift

Segmentation using graph cuts



# Today's agenda

- Introduction to recognition
- A object recognition pipeline
- Choosing the right features
- A training algorithm: KNN
- Testing an algorithm
- Challenges with kNN
- Dimensionality reduction: PCA

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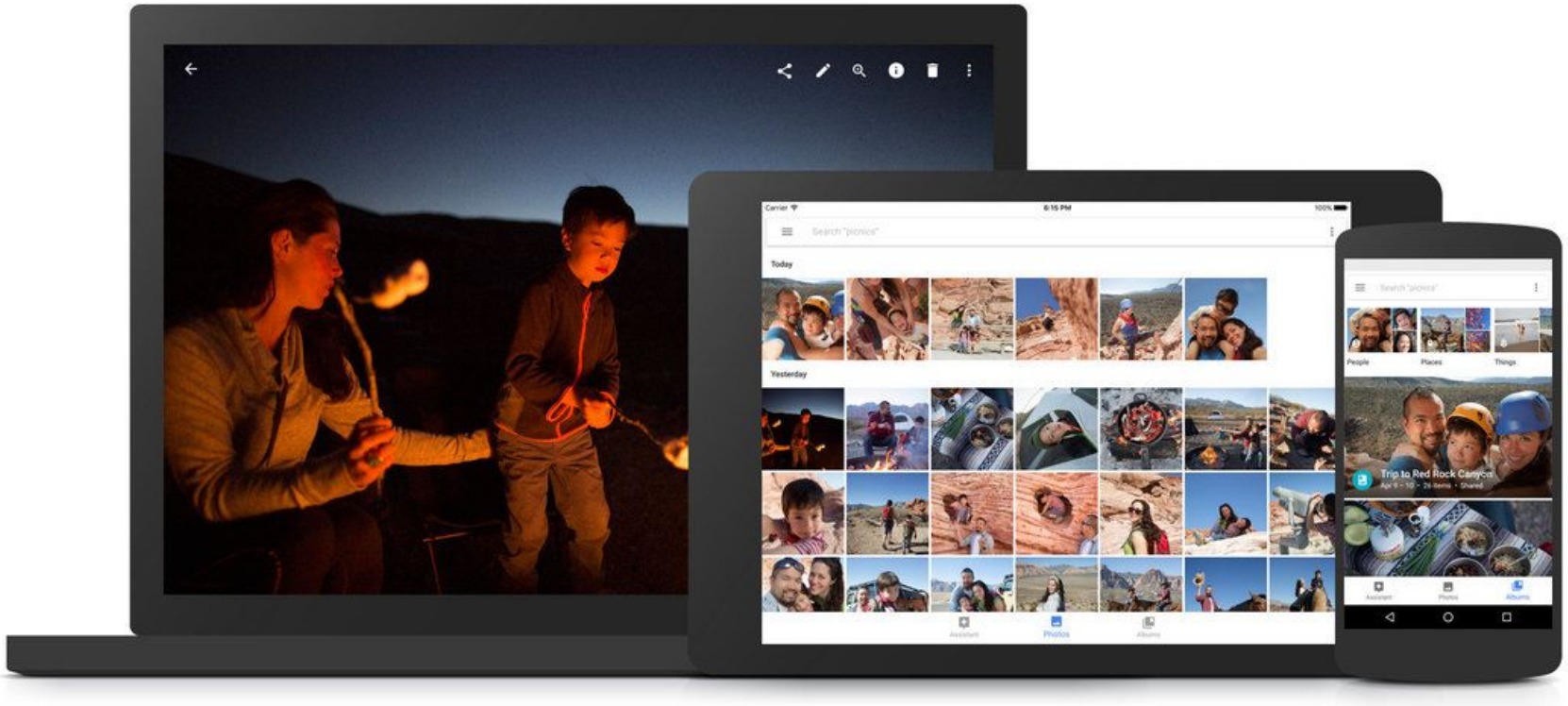
# What do we mean by recognition?

**Classification**: Does this image contain a building? [yes/no]

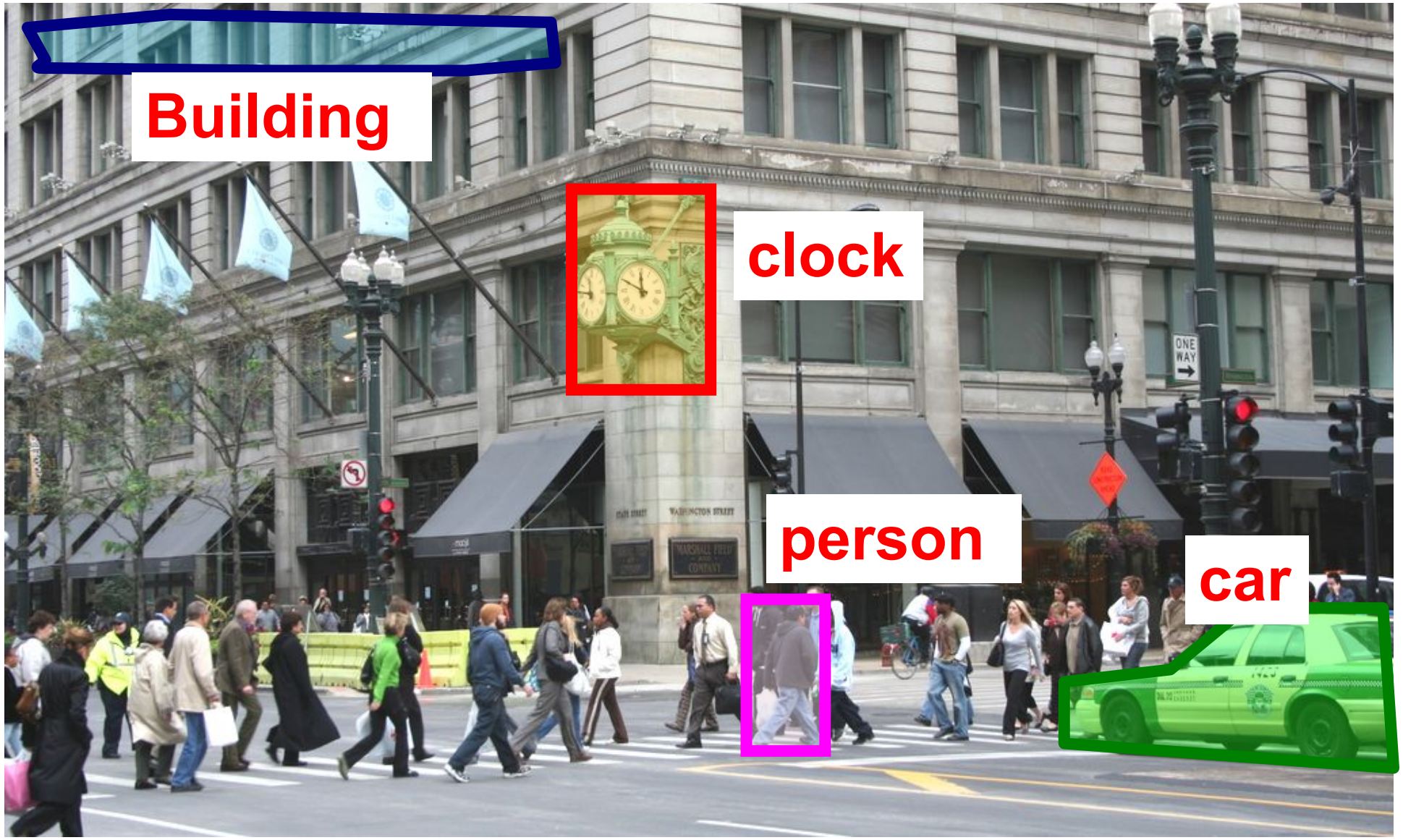
# **Classification:** Is this an beach?

# Applications: Image Search & Organizing photo collections



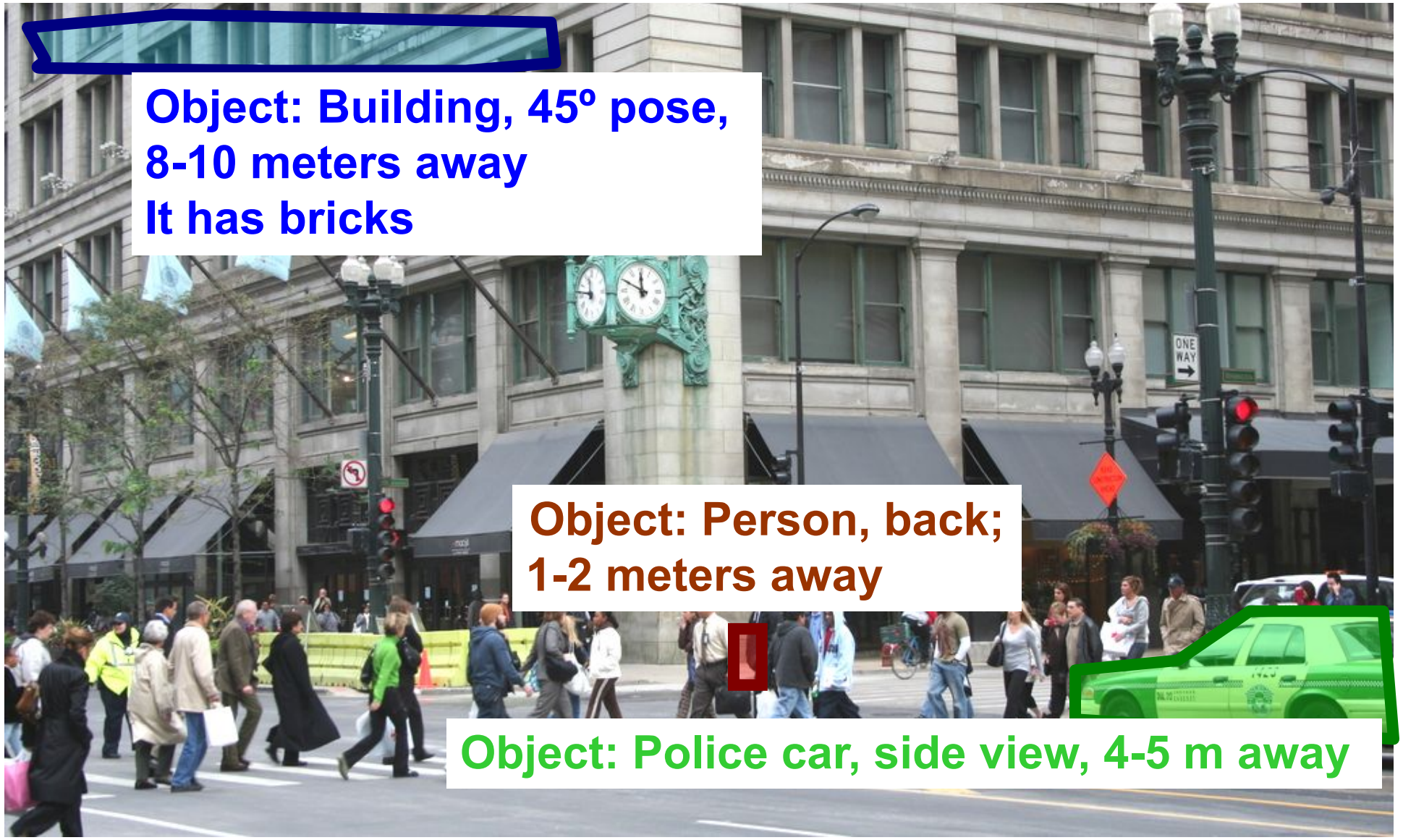
# **Detection:** Does this image contain a car? [where?]

**Detection:** Which object does this image contain? [where?]



# **Detection:** Accurate localization (segmentation)

# **Detection:** Estimating object semantic & geometric attributes



# Levels of recognition: Category-level vs instance-level

Does this image contain the Chicago Macy's building?

# Categorization vs Single instance recognition

We have seen a form of single instance categorization already: **Where is the crunchy nut?**


# Applications of computer vision

Recognizing landmarks
in mobile devices

# **Activity recognition:** What are these people doing?

# Visual Recognition

- Design algorithms that can:
  - Classify images or videos
  - Detect and localize objects
  - Estimate semantic and geometrical attributes
  - Classify human activities and events

Why is this challenging?

**How many object categories are there?**

# **Challenges**: viewpoint variation

Michelangelo 1475-1564

# **Challenges**: illumination

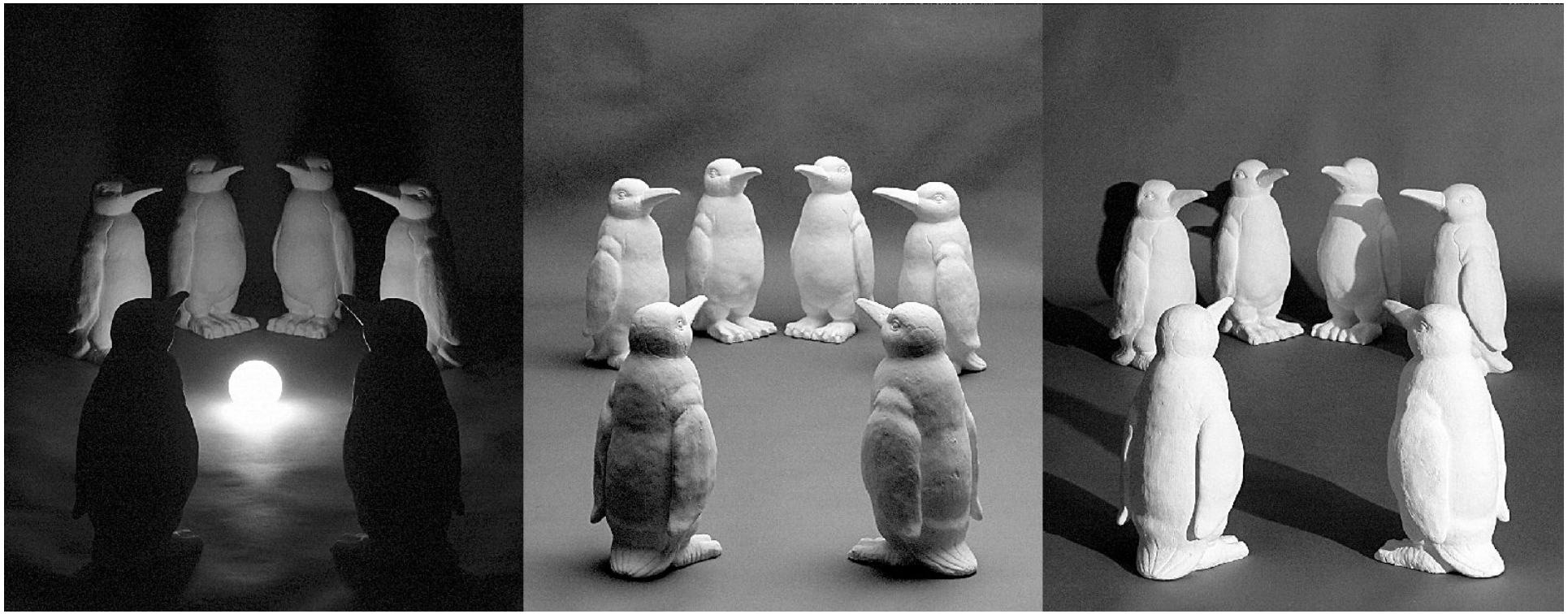
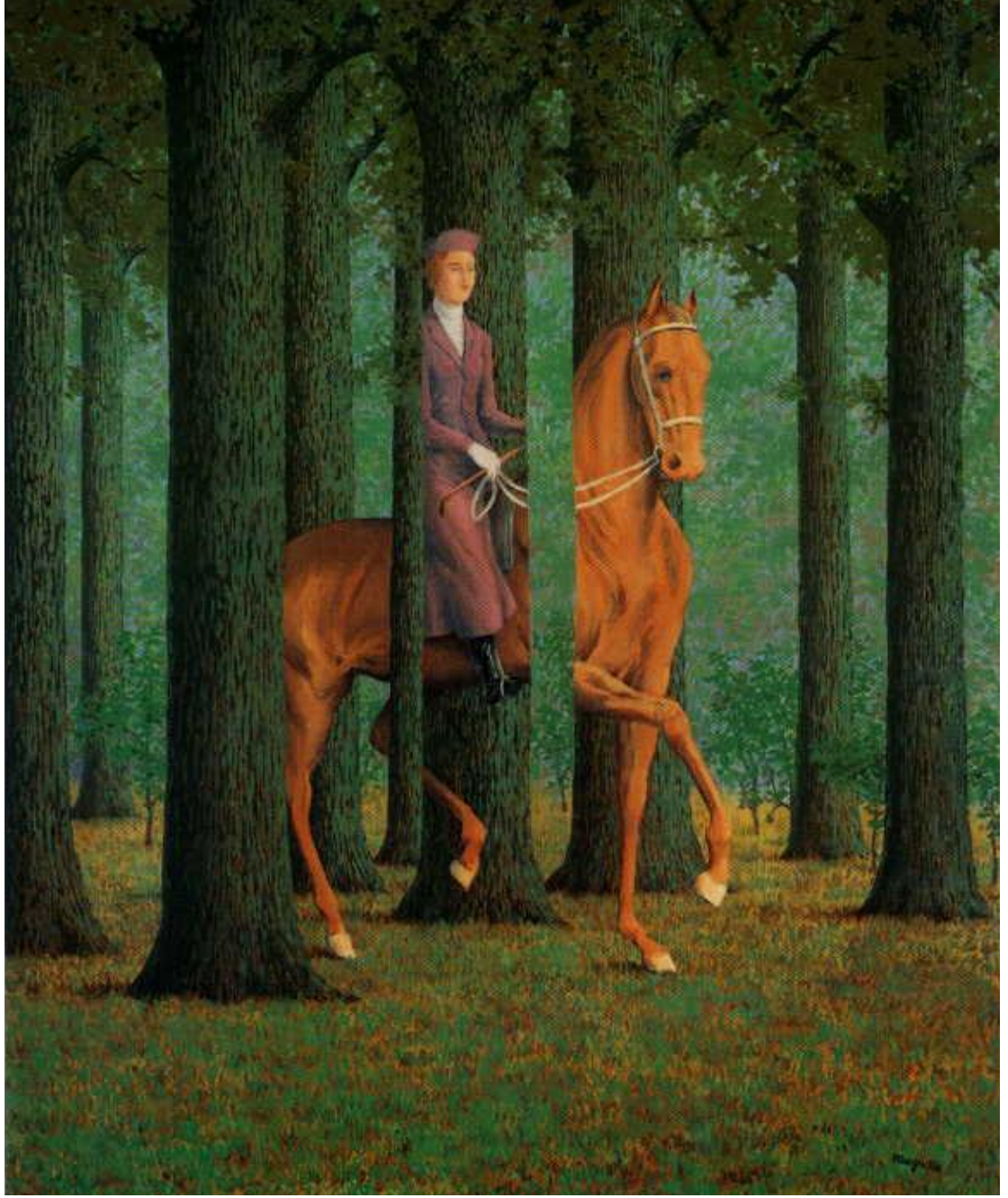


image credit: J. Koenderink

**Challenges**: scale

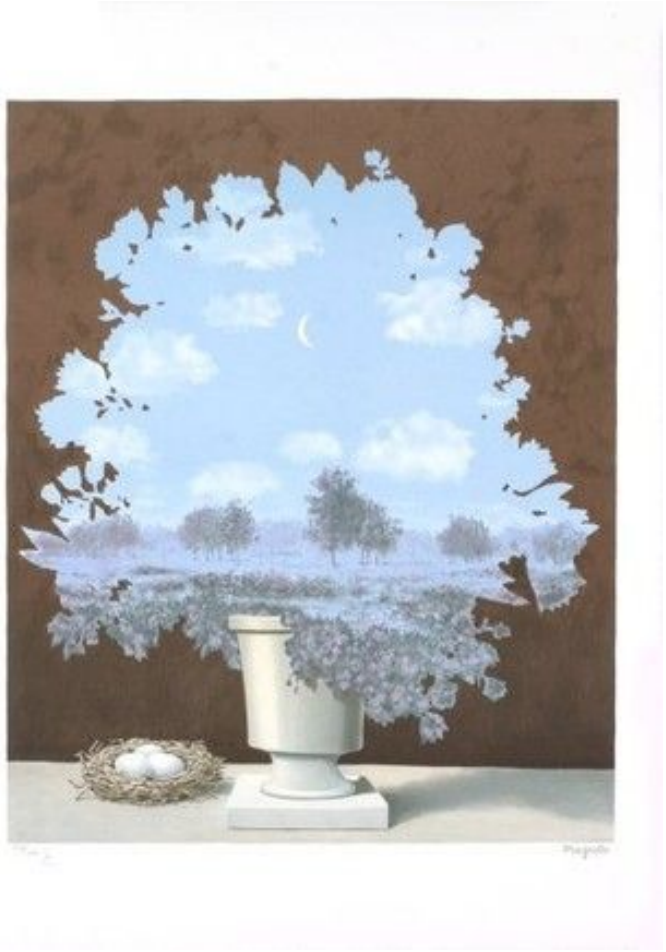
# **Challenges**: deformation

**Challenges**: occlusion



Magritte, 1957

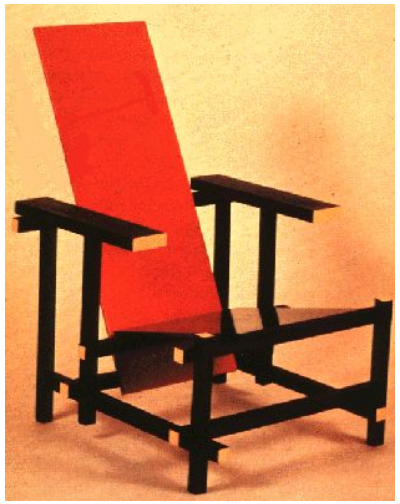
# Art Segway - Magritte



**Challenges**: background clutter

Kilmeny Niland. 1995

# **Challenges**: intra-class variation



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# Object recognition:
# a classification framework

- Apply a prediction function to a feature representation of the image to get the desired output:

f(  ) = “apple”

f(  ) = “tomato”

f(  ) = “cow”

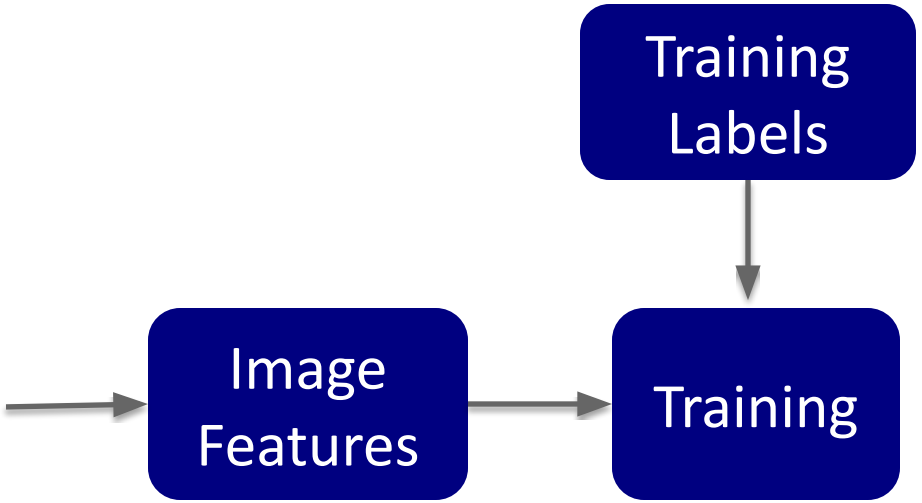
# A simple pipeline - Training

Training Images

Image Features

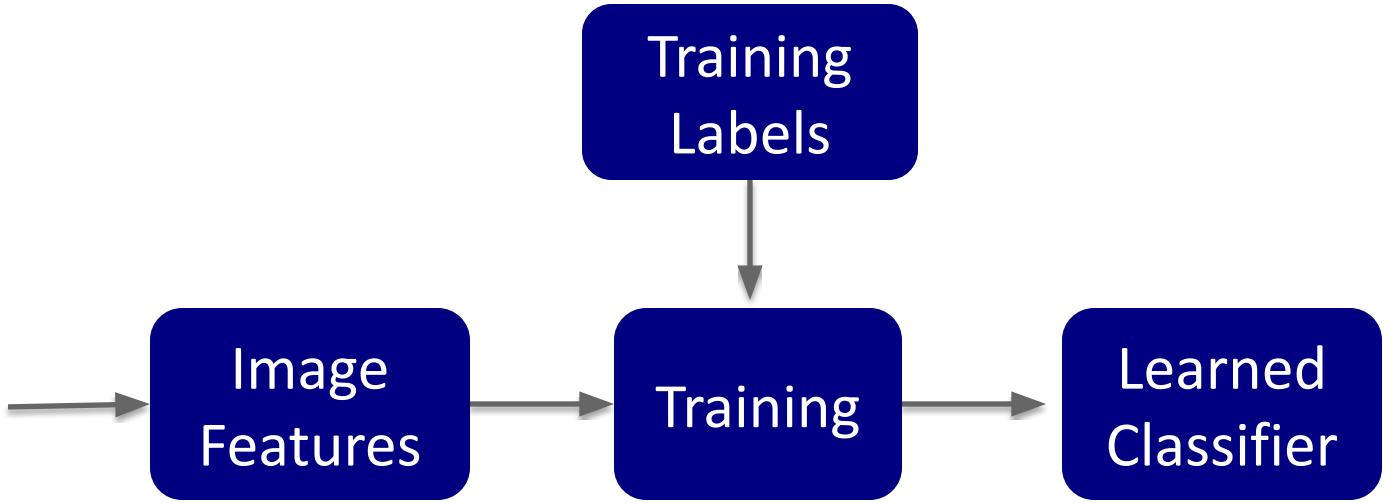
# A simple pipeline - Training

Training Images



# A simple pipeline - Training

Training Images



# A simple pipeline - Training

Training Images

Training Labels

Image Features

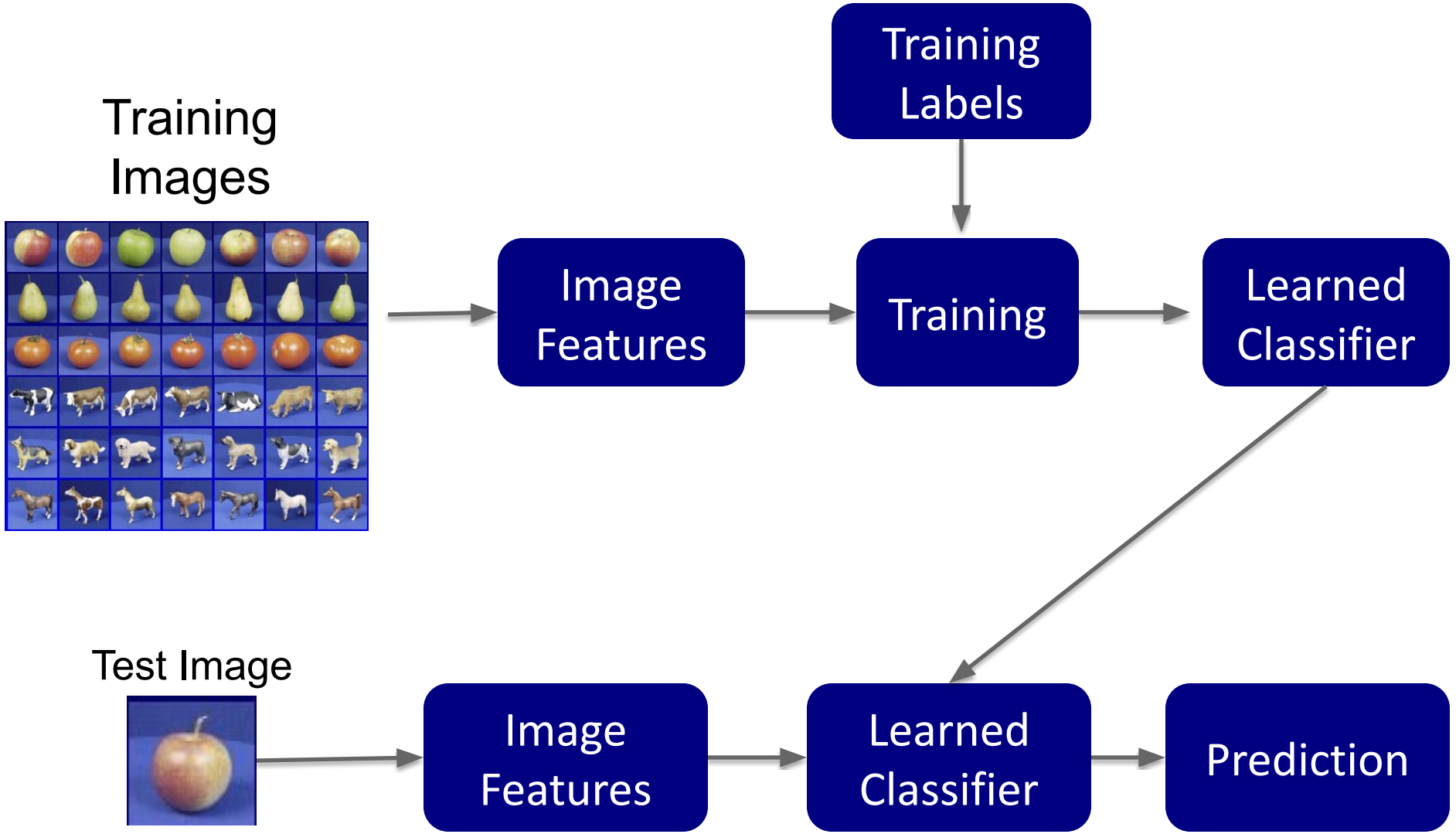
Training

Learned Classifier

Test Image

Image Features

# A simple pipeline - Training



# What we will learn today?

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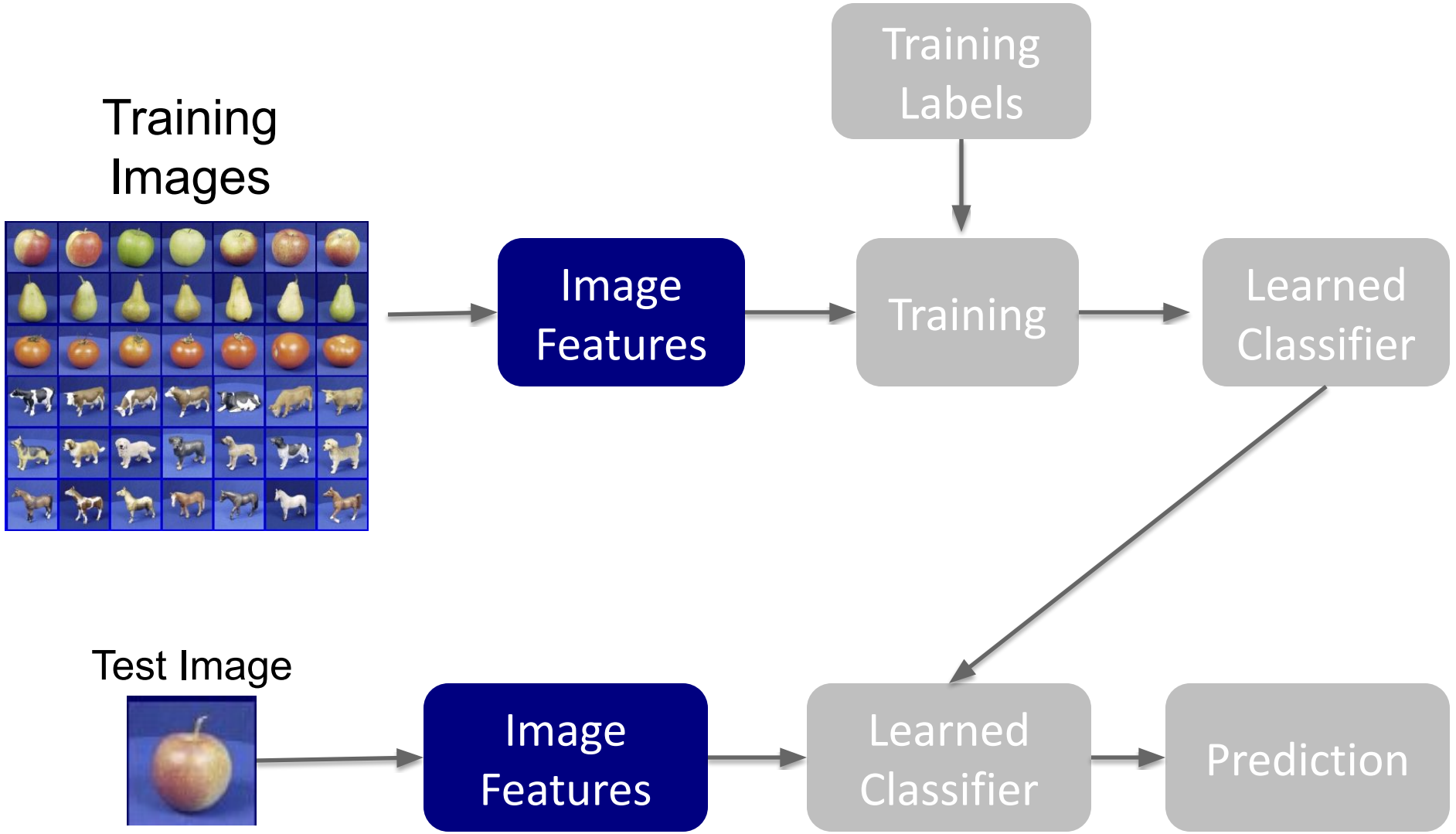
# A simple pipeline - Training



# Choices of features

| | Invariances | | | | | | |
|---|---|---|---|---|---|---|---|
| | **Translation** | **Scale** | **Rotation** (relative to camera plane) | **Rotation** (unconstrained) | **Partial Occlusion** | **Illumination** | **Gaussian Noise** |
| **RGB-histogram** | ? | | | | | | |
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# Choices of features

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| | **Translation** | **Scale** | **Rotation** (relative to camera plane) | **Rotation** (unconstrained) | **Partial Occlusion** | **Illumination** | **Gaussian Noise** |
| **RGB-histogram** | ✅ | | | | | | |
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✓ (global color counts don’t change if the image shifts)

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | | | | | |
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✓ (if the *entire* image is uniformly scaled, the color distribution remains the same)

# Choices of features

| | **Invariances** | | | | | | |
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| | **Translation** | **Scale** | **Rotation** (relative to camera plane) | **Rotation** (unconstrained) | **Partial Occlusion** | **Illumination** | **Gaussian Noise** |
| **RGB-histogram** | ✅ | ❌ | ? | ? | | | |
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# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | | | |
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✓ (rotating the entire image does not change overall color distribution)
✗ (appearance/colors can change if out-of-plane rotation reveals different surfaces)

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ? | | |
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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | | |
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✗ (removing part of the image can significantly alter color histogram)

# Choices of features

| | **Invariances** | | | | | | |
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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ? | ? |
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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
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✗ (shifts in illumination change color intensities/distribution)
✗ (noise directly alters pixel distribution)

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ? | ? | ? | ? | | | |
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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | | | |
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✗ (local bins move)
✗ (needs re-computation at multiple scales)
✗ (oriented gradients are tied to an image grid)
✗ (same reason as the ^)

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ? | ? | ? |
| | | | | | | | |
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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| | | | | | | | |
| | | | | | | | |

✗ (partial occlusion would result in no match)
✓ (gradients are more stable under monotonic intensity changes)
✗ (gradient orientations can be disrupted by significant noise)

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ? | ? | ? | ? | | | |
| | | | | | | | |

# Choices of features

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| | **Translation** | **Scale** | **Rotation** (relative to camera plane) | **Rotation** (unconstrained) | **Partial Occlusion** | **Illumination** | **Gaussian Noise** |
| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | | | |
| | | | | | | | |

✓ (keypoint-based, unaffected by shift)
✓ (built-in scale normalization)
✓ (SIFT normalizes orientation)
✗ (local keypoints might disappear if the object rotates)

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | ? | ? | ? |
| | | | | | | | |

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | ✅ | ✅ | ✅ |
| **Deep learning** | | | | | | | |

✓ (local keypoints can still match if some are visible)
✓ (gradient-based + normalization)
✓ (SIFT is relatively robust to moderate noise)

# Choices of features

| | Invariances | | | | | | |
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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | ✅ | ✅ | ✅ |
| **Deep learning** | ? | ? | ? | ? | | | |

# Choices of features

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| | **Translation** | **Scale** | **Rotation** (relative to camera plane) | **Rotation** (unconstrained) | **Partial Occlusion** | **Illumination** | **Gaussian Noise** |
| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | ✅ | ✅ | ✅ |
| **Deep learning** | usually | usually | usually | ❌ | | | |

~ Deep learning features are usually invariant to translation, scale, and planar rotation if the training data has these translations. It is not invariant to other rotations.

**Aside**: ImageNet has objects centered in the middle of images. So models trained on ImageNet are not translation or scale invariant.

# Choices of features

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| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | ✅ | ✅ | ✅ |
| **Deep learning** | usually | usually | usually | ❌ | ? | ? | ? |

# Choices of features

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| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | ✅ | ✅ | ✅ |
| **Deep learning** | usually | usually | usually | ❌ | ❌ | ✅ | ✅ |

✗ (standard CNNs are not strictly occlusion-invariant; partial robustness depends on training)
✓ (can learn robustness if trained on varied lighting)
✓ (CNNs can learn to be noise-robust with proper training)

# So, which features should we choose?

| | Invariances | | | | | | |
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| | **Translation** | **Scale** | **Rotation** (relative to camera plane) | **Rotation** (unconstrained) | **Partial Occlusion** | **Illumination** | **Gaussian Noise** |
| **RGB-histogram** | ✅ | ❌ | ✅ | ❌ | ❌ | ❌ | ❌ |
| **HoG** | ❌ | ❌ | ❌ | ❌ | ❌ | ✅ | ❌ |
| **SIFT** | ✅ | ✅ | ✅ | ❌ | ✅ | ✅ | ✅ |
| **Deep learning** | usually | usually | usually | ❌ | ❌ | ✅ | ✅ |

# What we will learn today?

- Introduction to recognition
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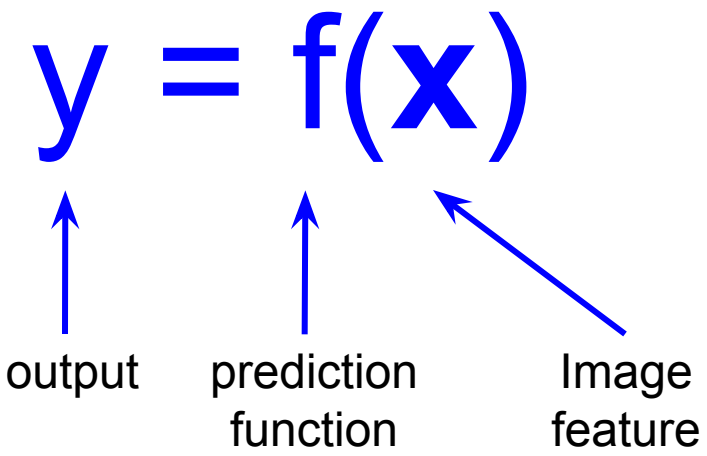
- Challenges with kNN
- Dimensionality reduction: PCA

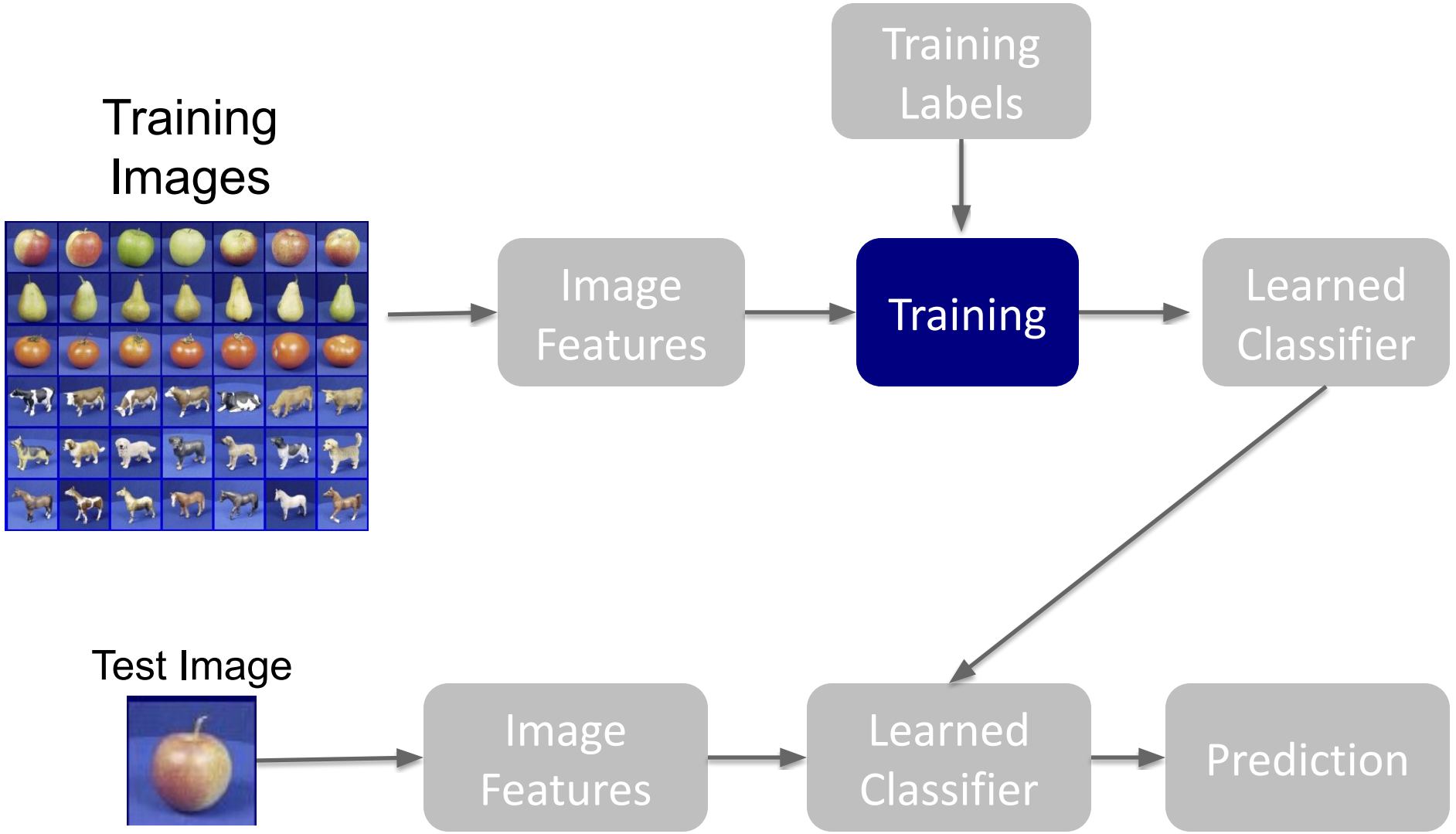
# Learning a classifier to map inputs to outputs




- **Training:** given a *training set* of labeled examples $\{(\mathbf{x}_1,y_1), \ldots, (\mathbf{x}_N,y_N)\}$, estimate the prediction function $f$ by minimizing the prediction error on the training set
- **Testing:** apply $f$ to a never before seen *test example* $\mathbf{x}$ and output the predicted value $y = f(\mathbf{x})$

Slide credit: L. Lazebnik

# A simple pipeline - Training



# Many classifiers to choose from

- **K-nearest neighbor**
- SVM
- Neural networks
- Naïve Bayes
- Bayesian network
- Logistic regression
- Randomized Forests
- Boosted Decision Trees
- RBMs
- Etc.

Which is the best one?

# An example training dataset

Training set (labels known)

Apples

Pear

Tomatos

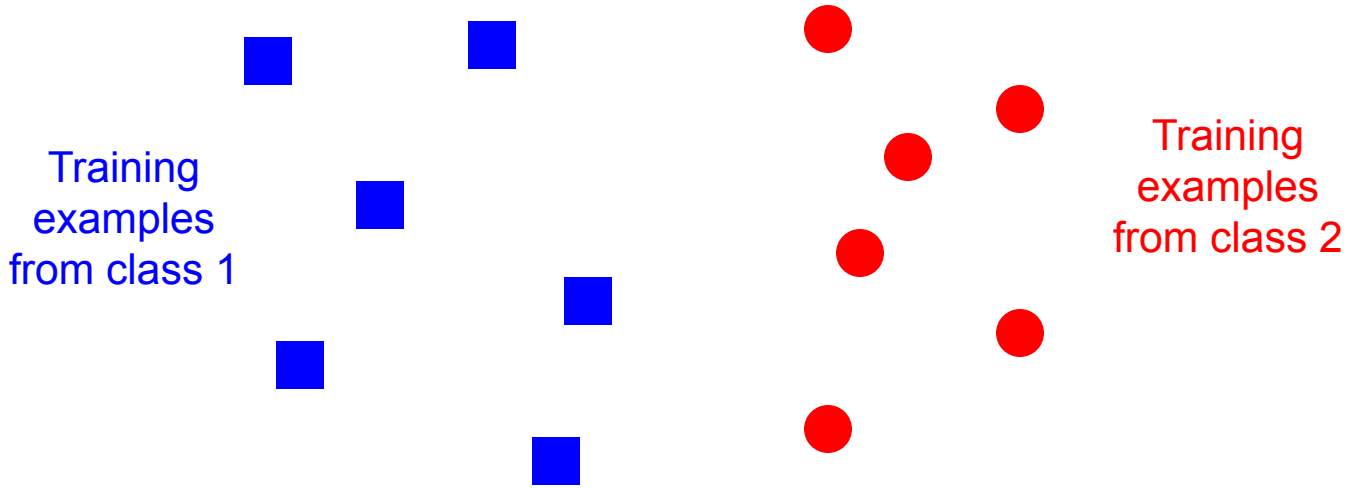
Cow

Dog

Horse

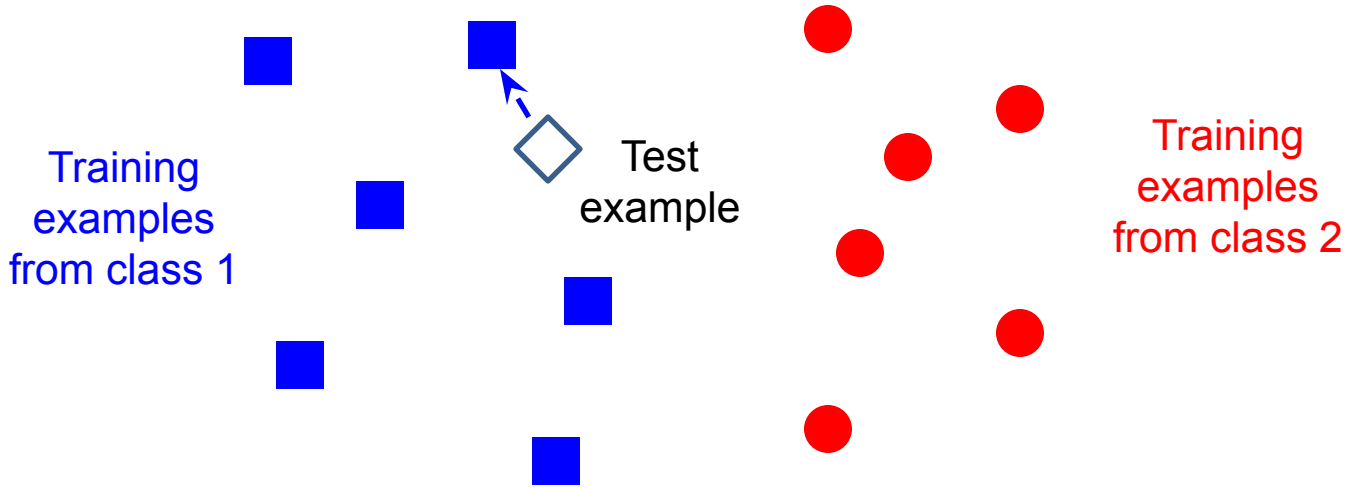
For kNN classifier, training simply means to store all training data.

# A stored training set




Slide credit: L. Lazebnik

# During testing, we assign the label of the nearest neighbot in feature space




Slide credit: L. Lazebnik

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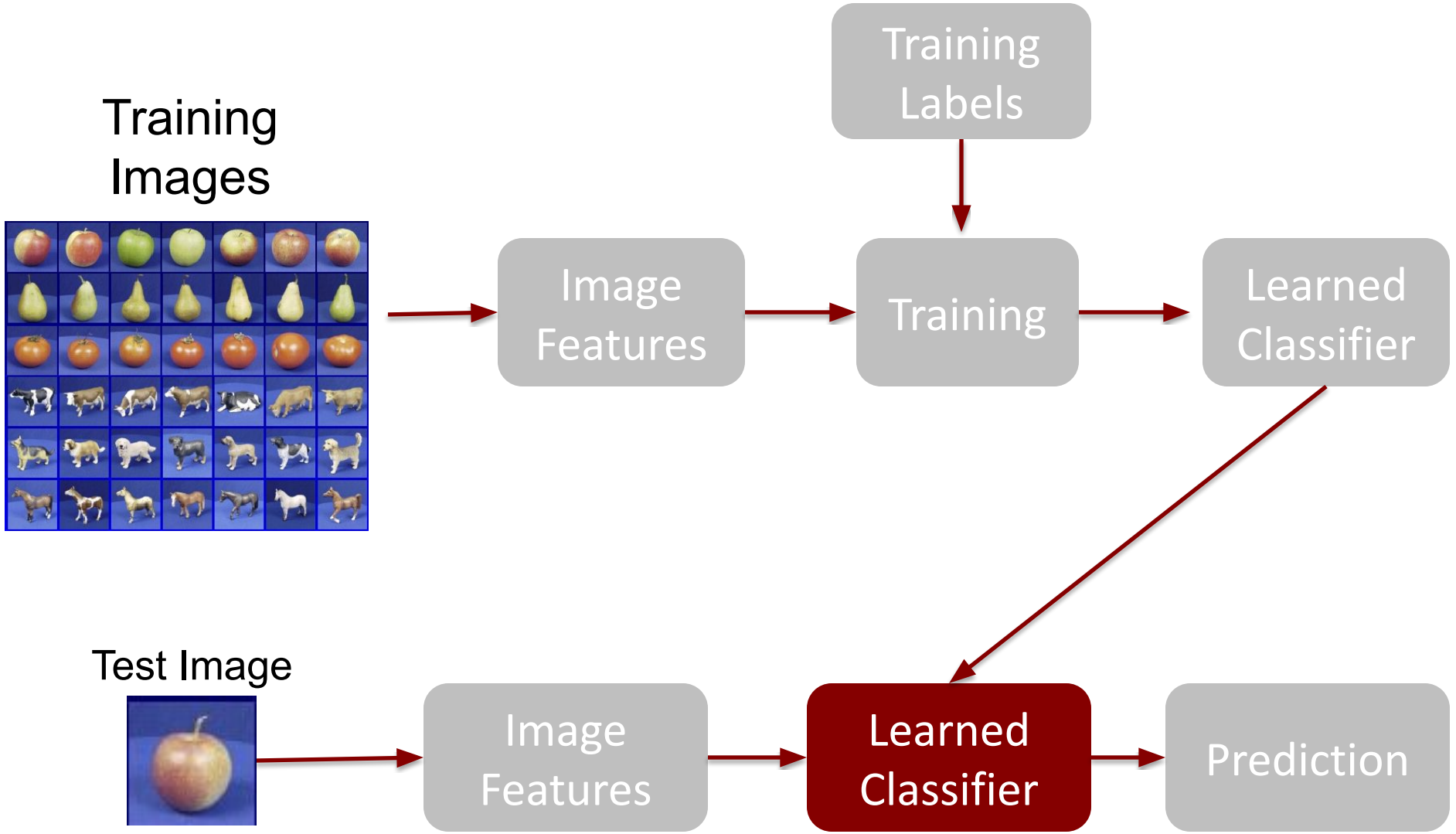
- Dimensionality reduction

# A simple pipeline - Training



# Generalization

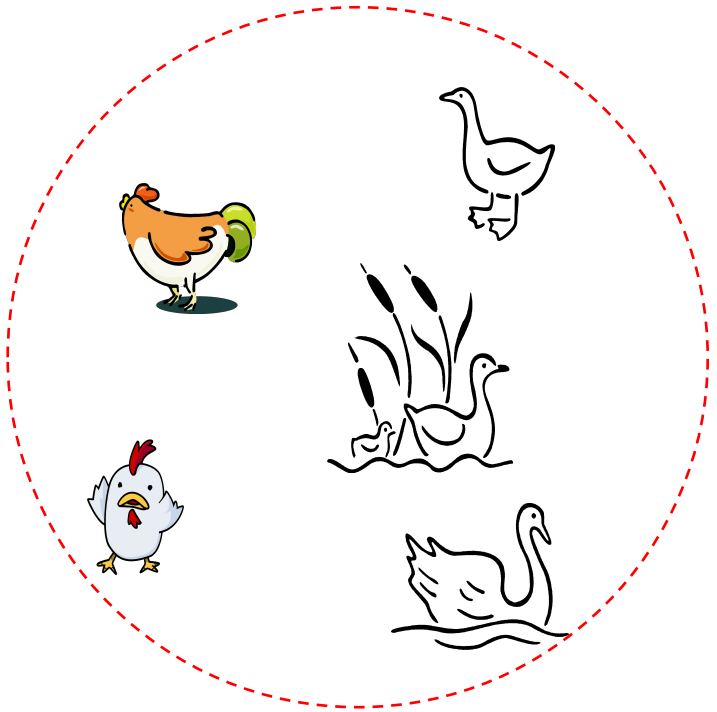
Training set (labels known)

Test set (labels unknown)

- How well does a learned model generalize from the data it was trained on to a new test set?

# Intuition for Nearest Neighbor Classifier

Given a training dataset, simply store each image's features and their corresponding label.

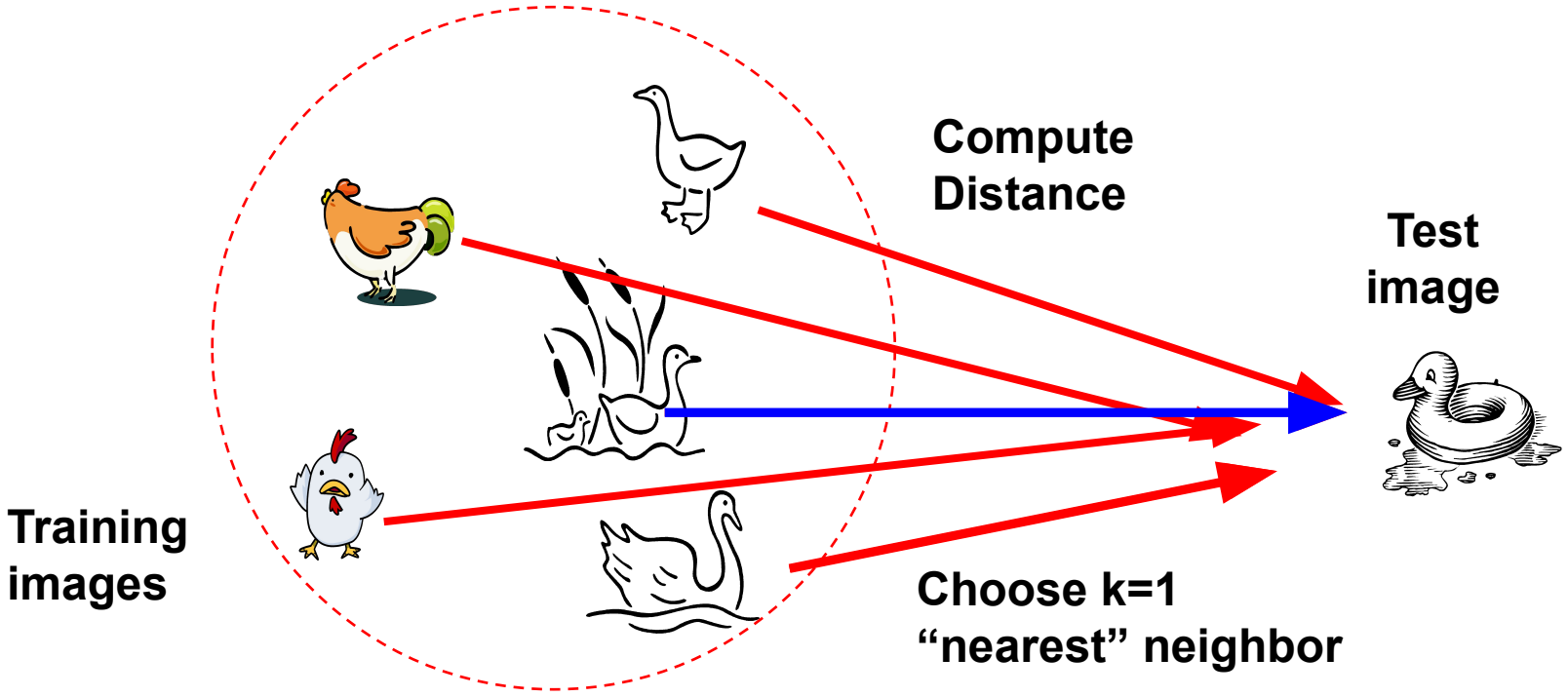


**Training images**

Source: N. Goyal

# Intuition for Nearest Neighbor Classifier

Given a training dataset, simply store each image's features and their corresponding label.

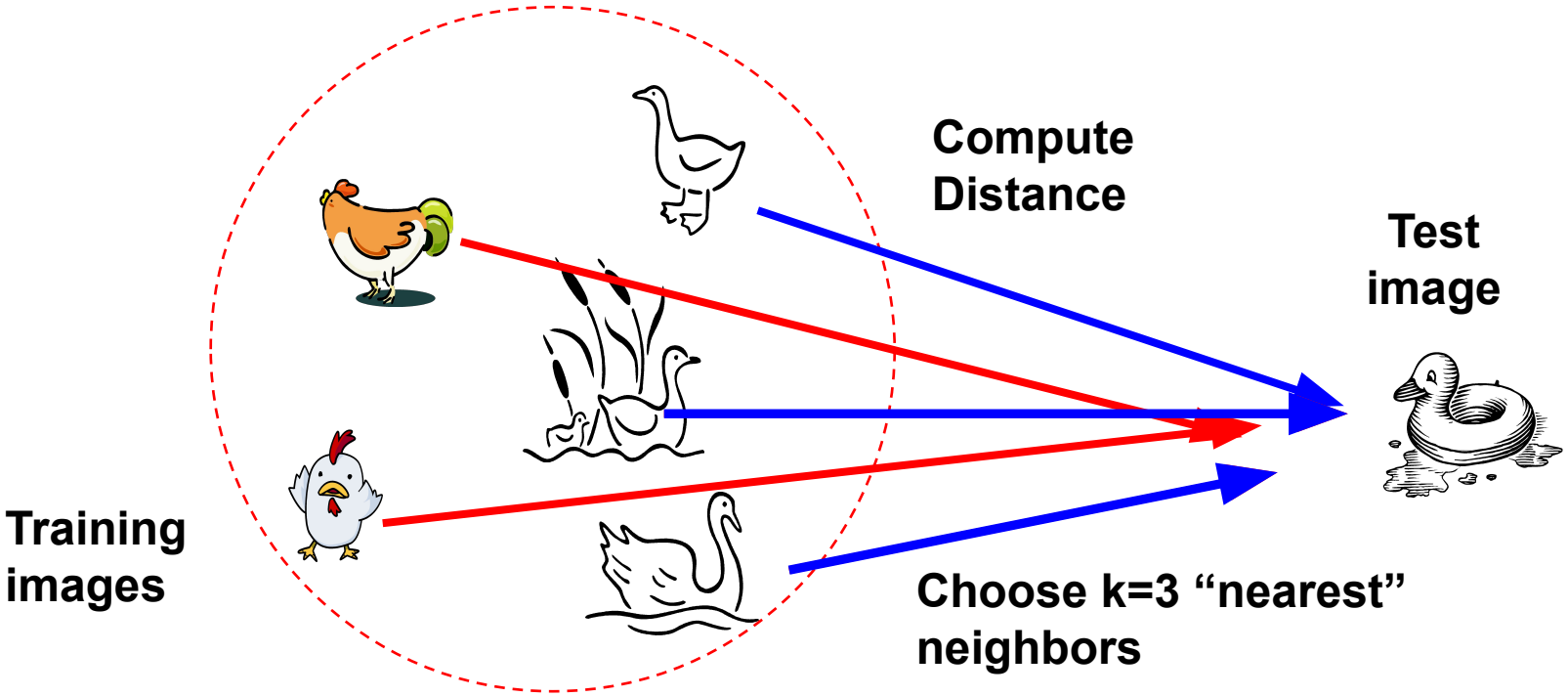





Source: N. Goyal

# Nearest Neighbor Classifier

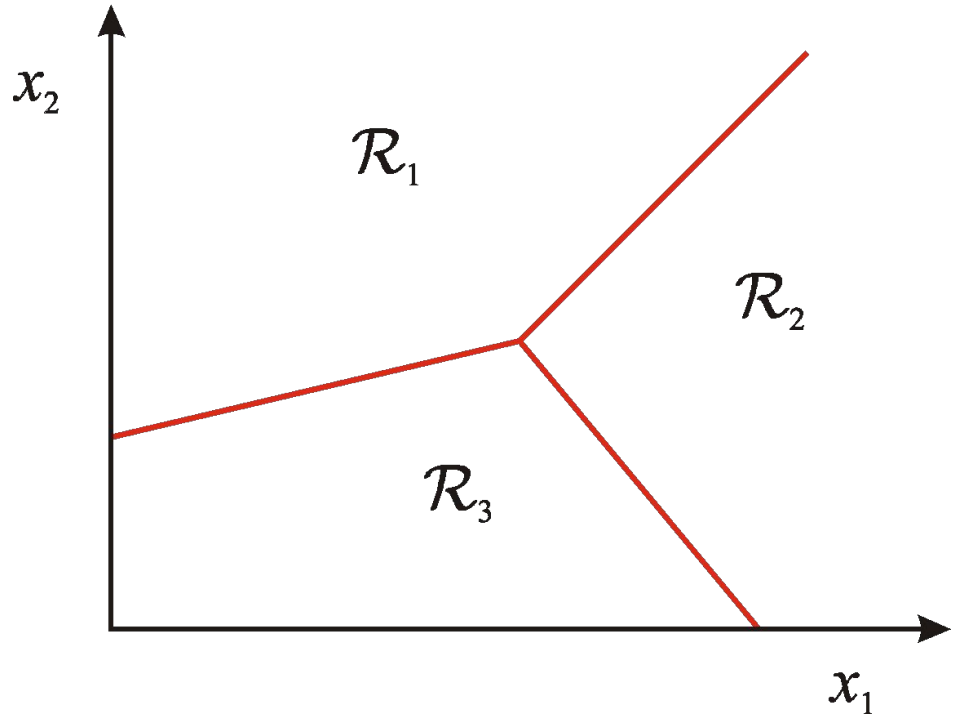
- Assign label of majority of K=3 nearest neighbors




Source: N. Goyal

# Classification

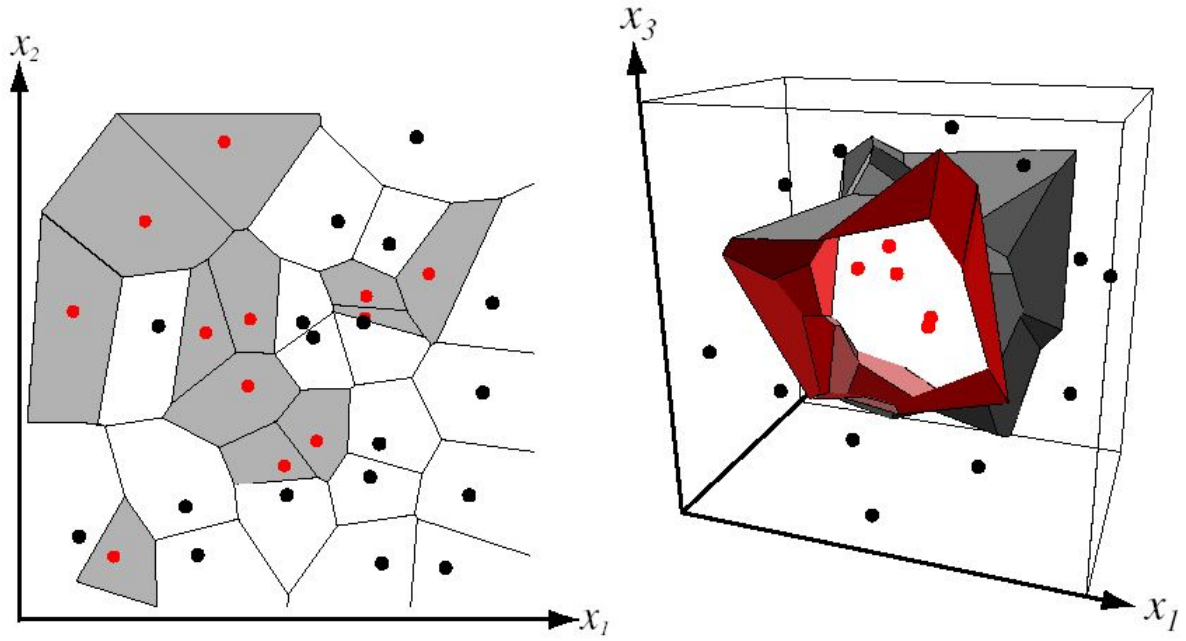
- Assign input vector to one of many classes (categories)
- **Geometric interpretation** of classifiers: A classifier divides input space into *decision regions* separated by *decision boundaries*




Slide credit: L. Lazebnik

# Nearest Neighbor Classifier

- Assign label of nearest training data point to each test data point




from Duda *et al.*

Partitioning of feature space
for two-category 2D and 3D data

Source: D. Lowe

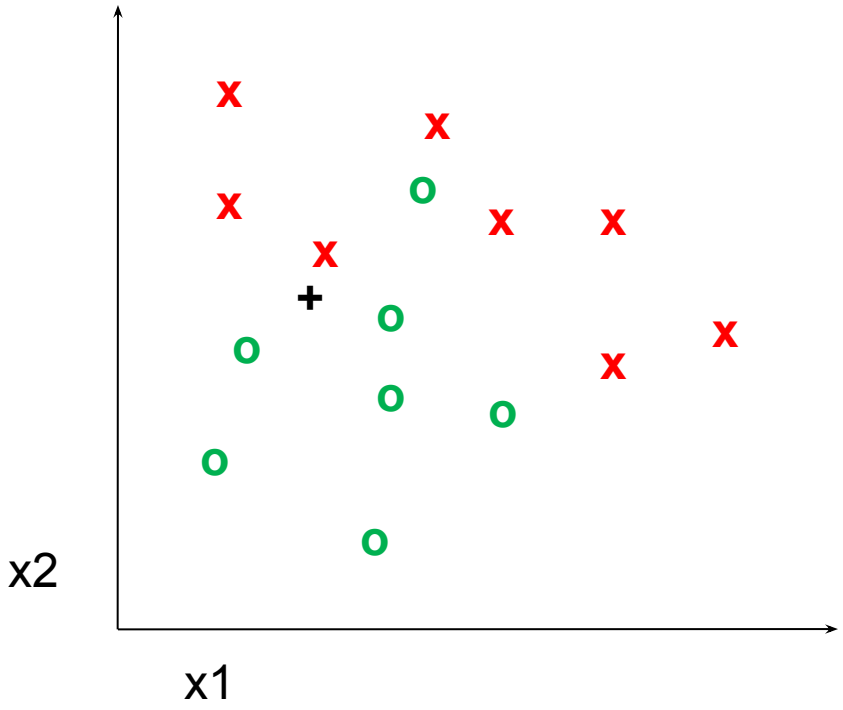
# How do we find the nearest neighbors in feature space?

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where $X^n$ and $X^m$ are the n-th and m-th data points



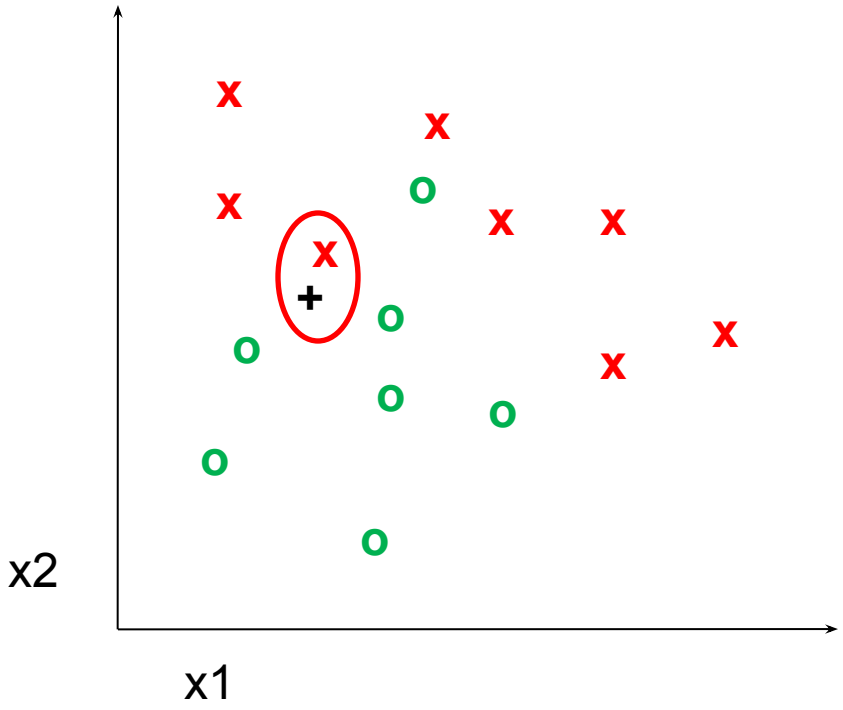
# 1-nearest neighbor

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where $X^n$ and $X^m$ are the n-th and m-th data points



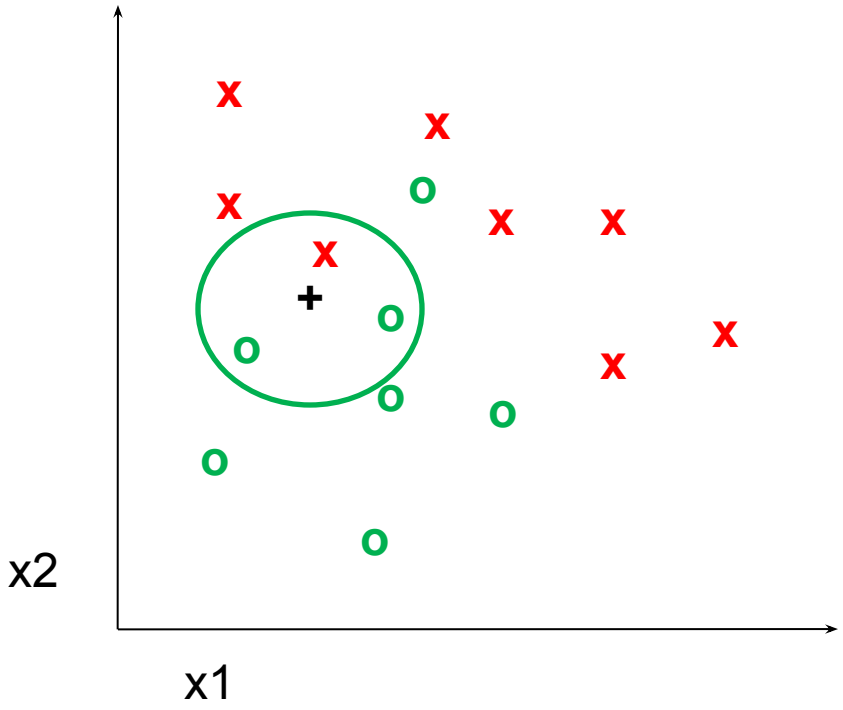
# 3-nearest neighbor

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where $X^n$ and $X^m$ are the n-th and m-th data points



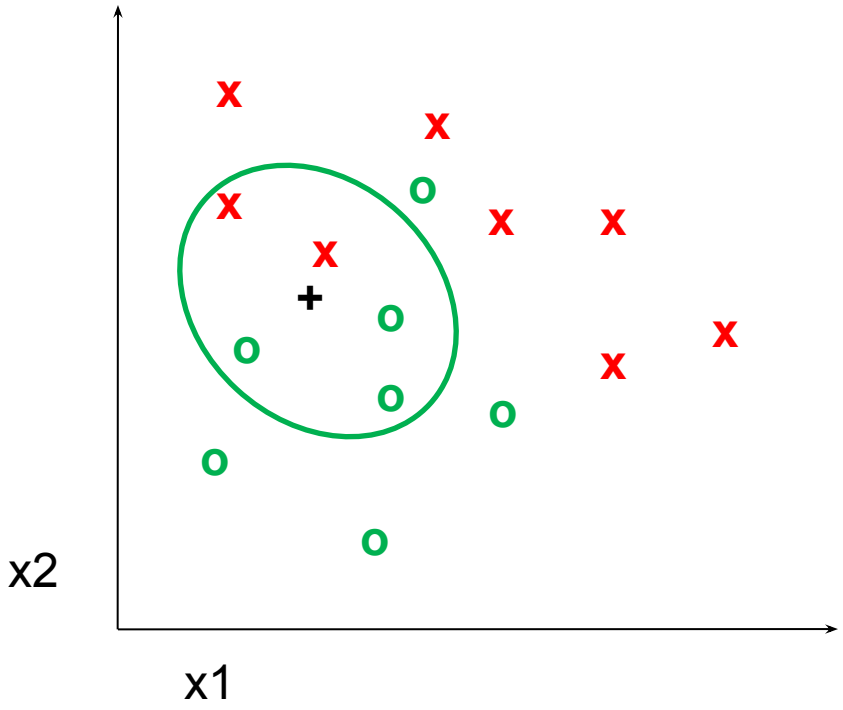
# 5-nearest neighbor

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where $X^n$ and $X^m$ are the n-th and m-th data points



# Choosing the right features is important but dataset-dependent

| | Color | $D_x D_y$ | Mag-Lap | PCA Masks | PCA Gray | Cont. Greedy | Cont. DynProg | Avg. |
|---|---|---|---|---|---|---|---|---|
| apple | 57.56% | **85.37%** | 80.24% | 78.78% | **88.29%** | 77.07% | 76.34% | 77.66% |
| pear | 66.10% | 90.00% | 85.37% | **99.51%** | **99.76%** | 90.73% | 91.71% | 89.03% |
| tomato | **98.54%** | 94.63% | **97.07%** | 67.80% | 76.59% | 70.73% | 70.24% | 82.23% |
| cow | 86.59% | 82.68% | **94.39%** | 75.12% | 62.44% | 86.83% | 86.34% | 82.06% |
| dog | 34.63% | 62.44% | 74.39% | 72.20% | 66.34% | **81.95%** | **82.93%** | 67.84% |
| horse | 32.68% | 58.78% | 70.98% | 77.80% | 77.32% | **84.63%** | **84.63%** | 69.55% |
| cup | 79.76% | 66.10% | 77.80% | **96.10%** | **96.10%** | **99.76%** | **99.02%** | 87.81% |
| car | 62.93% | **98.29%** | 77.56% | **100.0%** | **97.07%** | **99.51%** | **100.0%** | 90.77% |
| total | 64.85% | 79.79% | 82.23% | 83.41% | 82.99% | 86.40% | 86.40% | 80.87% |

Dataset: ETH-80, by B. Leibe, 2003

# Results

| Category | Primary feature(s) | Secondary feature(s) |
|---|---|---|
| apple | PCA Gray | Texture $D_x D_y$ |
| pear | PCA Gray / Masks | |
| tomato | Color | Texture Mag-Lap |
| cow | Texture Mag-Lap | Contour / Color |
| dog | Contour | |
| horse | Contour | |
| cup | Contour | PCA Gray / Masks |
| car | PCA Masks / Contour | Texture $D_x D_y$ |

Dataset: ETH-80, by B. Leibe, 2003

# K-NN: a very useful algorithm

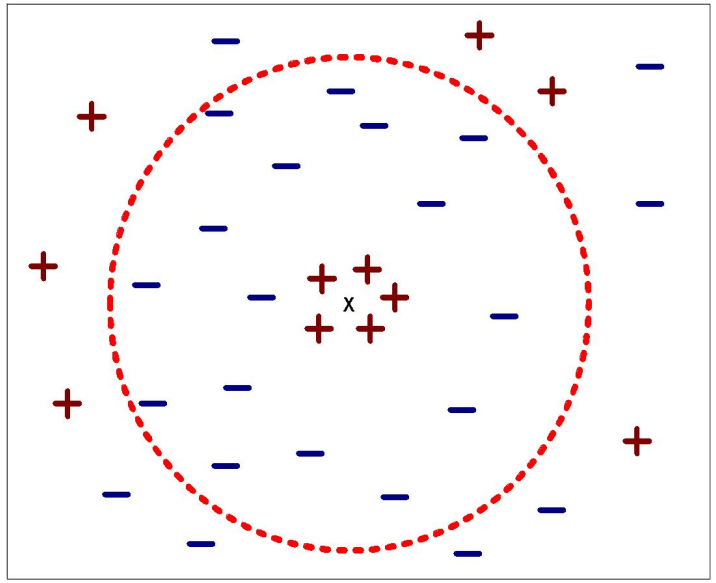
- Simple, a good one to try first
- Very flexible decision boundaries

# What we will learn today?

- Introduction to recognition
- A simple Object Recognition pipeline
- Choosing the right features
- A training algorithm: kNN
- Testing an algorithm
- Challenges with kNN
- Dimensionality reduction

# K-NN: issues to keep in mind

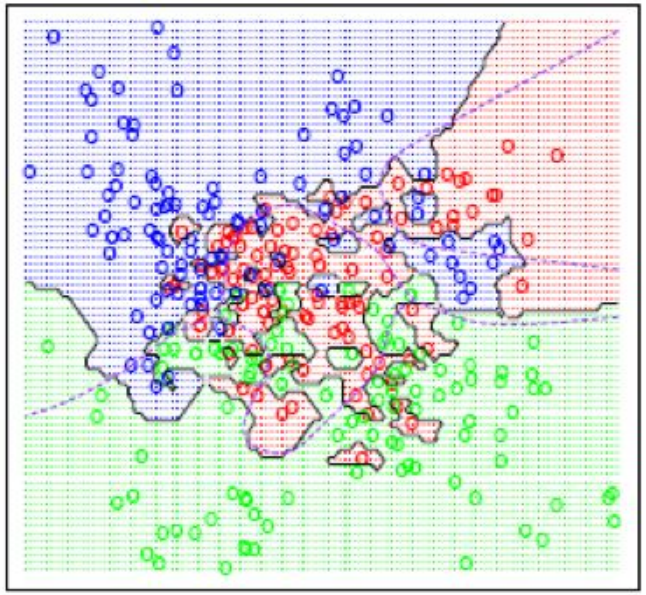
- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes



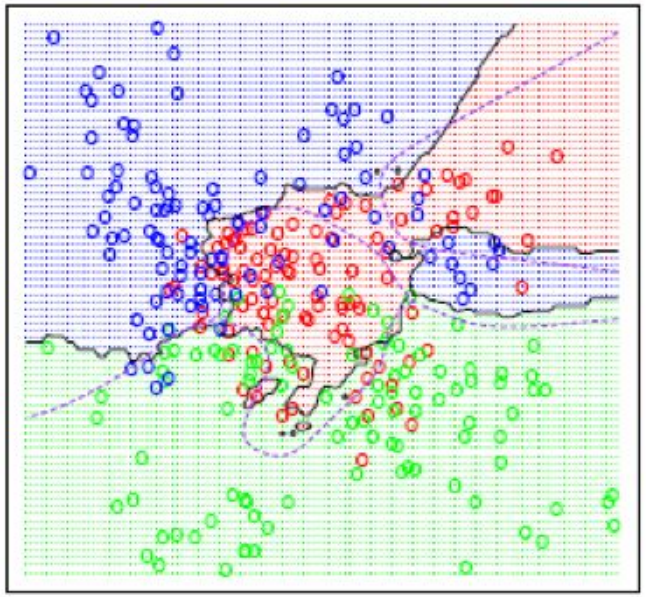
# K-NN: issues to keep in mind

- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes

K=1

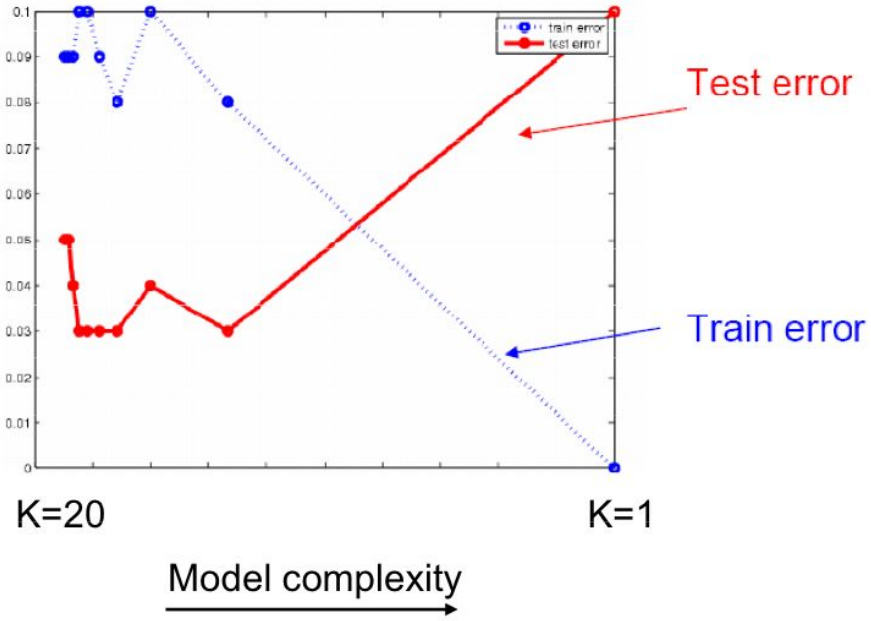


K=15

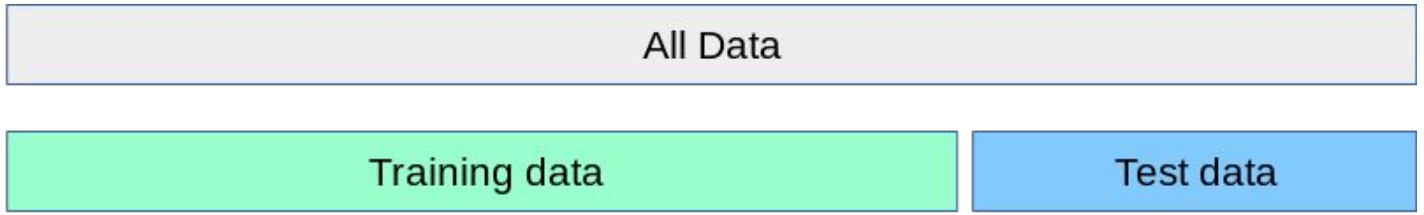


# K-NN: issues to keep in mind

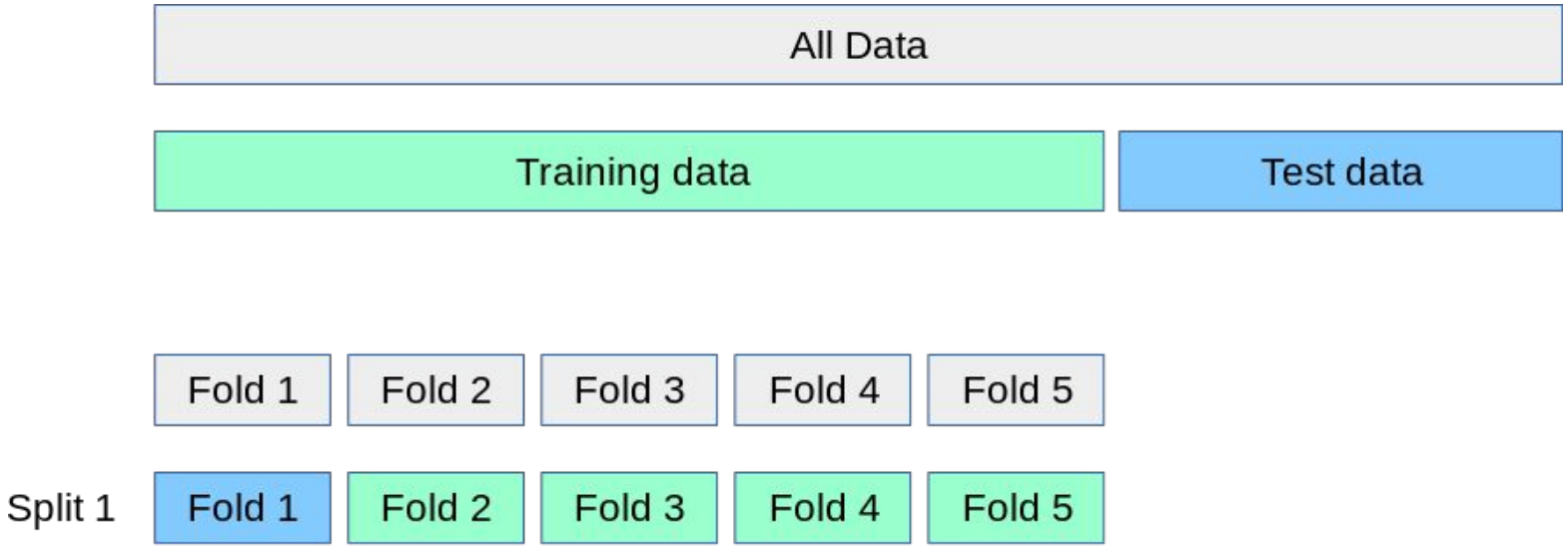
- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes
  - Solution: Cross validate



# Cross validation

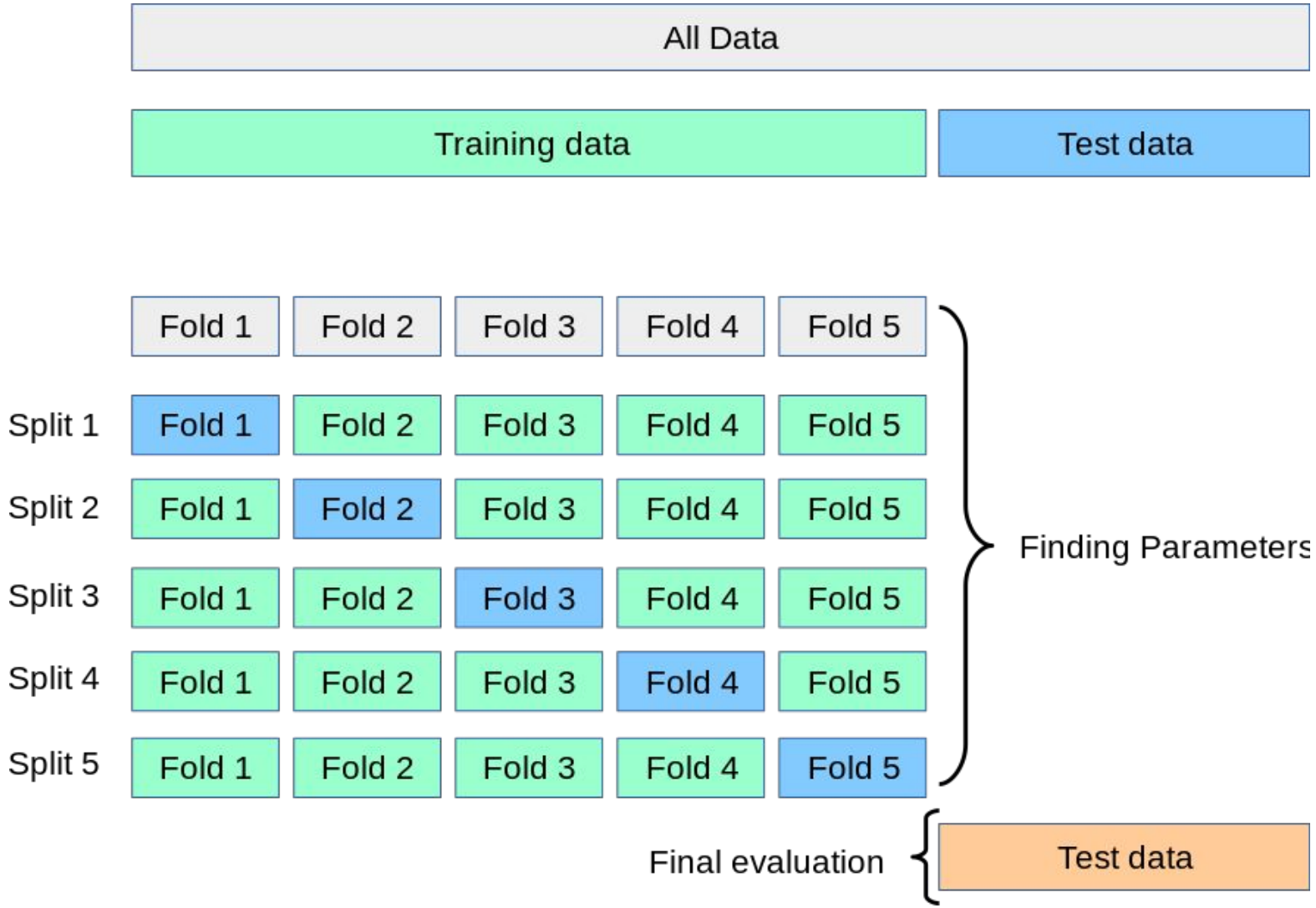


# Cross validation



# Cross validation

# Cross validation



# K-NN: issues to keep in mind

- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes
  - Solution: cross validate!
- Curse of Dimensionality

# Curse of dimensionality

- As the dimensionality increases, the number of data points required for good performance increases exponentially.
- Let's say that for a model to perform well, we need at least 10 data points for each combination of feature values.

**Need for Data Points with Increase in Dimensions**

| | | |
|---|---|---|
| 1 Binary feature → | $2^1$ unique values → | $2^1 \times 10 = 20$ data points |
| 2 Binary features → | $2^2$ unique values → | $2^2 \times 10 = 40$ data points |
| 3 Binary features → | $2^3$ unique values → | $2^3 \times 10 = 80$ data points |
| . | . | . |
| . | . | . |
| . | . | . |
| k Binary features → | $2^k$ unique values → | $2^k \times 10$ data points |

# K-NN: issues to keep in mind

- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes
  - Solution: cross validate!
- Curse of Dimensionality
  - Solution: dimensionality reduction

# What we will learn today

- Introduction to recognition
- A simple Object Recognition pipeline
- Choosing the right features
- A training algorithm: kNN
- Testing an algorithm
- Challenges with kNN
- Dimensionality reduction

# Singular Value Decomposition (SVD)

$$\mathbf{U\Sigma V^T = A}$$

- Where **U** and **V** are rotation matrices, and **Σ** is a scaling matrix. For example:

$$\underset{U}{\begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}}$$

# Singular Value Decomposition (SVD)

- Beyond 2x2 matrices:
  - In general, if **A** is *m* x *n*, then **U** will be *m* x *m*, **Σ** will be *m* x *n*, and $\mathbf{V}^{\mathbf{T}}$ will be *n* x *n*.
  - (Note the dimensions work out to produce *m* x *n* after multiplication)

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

# Singular Value Decomposition (SVD)

- **U** and **V** are always rotation matrices.
  - Geometric rotation may not be an applicable concept, depending on the matrix. So we call them “unitary” matrices – each column is a unit vector.
- **Σ** is a diagonal matrix
  - The number of nonzero entries = rank of **A**
  - The algorithm always sorts the entries high to low

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

# SVD Applications

- We've discussed SVD in terms of geometric transformation matrices
- But SVD of an image matrix can also be very useful
- To understand this, we'll look at a less geometric interpretation of what SVD is doing

# What is SVD actually doing for images?

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

- Look at how the multiplication works out, left to right:
- Column 1 of **U** gets scaled by the first value from **Σ**.

$$\underset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}}$$

# What is SVD actually doing for images?

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

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# What is SVD actually doing for images?

$$\underset{U}{\begin{bmatrix}-.39 & -.92\\ -.92 & .39\end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix}9.51 & 0 & 0\\ 0 & .77 & 0\end{bmatrix}} \times \underset{V^T}{\begin{bmatrix}-.42 & -.57 & -.70\\ .81 & .11 & -.58\\ .41 & -.82 & .41\end{bmatrix}} = \underset{A}{\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}}$$

- Look at how the multiplication works out, left to right:
- Column 1 of **U** gets scaled by the first value from **Σ**.

$$\underset{U\Sigma}{\begin{bmatrix}-3.67 & -.71 & 0\\ -8.8 & .30 & 0\end{bmatrix}} \times \underset{V^T}{\begin{bmatrix}-.42 & -.57 & -.70\\ .81 & .11 & -.58\\ .41 & -.82 & .41\end{bmatrix}} \quad \underset{A_{partial}}{\begin{bmatrix}1.6 & 2.1 & 2.6\\ 3.8 & 5.0 & 6.2\end{bmatrix}}$$

- The resulting vector gets scaled by row 1 of $\mathbf{V^T}$ to produce a contribution to the columns of **A**

# SVD is a type dimensionality reduction

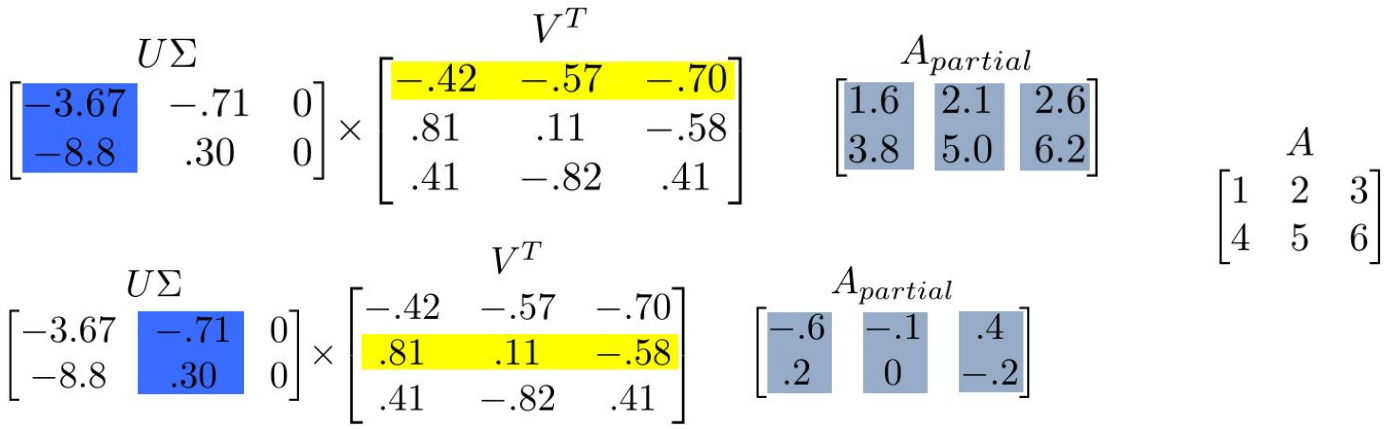
$$\underset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

$$+ \underset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A_{partial}}{\begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

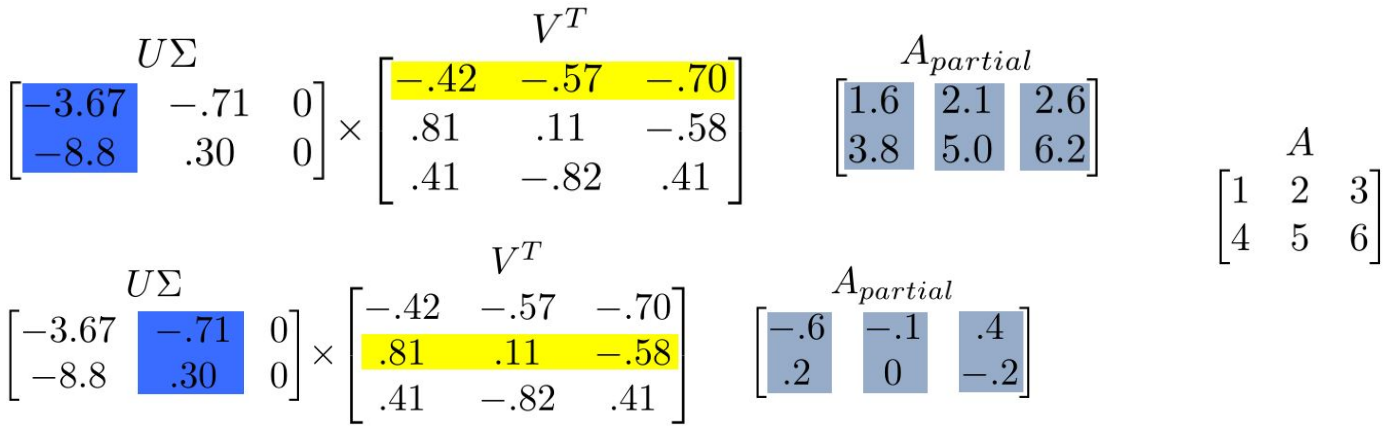
- Each product of (*column i of* ***U***)·(*value i from* ***Σ***)·(*row i of* $\boldsymbol{V^T}$) produces a component of the final **A**.

# SVD is a type dimensionality reduction




- We're building **A** as a linear combination of the columns of ***U***
- Using all columns of ***U***, we'll rebuild the original matrix perfectly
- But, in real-world data, often we can just use the first few columns of ***U*** and we'll get something close (e.g. the first $\boldsymbol{A_{partial}}$, above)

# SVD is a type dimensionality reduction




- We can call those first few columns of ***U*** the Principal Components of the data
- They show the major patterns that can be added to produce the columns of the original matrix
- The rows of $\mathbf{V^T}$ show how the *principal components* are mixed to produce the columns of the matrix

# SVD is a type dimensionality reduction

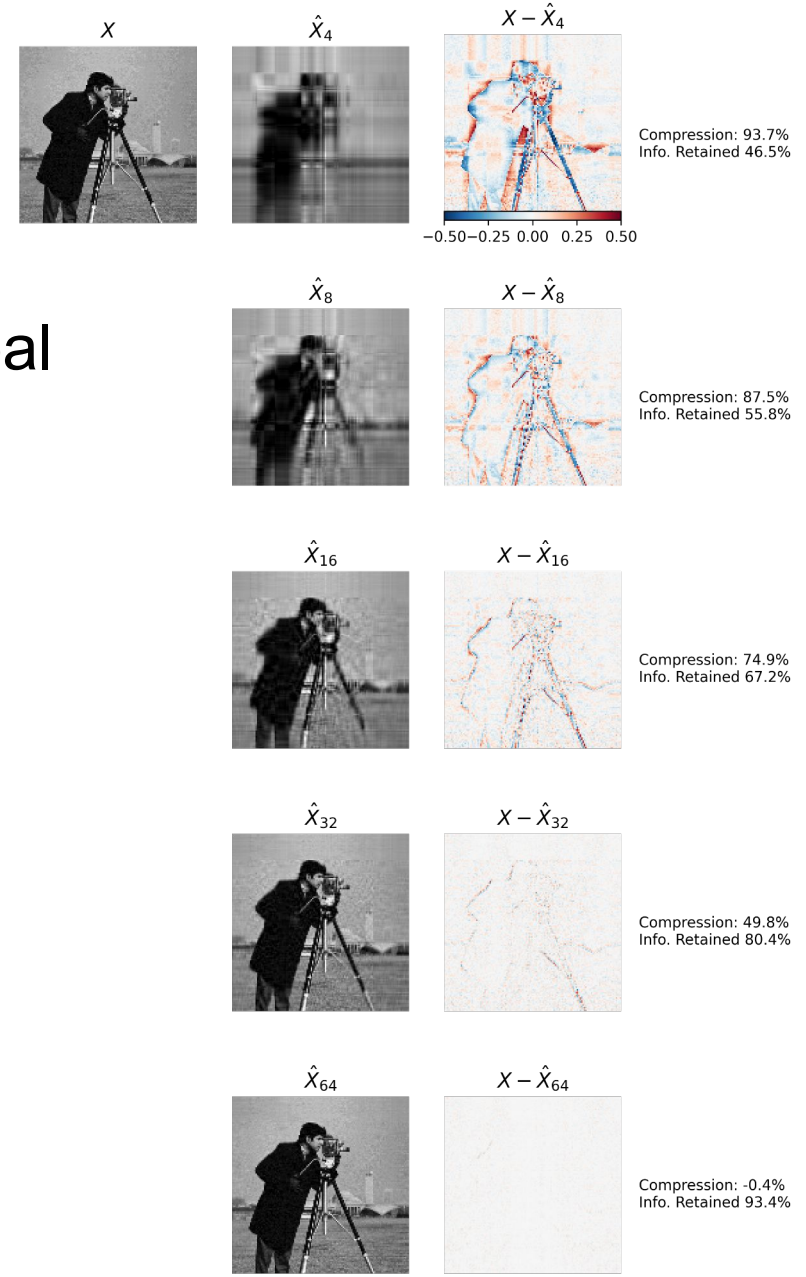
$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

We can look at **Σ** to see that the first column has a large effect

while the second column has a much smaller effect in this example

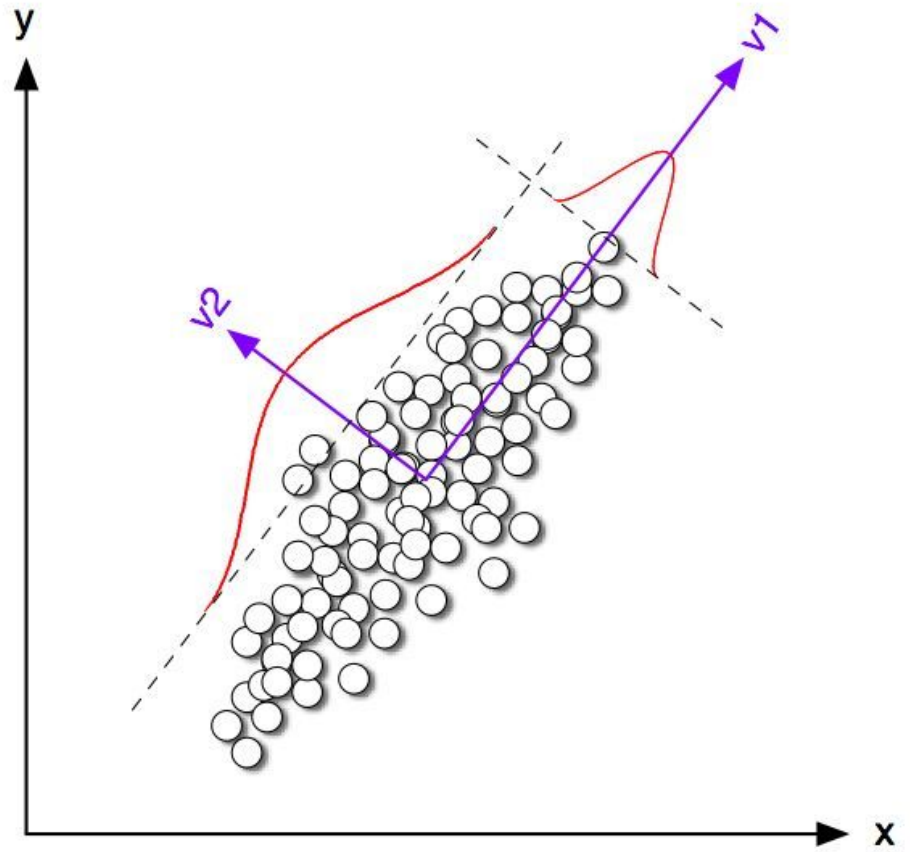
# Image compression

- For this image, using **only the first 16** of 300 principal components produces a recognizable reconstruction
- Using the first 64 almost perfectly reconstructs the image

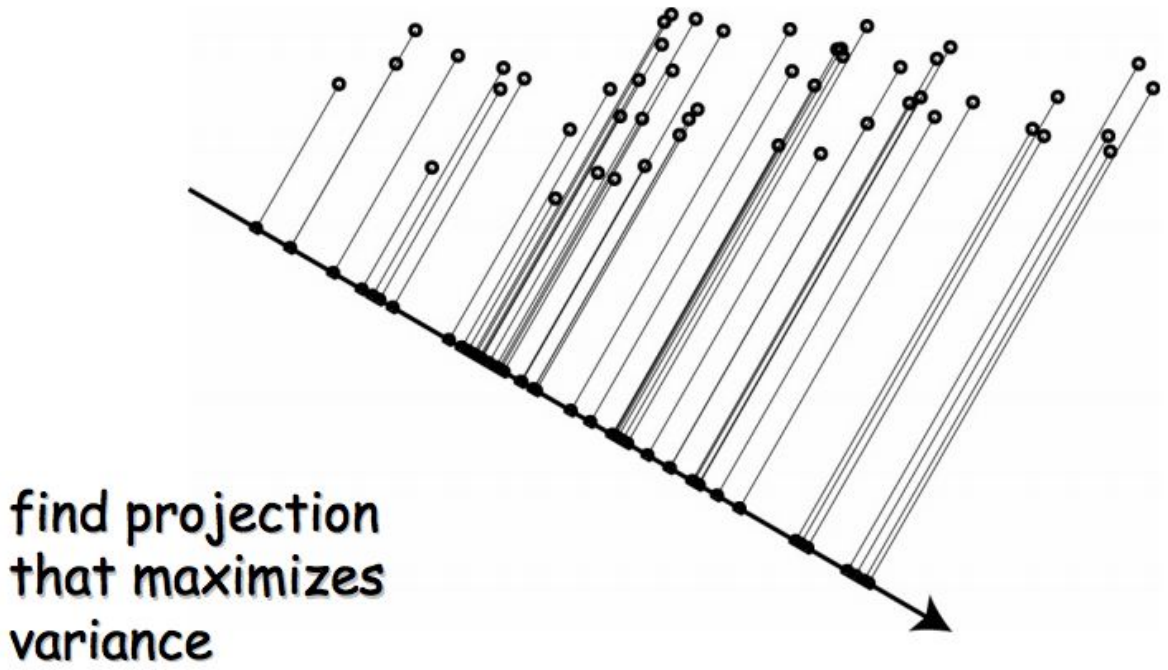


# Intuition behind PCA: high dimensional data usually lives in some lower dimensional space

**Covariance** between the two dimensions of features is high. Can we reduce the number of dimensions to just 1?

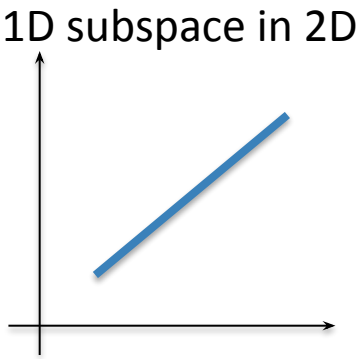


# Geometric interpretation of PCA



# Geometric interpretation of PCA

- Let's say we have a set of 2D data points x. But we see that all the points lie on a line in 2D.
- So, 2 dimensions are redundant to express the data. We can express all the points with just one dimension.



# PCA: Principal Component Analysis

- Given a dataset of images, can we compressed them like we can compress a single image?
  - Yes, the trick is to look into the correlation between the dimensions of the image
  - The tool for doing this is called PCA

PCA can be used to compress image RGB pixel values or also be used to compress their features!

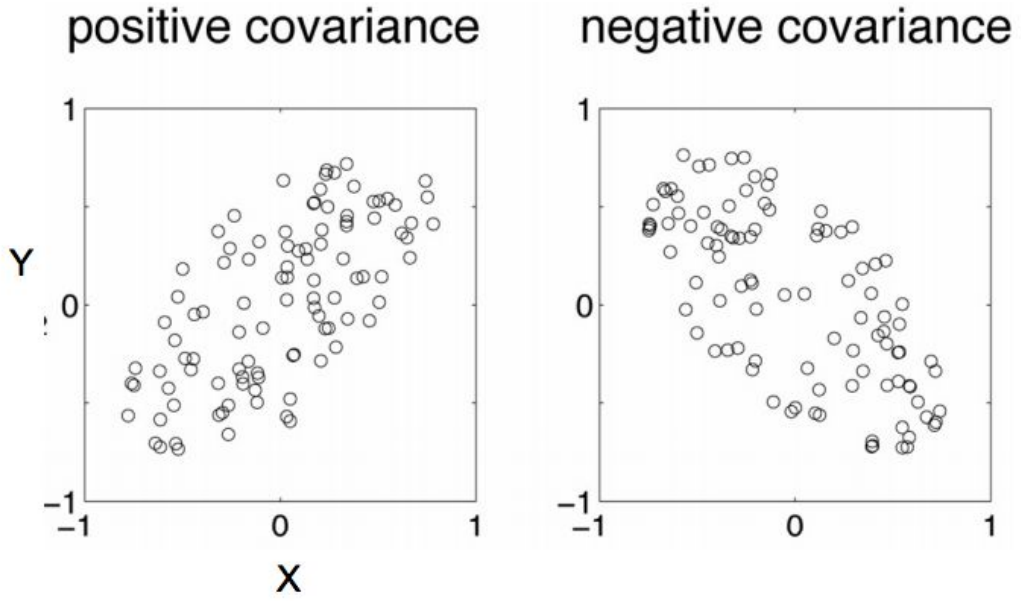
# Toy example to explain covariance

- What is covariance between dimensions?
- Let's say we have a dataset of students
  - each student is represented with 3 dimensions
  - **x**: number of hours studied for a class
  - **y**: grades obtained in that class
  - **z**: number of lectures attended
- covariance value between **x and y is say: 104.53**
  - what does this value mean?

# Covariance interpretation

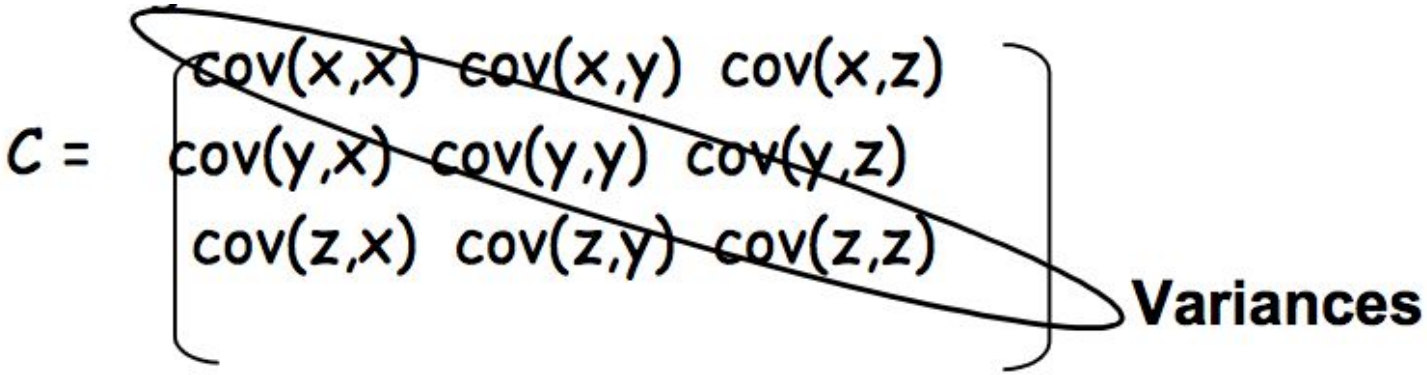
- **x**: number of hours studied for a subject
- **y**: marks obtained in that subject

- covariance value between **x and y is say: 104.53**
  - what does this value mean?



# Visualizing this covariance matrix

- We can represent these covariance correlation numbers in a matrix
- e.g. for 3 dimensions:




- Diagonal is the **variances** of x, y and z
- **cov(x,y) = cov(y,x)** hence **C is symmetrical** about the diagonal
- N-dimensional data will result in NxN covariance matrix

# Covariance interpretation

- Exact value is not as important as it's sign.
- A **positive value** of covariance indicates both dimensions increase or decrease together e.g. as the number of hours studied increases, the marks in that subject increase.
- A **negative value** indicates while one increases the other decreases, or vice-versa e.g. active social life at PSU vs performance in CS dept.
- If **covariance is zero**: the two dimensions are independent of each other e.g. heights of students vs the marks obtained in a subject

# PCA by SVD

- To relate this to PCA, we consider the image (or feature) matrix

$$X = \begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix}$$

- The sample mean of this dataset (or in plain english, the average image) is:

$$\mu = \frac{1}{n}\sum_i x_i = \frac{1}{n}\begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{n}X1$$

# PCA by SVD

- Center the data by subtracting the mean to each column of X
- The centered dataset matrix is

$$X_c = \begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix} - \begin{bmatrix} | & & | \\ \mu & \dots & \mu \\ | & & | \end{bmatrix}$$

# PCA by SVD

- The sample covariance matrix is

$$C = \frac{1}{n}\sum_i (x_i - \mu)(x_i - \mu)^T = \frac{1}{n}\sum_i x_i^c (x_i^c)^T$$

where $x_i^c$ is the $i^{th}$ column of $X_c$

- This can be written as

$$C = \frac{1}{n}\begin{bmatrix} | & & | \\ x_1^c & \cdots & x_n^c \\ | & & | \end{bmatrix}\begin{bmatrix} - & x_1^c & - \\ & \vdots & \\ - & x_n^c & - \end{bmatrix} = \frac{1}{n} X_c X_c^T$$

# PCA by SVD

- The matrix

$$X_c^T = \begin{bmatrix} - & x_1^c & - \\ & \vdots & \\ - & x_n^c & - \end{bmatrix}$$

is real (n x d). Assuming n>d it has SVD decomposition

$$X_c^T = U\Sigma V^T \qquad U^T U = I \qquad V^T V = I$$

and

$$C = \frac{1}{n} X_c X_c^T$$

# Calculating covariance matrix

$$
\begin{aligned}
C &= \frac{1}{n} X_c X_c^T \\
&= \frac{1}{n} U \Sigma V^T (U \Sigma V^T)^T \\
&= \frac{1}{n} U \Sigma V^T V \Sigma U^T \\
&= \frac{1}{n} U \Sigma^2 U^T
\end{aligned}
$$

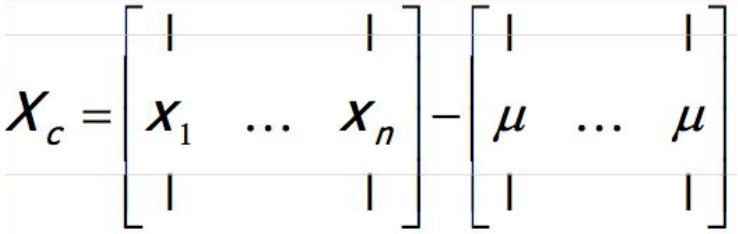
# PCA by SVD

$$C = \frac{1}{n} U \Sigma^2 U^T$$

- Note that U is (d x d) and orthonormal, and $\Sigma^2$ is diagonal. **This is just the eigenvalue decomposition of C**
- This means that we can calculate the eigenvectors of C using the eigenvectors of $X_c$
- It follows that
  - The eigenvectors of C are the columns of U
  - The eigenvalues of C are the diagonal entries of $\Sigma^2$: $\lambda_i^2$

# PCA by SVD

- In summary, computation of PCA by SVD
- Given X with one image (or feature) per column
  - Create the centered data matrix



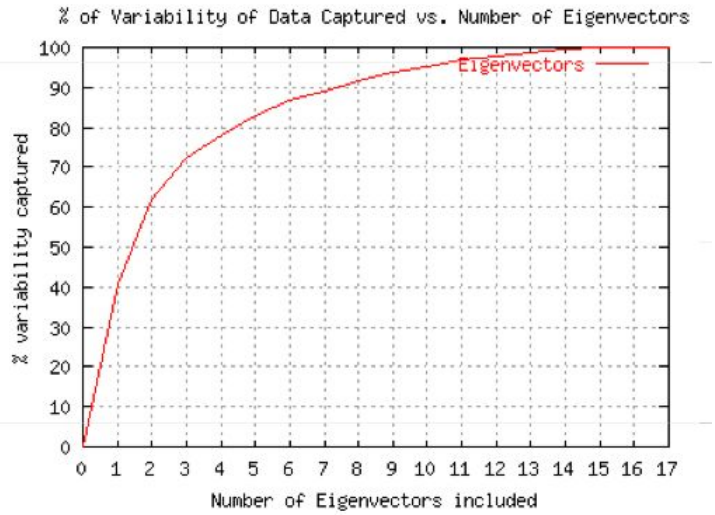
  - Compute its SVD

$$X_c^T = U\Sigma V^T$$

  - Principal components of the covariance matrix C are columns of U

# To compress an image dataset, pick the largest eigenvalues and their corresponding eigenvectors

- Pick the eigenvectors that explain p% of the image data variability
  - Can be done by plotting the ratio $r_k$ as a function of k



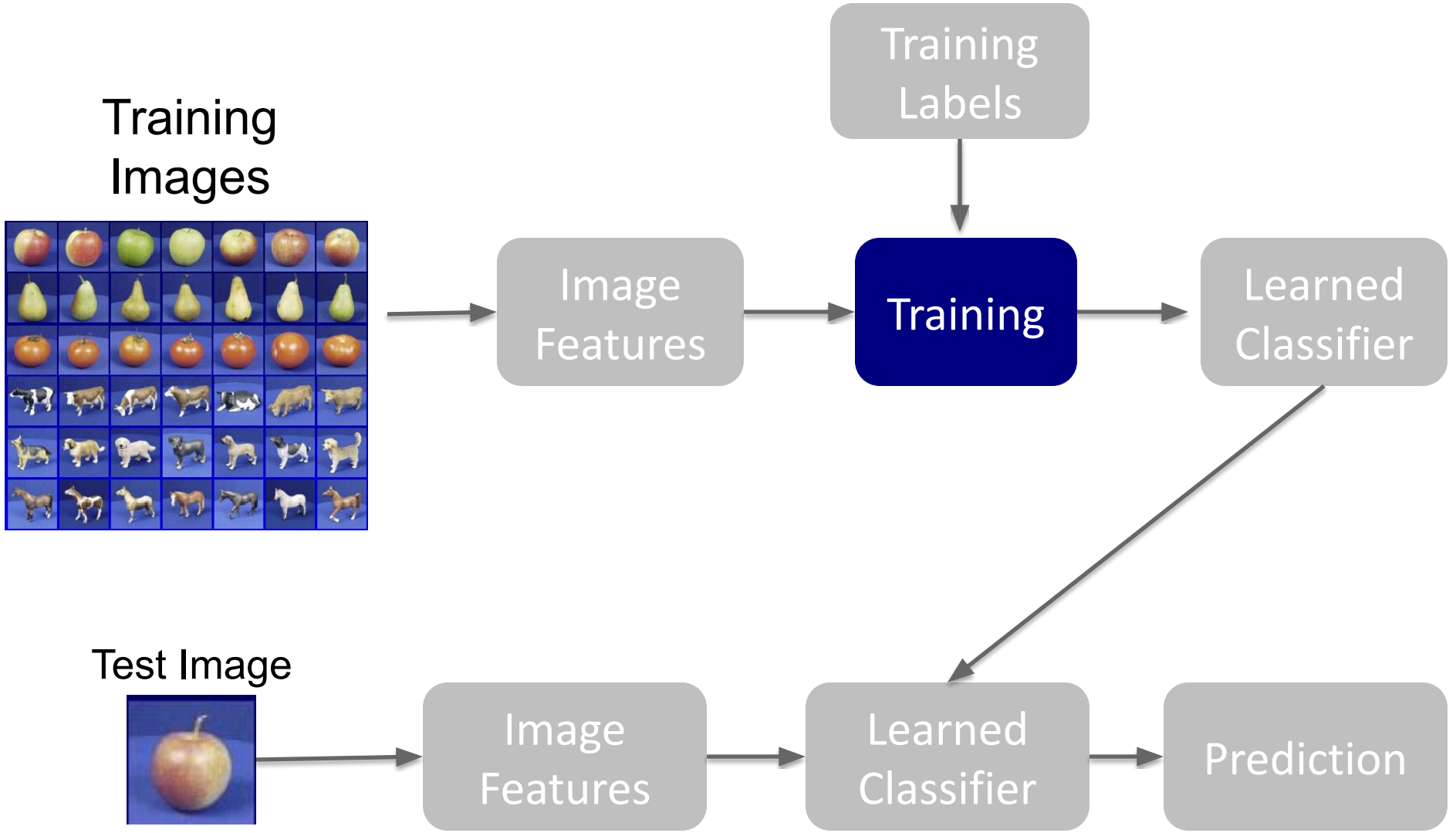

$$r_k = \frac{\sum_{i=1}^{k} \lambda_i^2}{\sum_{i=1}^{n} \lambda_i^2}$$

  - E.g. we need k=3 eigenvectors to cover 70% of the variability of this dataset

# What exactly is the covariance

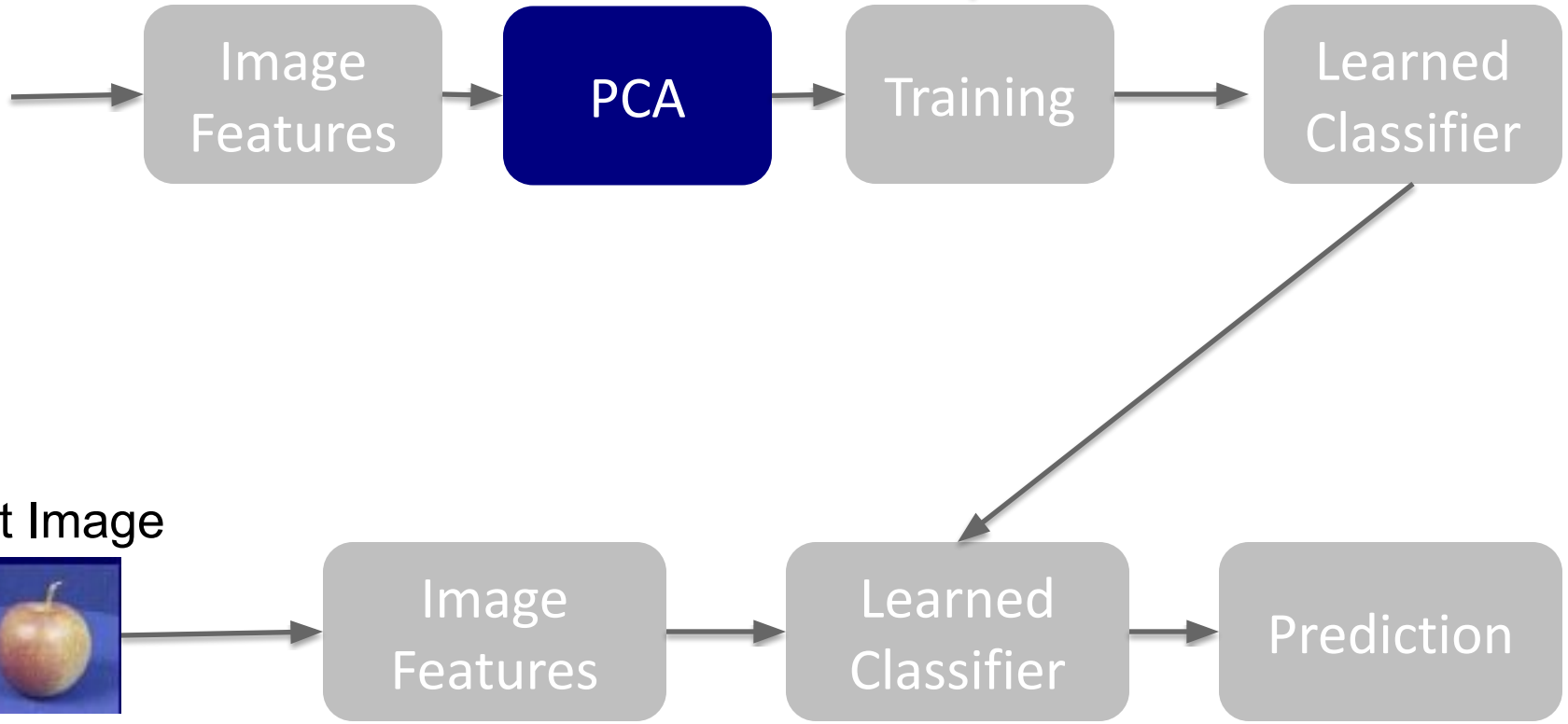
- Variance and Covariance are a measure of the **“spread”** of a set of points around their center of mass (mean)
- **Variance** – measure of the deviation from the mean for points in one dimension e.g. heights
- **Covariance** as a measure of how much each of the dimensions vary from the mean with respect to each other.
- Covariance is measured between 2 dimensions to see if there is a relationship between the 2 dimensions e.g. number of hours studied & marks obtained.
- The covariance between one dimension and itself is the variance

# What happens with PCA during training?



# What happens with PCA during training?

Training Images



# PCA during training

Let's say that we choose k top eigenvalues and their corresponding eigenvectors: [$u_1$, …, $u_k$]

Replace all image features x with:

$$\hat{x} = \begin{bmatrix} u_1^T x \\ u_2^T x \\ \dots \\ u_k^T x \end{bmatrix}$$

# What happens with PCA during testing?

Training Images

Training Labels

Image Features

PCA

Training

Learned Classifier

Test Image

Image Features

Learned Classifier

Prediction

# What happens with PCA during testing?

Training Images

Training Labels

Image Features → PCA → Training → Learned Classifier

Test Image

Image Features → PCA → Learned Classifier → Prediction

# How PCA was originally used in vision: To identify celebrities using their faces

- An image is a point in a high dimensional space
  - In grayscale, an N x M image is a point in $R^{NM}$
  - E.g. 100x100 images lives in a 10,000-dimensional space

# SVD for symmetric matrices

- If A is a symmetric matrix, it can be decomposed as the following:

- Compared to a traditional $A = \Phi\Sigma\Phi^T$ $V^T$ and is an orthogonal matrix.

# What we have learned today?

- Introduction to recognition
- A simple Object Recognition pipeline
- Choosing the right features
- A training algorithm: kNN
- Testing an algorithm
- Challenges with kNN
- Dimensionality reduction: PCA

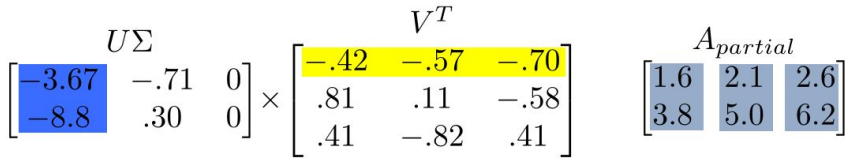
# Next lecture

Object detection

# Extra slides (out of scope)

for those of you curious about how SVD is calculated
and what PCA is usually used for outside of computer vision

# Principal Component Analysis




- Remember, columns of ***U*** are the *Principal Components* of the data: the major patterns that can be added to produce the columns of the original matrix
- One use of this is to construct a matrix where each column is a separate data sample
- Run SVD on that matrix, and look at the first few columns of ***U*** to see patterns that are common among the columns
- This is called *Principal Component Analysis* (or PCA) of the data samples

# Principal Component Analysis

$$\underset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \underset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- Often, raw data samples have a lot of redundancy and patterns
- PCA can allow you to represent data samples as weights on the principal components, rather than using the original raw form of the data
- By representing each sample as just those weights, you can represent just the “meat” of what’s different between samples.
- This minimal representation makes machine learning and other algorithms much more efficient

# How is SVD computed?

- For this class: tell PYTHON to do it. Use the result.
- But, if you're interested, one computer algorithm to do it makes use of Eigenvectors!

# Eigenvector definition

- Suppose we have a square matrix **A**. We can solve for vector x and scalar λ such that Ax= λx
- In other words, find vectors where, if we transform them with **A**, the only effect is to scale them with no change in direction.
- These vectors are called eigenvectors (German for “self vector” of the matrix), and the scaling factors λ are called eigenvalues
- An *m* x *m* matrix will have ≤ *m* eigenvectors where λ is nonzero

# Finding eigenvectors

- Computers can find an x such that $Ax = \lambda x$ using this iterative algorithm:
  - X = random unit vector
  - while(x hasn't converged)
    - X = Ax
    - normalize x
- x will quickly converge to an eigenvector
- Some simple modifications will let this algorithm find all eigenvectors

# Finding SVD

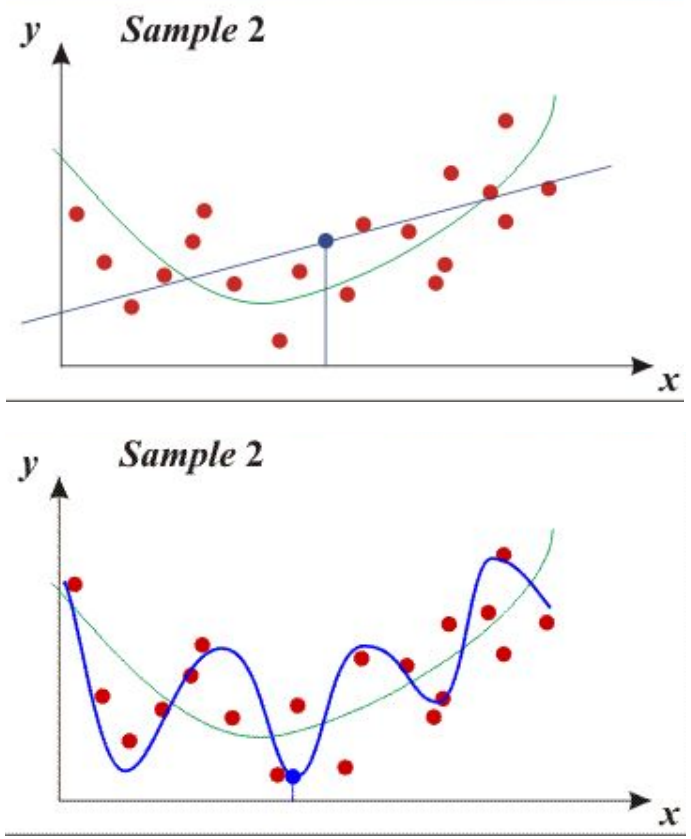
- Eigenvectors are for square matrices, but SVD is for all matrices
- To do svd(A), computers can do this:
  - Take eigenvectors of $AA^T$ (matrix is always square).
    - These eigenvectors are the columns of **U**.
    - Square root of eigen*values* are the singular values (the entries of **Σ**).
  - Take eigenvectors of $A^TA$ (matrix is always square).
    - These eigenvectors are columns of **V** (or rows of $\mathbf{V^T}$)

# Finding SVD

- Moral of the story: SVD is fast, even for large matrices
- It's useful for a lot of stuff
- There are also other algorithms to compute SVD or part of the SVD
  - Python's np.linalg.svd() command has options to efficiently compute only what you need, if performance becomes an issue

A detailed geometric explanation of SVD is here:
http://www.ams.org/samplings/feature-column/fcarc-svd

# Bias-Variance Trade-off




- Models with too few parameters are inaccurate because of a large bias (not enough flexibility).
- Models with too many parameters are inaccurate because of a large variance (too much sensitivity to the sample).

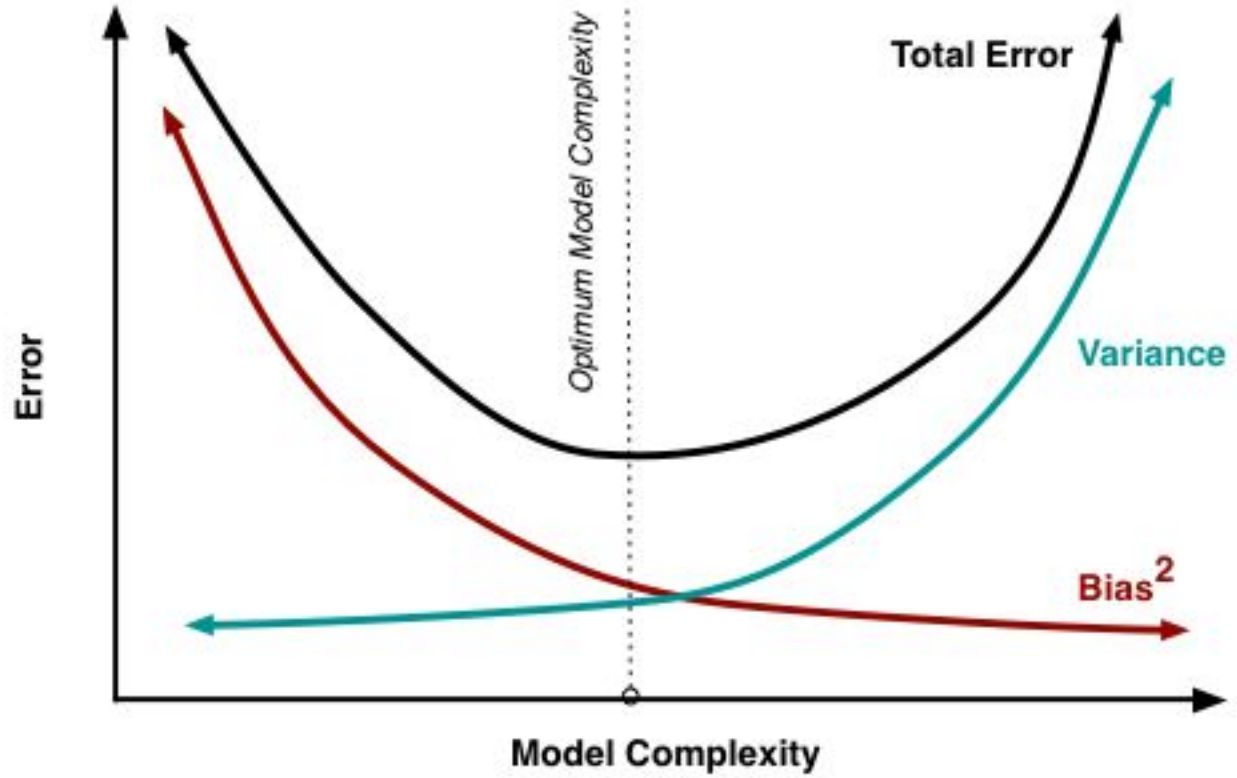
Slide credit: D. Hoiem

# Bias versus variance

- Components of generalization error
  - **Bias:** how much the average model over all training sets differ from the true model?
    - Error due to inaccurate assumptions/simplifications made by the model
  - **Variance:** how much models estimated from different training sets differ from each other
- **Underfitting:** model is too “simple” to represent all the relevant class characteristics
  - High bias and low variance
  - High training error and high test error
- **Overfitting:** model is too “complex” and fits irrelevant characteristics (noise) in the data
  - Low bias and high variance
  - Low training error and high test error


Slide credit: L. Lazebnik

# Bias versus variance trade off



# No Free Lunch Theorem


In a supervised learning setting, we can't tell which classifier will have best generalization

Slide credit: D. Hoiem

# Remember…

- No classifier is inherently better than any other: you need to make assumptions to generalize

- Three kinds of error
  - Inherent: unavoidable
  - Bias: due to over-simplifications
  - Variance: due to inability to perfectly estimate parameters from limited data



Slide credit: D. Hoiem

# How to reduce variance?

- Choose a simpler classifier
- Regularize the parameters
- Get more training data

How do you reduce bias?

Slide credit: D. Hoiem

# Last remarks about applying machine learning methods to object recognition

- There are machine learning algorithms to choose from
- Know your data:
  - How much supervision do you have?
  - How many training examples can you afford?
  - How noisy?
- Know your goal (i.e. task):
  - Affects your choices of representation
  - Affects your choices of learning algorithms
  - Affects your choices of evaluation metrics
- Understand the math behind each machine learning algorithm under consideration!

# PCA by SVD

- An alternative manner to compute the principal components, based on singular value decomposition
- Quick reminder: SVD
  - Any real n x m matrix (n>m) can be decomposed as

$$A = M \Pi N^T$$

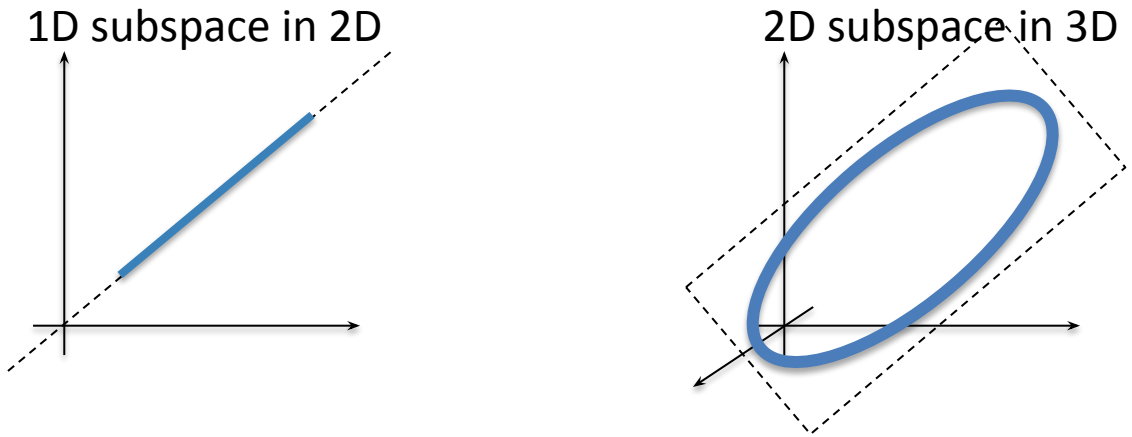
  - Where M is an (n x m) column orthonormal matrix of left singular vectors (columns of M)
  - Π is an (m x m) diagonal matrix of singular values
  - $N^T$ is an (m x m) row orthonormal matrix of right singular vectors (columns of N)

$$M^T M = I \qquad N^T N = I$$

Slide inspired by N. Vasconcelos

# PCA Formulation

- Basic idea:
  - If the images (or their features) live in a subspace, it is going to look very flat when viewed from the full feature space, e.g.



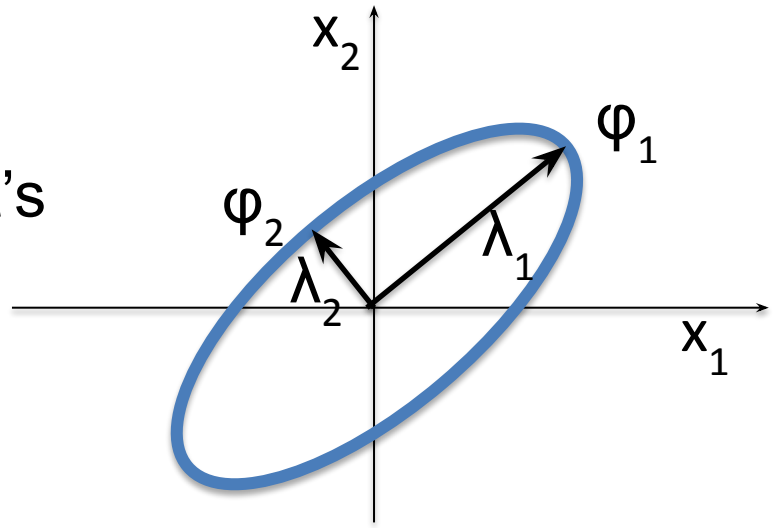

Slide inspired by N. Vasconcelos

# Alternative PCA Formulation

- Assume images x is Gaussian with covariance Σ.

- Recall that a gaussian is defined with it's mean and variance:

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$




- Recall that μ and Σ of a gaussian are defined as:

$$\boldsymbol{\mu} = \mathrm{E}[\mathbf{X}]$$

$$\boldsymbol{\Sigma} =: \mathrm{E}[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}}] = [\mathrm{Cov}[X_i, X_j]; 1 \leq i, j \leq k]$$

# Alternative PCA formulation

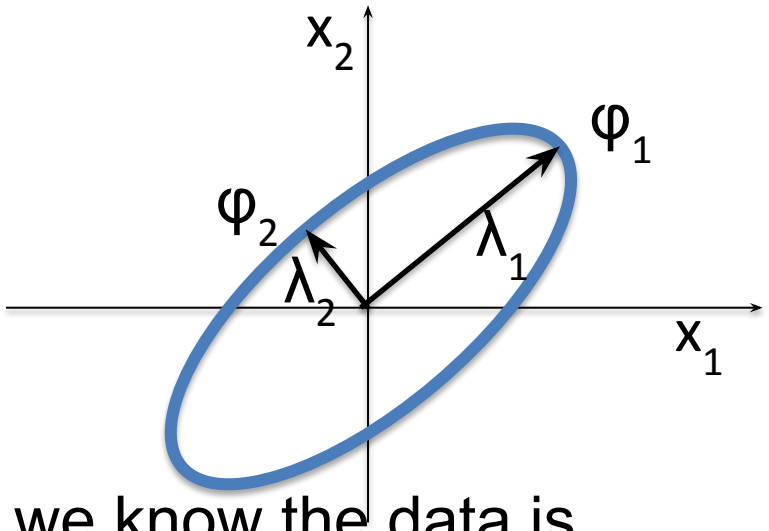
- Since gaussians are symmetric, it's covariance matrix is also a symmetric matrix. So we can express it as:
  - $\mathbf{\Sigma} = \mathbf{U\Lambda U}^T = \mathbf{U\Lambda}^{1/2}(\mathbf{U\Lambda}^{1/2})^T$

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \iff \mathbf{X} \sim \boldsymbol{\mu} + \mathbf{U}\boldsymbol{\Lambda}^{1/2}\mathcal{N}(0, \mathbf{I})$$

$$\iff \mathbf{X} \sim \boldsymbol{\mu} + \mathbf{U}\mathcal{N}(0, \boldsymbol{\Lambda}).$$

# Alternative PCA Formulation

- If x is Gaussian with covariance Σ,
  - Principal components $\varphi_i$ are the eigenvectors of Σ
  - Principal lengths $\lambda_i$ are the eigenvalues of Σ




- by computing the eigenvalues we know the data is
  - Not flat if $\lambda_1 \approx \lambda_2$
  - Flat if $\lambda_1 >> \lambda_2$

# Alternative PCA Algorithm (training)

- Given sample $\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, $x_i \in \mathcal{R}^d$

  - compute sample mean: $\hat{\mu} = \frac{1}{n}\sum_i(\mathbf{x}_i)$
  - compute sample covariance: $\hat{\Sigma} = \frac{1}{n}\sum_i(\mathbf{x}_i - \hat{\mu})(\mathbf{x}_i - \hat{\mu})^T$
  - compute eigenvalues and eigenvectors of $\hat{\Sigma}$

$$\hat{\Sigma} = \Phi\Lambda\Phi^T, \ \Lambda = diag(\sigma_1^2, \ldots, \sigma_n^2) \ \Phi^T\Phi = I$$

  - order eigenvalues $\sigma_1^2 > \ldots > \sigma_n^2$
  - if, for a certain $k$, $\sigma_k << \sigma_1$ eliminate the eigenvalues and eigenvectors above $k$.

# Alternative PCA Algorithm (testing)

- Given principal compoenents $\phi_i, i \in 1, \ldots, k$ and a test sample $\mathcal{T} = \{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$, $t_i \in \mathcal{R}^d$
  - subtract mean to each point $\mathbf{t}'_i = \mathbf{t}_i - \widehat{\mu}$
  - project onto eigenvector space $\mathbf{y}_i = \mathbf{A}\mathbf{t}'_i$ where

$$\mathbf{A} = \begin{bmatrix} \phi_1^T \\ \vdots \\ \phi_k^T \end{bmatrix}$$

  - use $\mathcal{T}' = \{\mathbf{y}_1, \ldots \mathbf{y}_n\}$ to estimate class conditional densities and do all further processing on $\mathbf{y}$.