Lecture 11

Geometry and Cameras

Administrative

A3 is out

- Due May 19th

A1 is graded

Administrative

Recitation

- Multiview geometry

Today's agenda

- How biological vision understands geometry
- Brief history of geometric vision
- Geometric transformations
- Pinhole camera
- The Pinhole camera transformation

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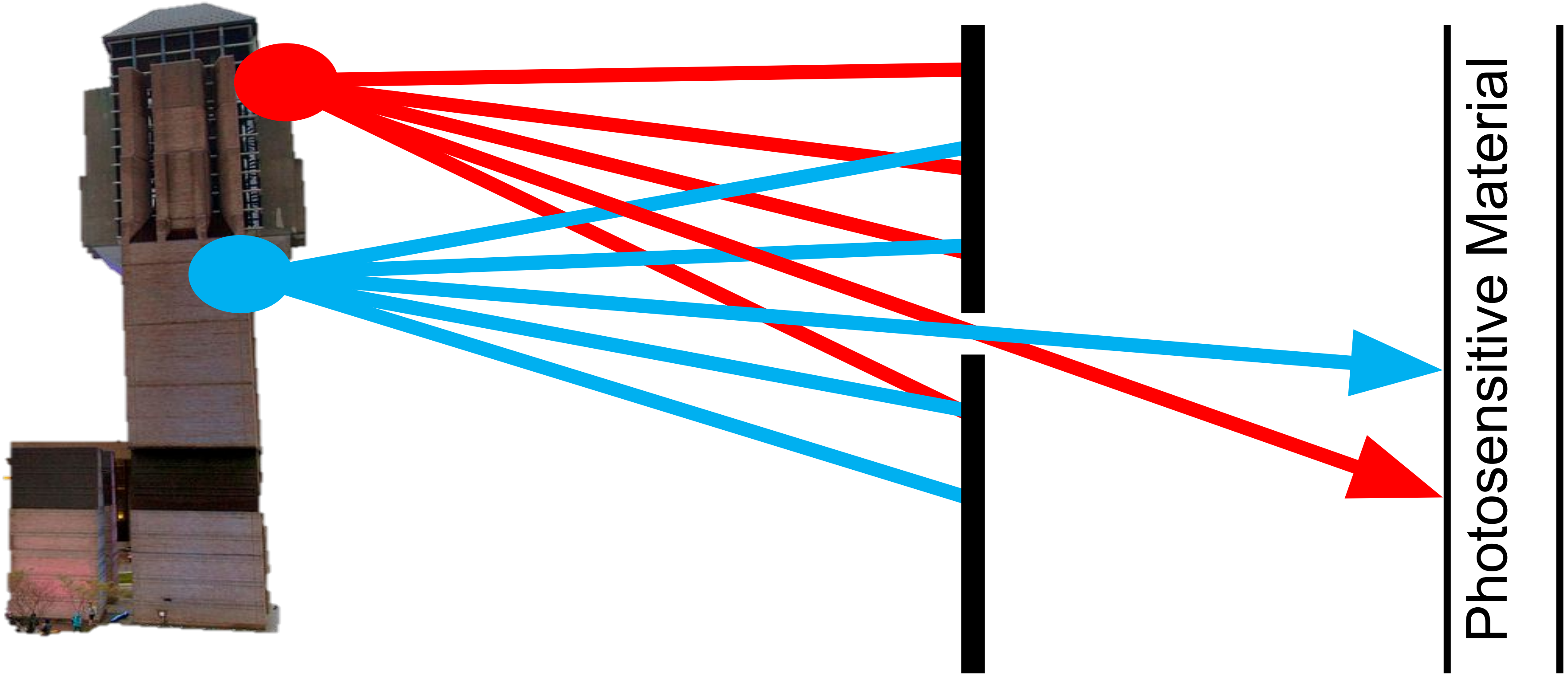
Our goal: Recover the 3D geometry of the world

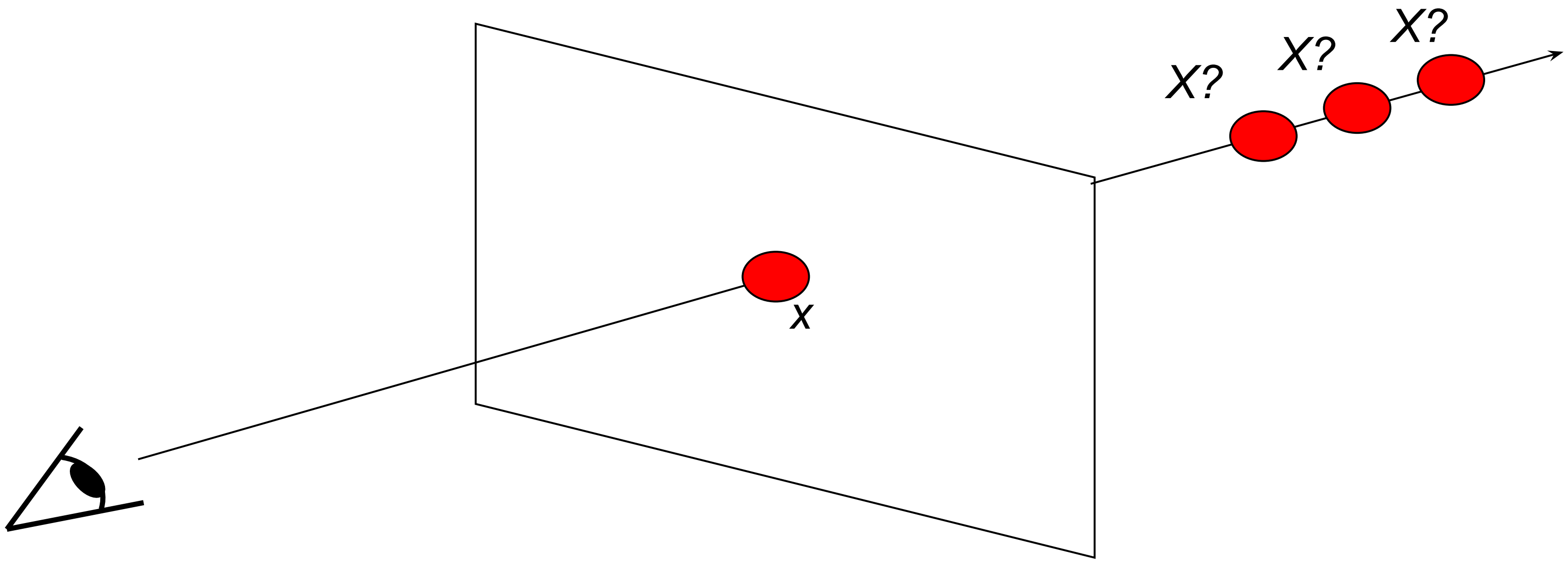
J. Vermeer, *Music Lesson*, 1662

A. Criminisi, M. Kemp, and A. Zisserman, Bringing Pictorial Space to Life: computer techniques for the analysis of paintings, *Proc. Computers and the History of Art*, 2002

Let's Take a Picture!

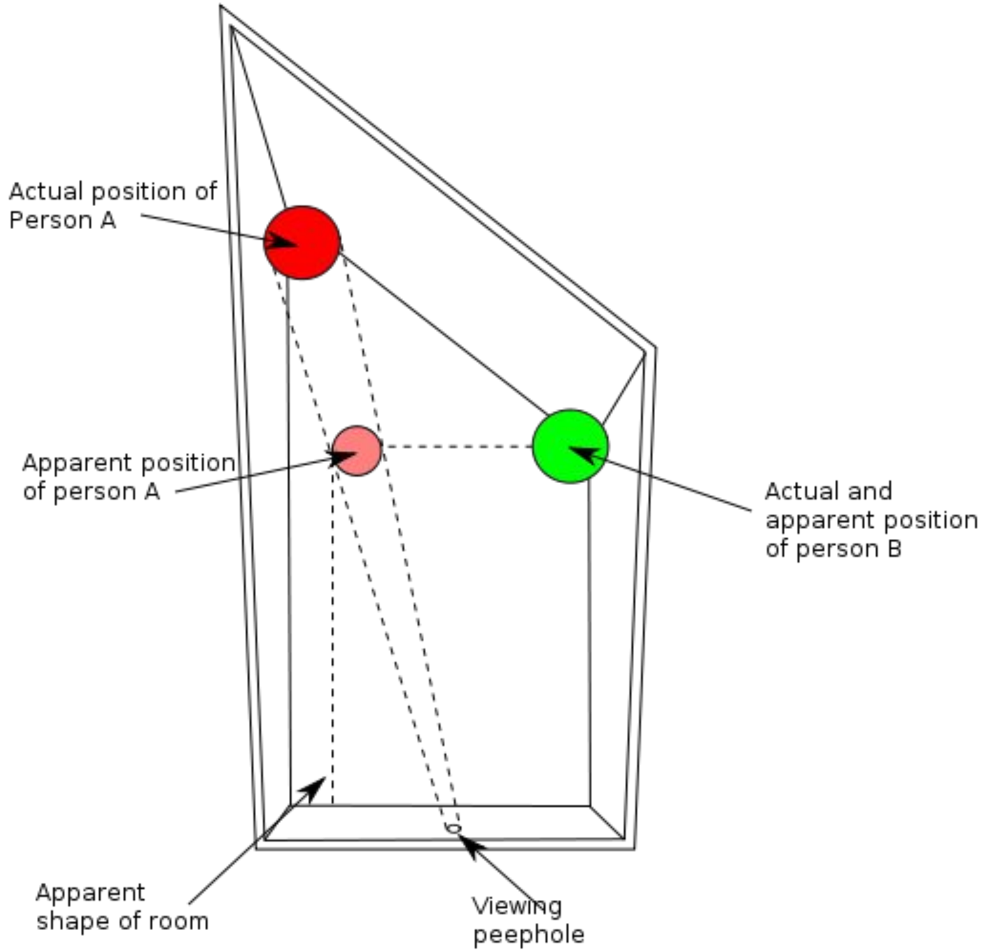


Single-view Ambiguity



- Given a camera and an image, we only know the ray corresponding to each pixel.
- We don’t know how far away the object the ray was reflected from
  - We don’t have enough constraints to solve for X (depth)

Single-view Ambiguity



http://en.wikipedia.org/wiki/Ames_room

Slide Credit: J. Hays

Single-view Ambiguity

Diagram credit: J. Hays

Resolving Single-view Ambiguity

- Shoot light (lasers etc.) out of your eyes!
- Con: not so biologically plausible, dangerous?

Resolving Single-view Ambiguity

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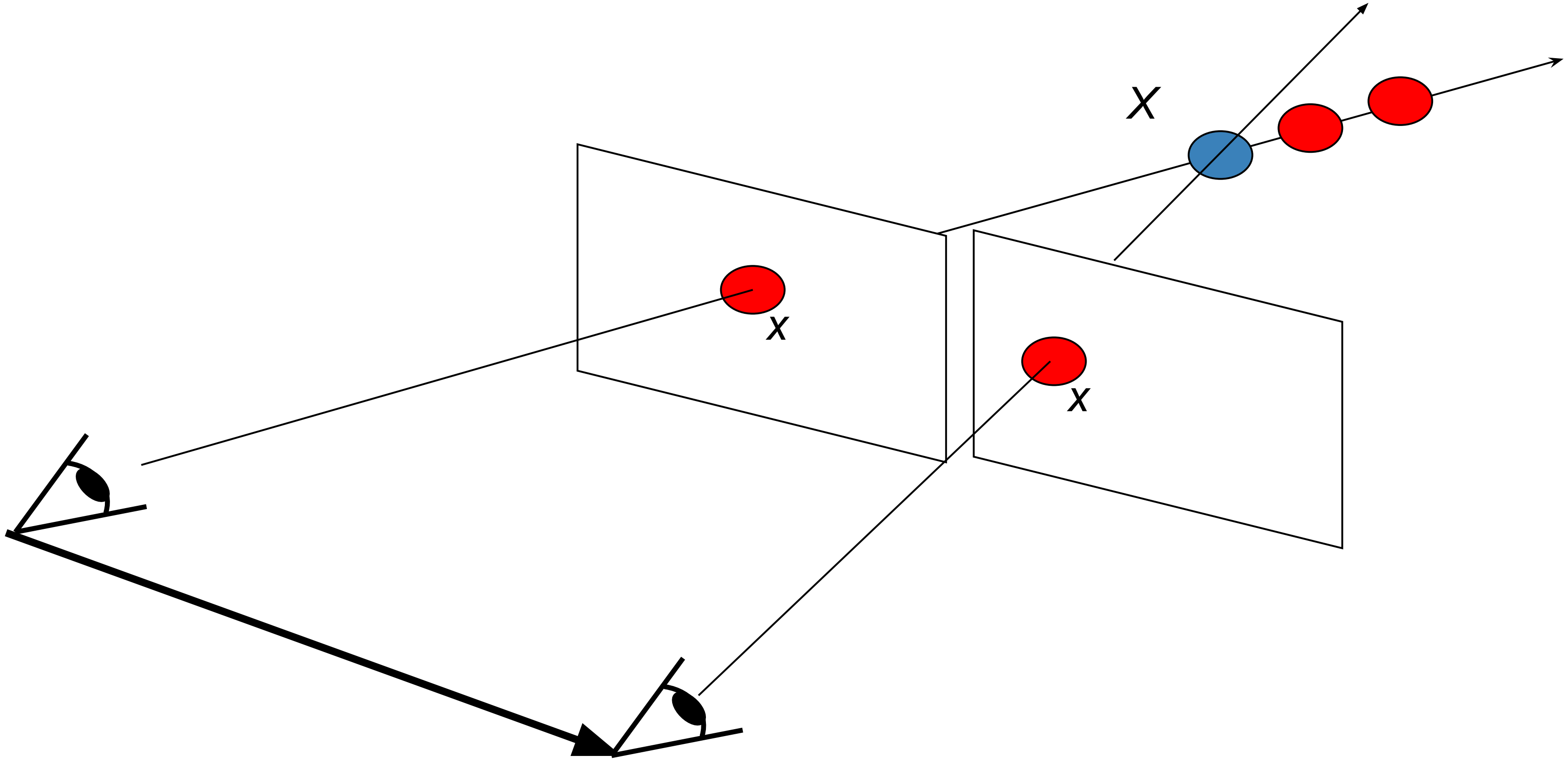
- Con: not so biologically plausible, dangerous?

How do humans estimate depth? Two eyes!



- Stereo: given 2 calibrated cameras in different views and correspondences, can solve for X

Stereo photography and stereo viewers

Take two pictures of the same subject from two slightly different viewpoints and display so that each eye sees only one of the images.

Invented by Sir Charles Wheatstone, 1838

Image from fisher-price.com

http://www.johnsonshawmuseum.org

Slide credit: J. Hays

http://www.well.com/~jimg/stereo/stereo_list.html

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Not all animals see stereo:

Prey animals are Stereoblind
(large field of view to spot predators)

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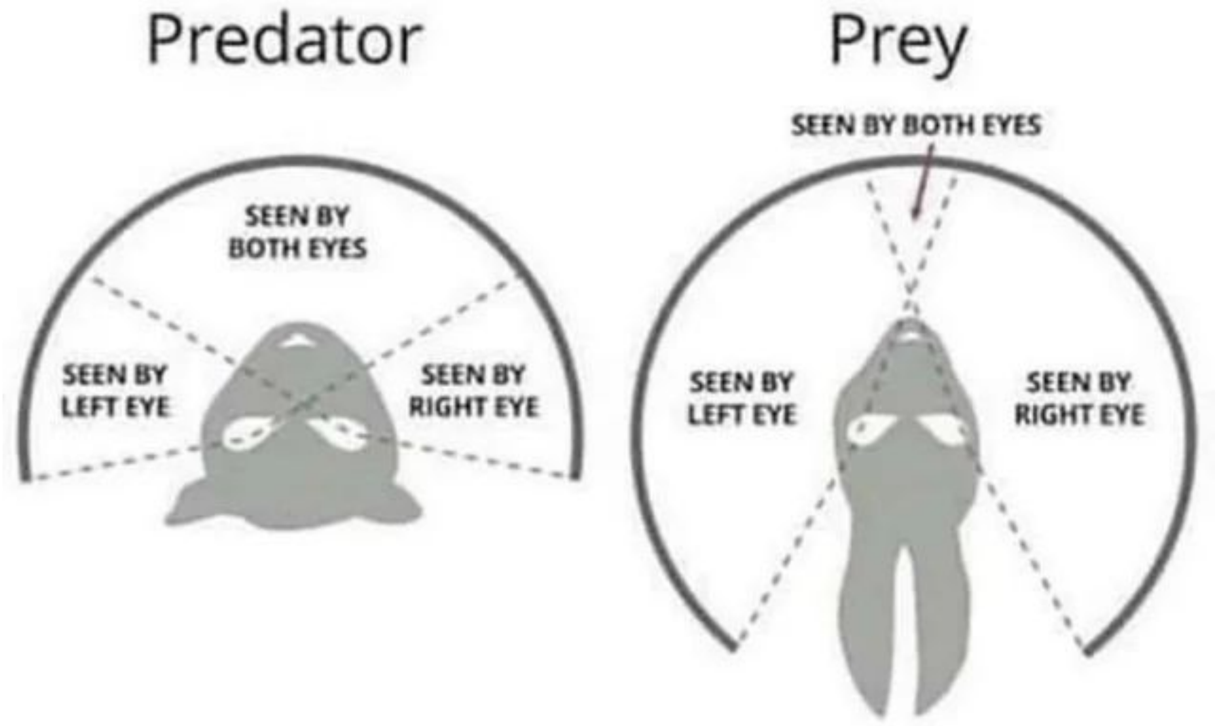
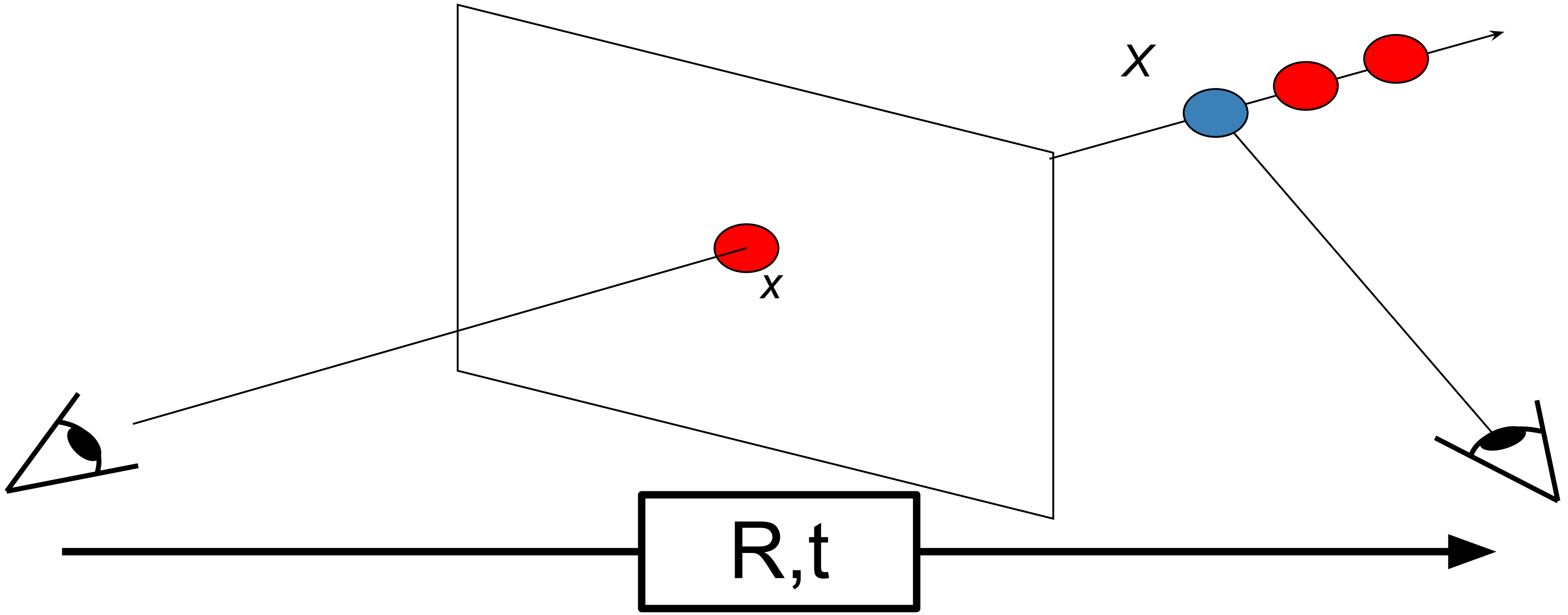


Resolving Single-view Ambiguity



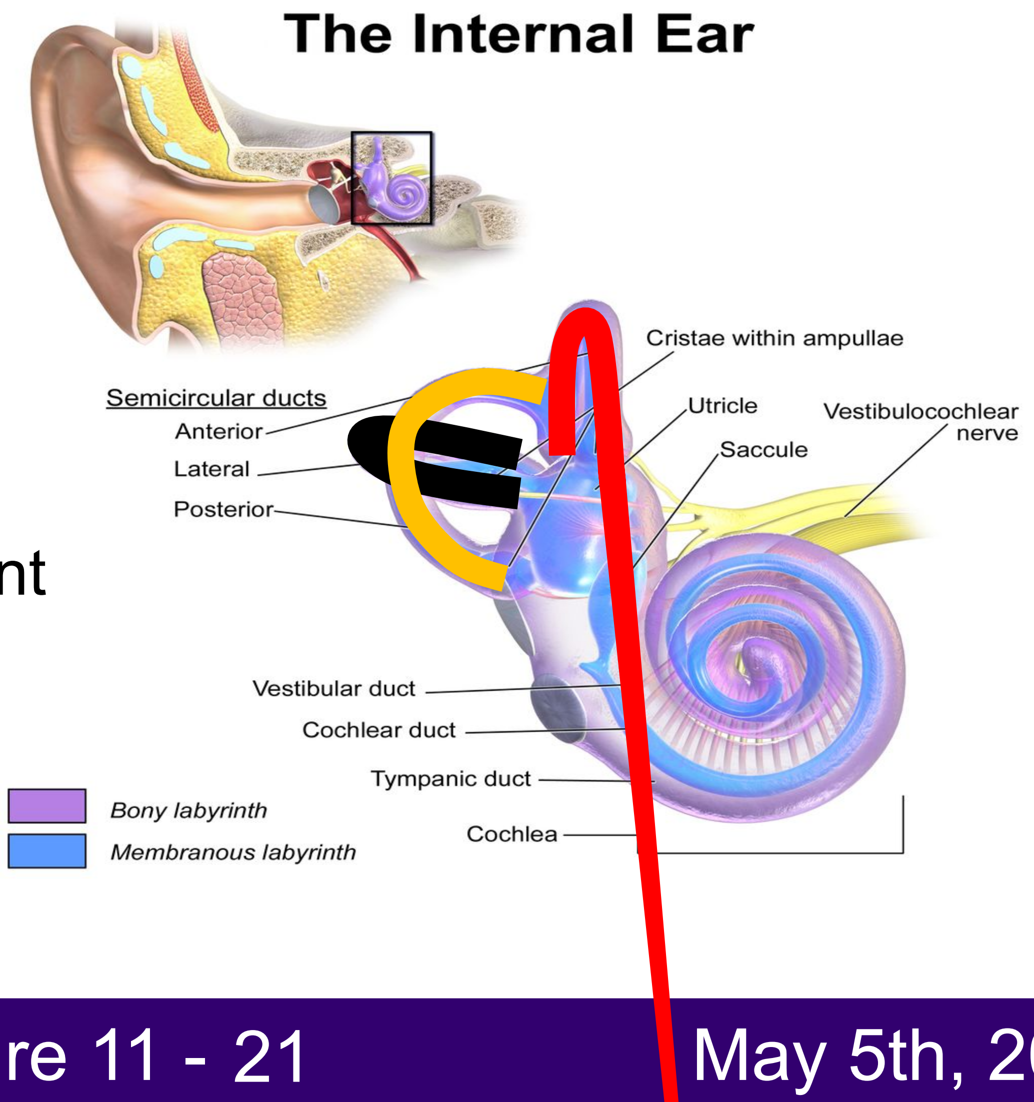
- One option: move the camera, find matching correspondences
- If you know how you moved in the physical world and have corresponding points in image space, you can solve for X

How do you estimate how much you moved in the physical world?

Can estimate using our eyes!

Can estimate using our ears!

- Our inner ears have 3 ducts
- Can estimate movement via signals sent to muscles

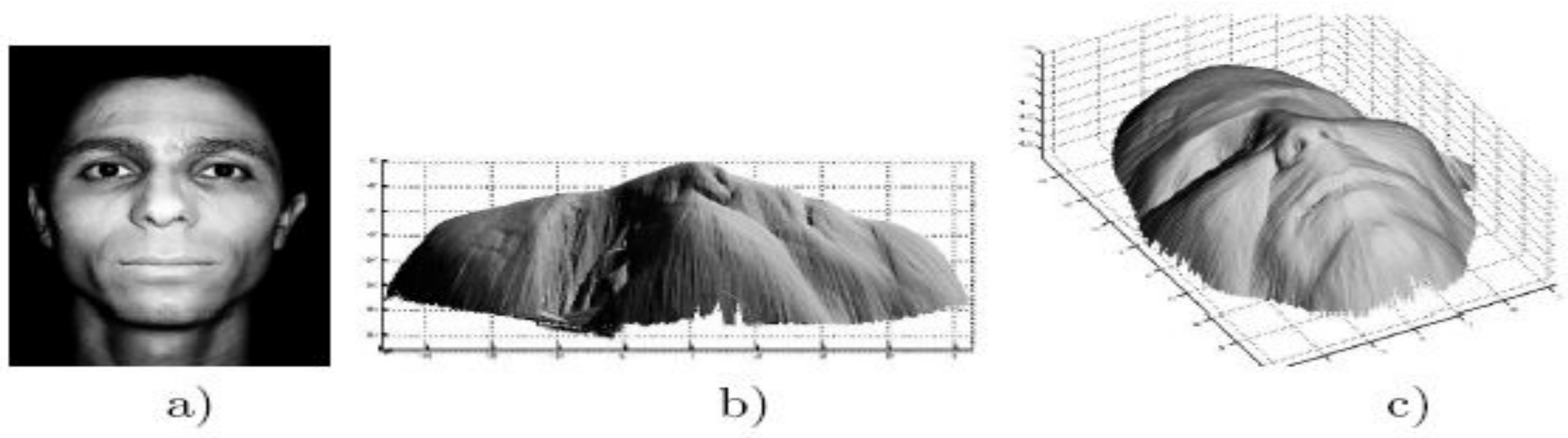


We can estimate depth from a single image. But how?

- You haven’t been here before, yet you probably have a fairly good understanding of this scene.

We use pictorial cues – such as familiar objects

We use pictorial cues – such as shading



[Figure from Prados & Faugeras 2006]

We use pictorial cues – such as perspective effects

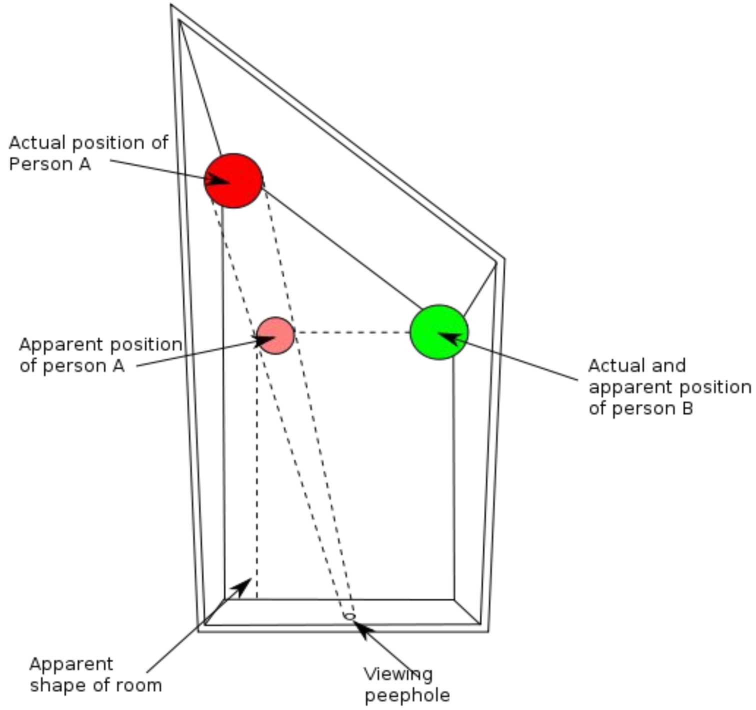
Reality of 3D Perception

- 3D perception is absurdly complex and involves integration of many cues:
  - **Learned cues** for 3D
  - **Stereo** between eyes
  - Stereo via **motion**
  - Integration of known motion signals to **muscles** (efferent copy), acceleration sensed via ears
  - Past experience of **touching** objects
- All connect: learned cues from 3D probably come from stereo/motion cues in large part

Really fantastic article on cues for 3D from Cutting and Vishton, 1995: https://pmvish.people.wm.edu/cutting%26vishton1995.pdf

Regardless, illusions can still fool this complex system

Ames illusion persists (in a weaker form) even if you have stereo vision –guessing the texture is rectilinear is usually incredibly reliable



Gehringer and Engel, Journal of Experimental Psychology: Human Perception and Performance, 1986

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Simplified Image Formation

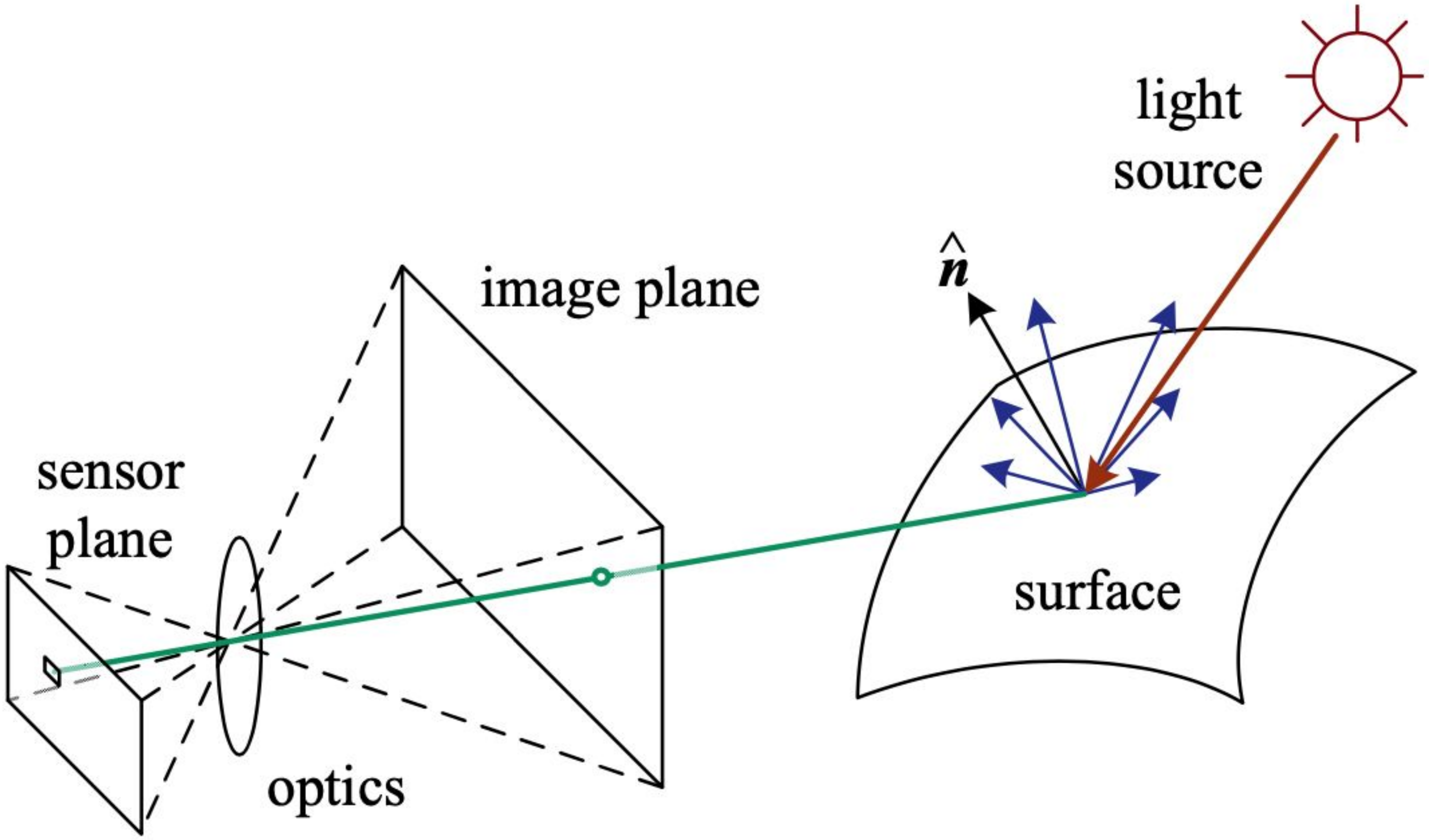
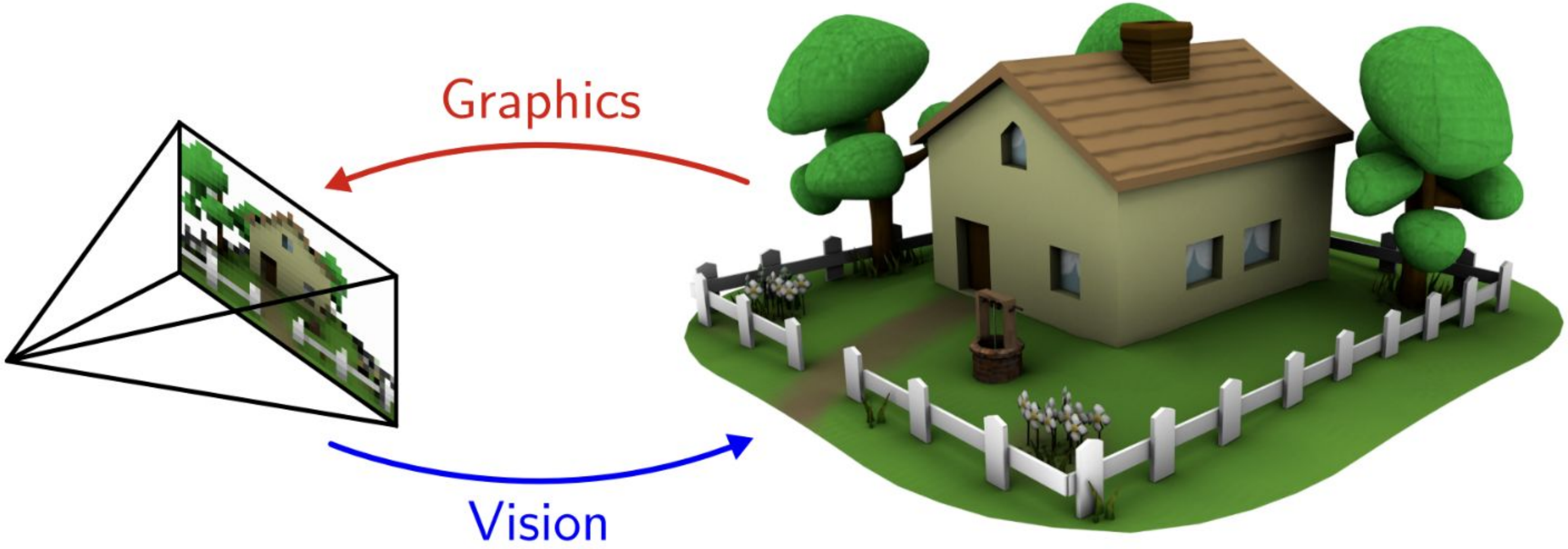


Figure: R. Szeliski

Geometric vision is an **ill-posed inverse problem**

2D Image | **3D Scene**



Pixel Matrix

217	191	252	255	239
102	80	200	146	138
159	94	91	121	138
179	106	136	85	41
115	129	83	112	67
94	114	105	111	89

Objects | Material
Shape/Geometry | Motion
Semantics | 3D Pose

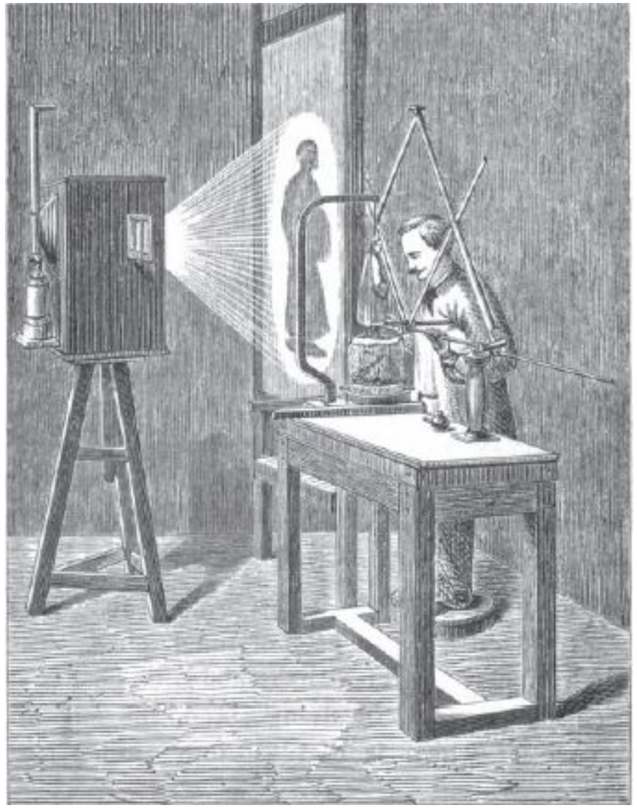
Slide credit: Andreas Geiger

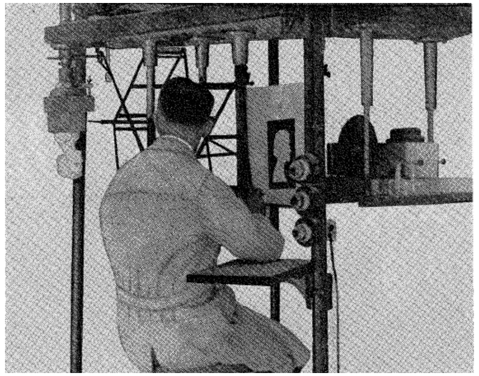
Brief History of Geometric Vision

- 2020-: geometry + learning
- 2010s: deep learning
- 2000s: local detectors and descriptors
- 1990s: digital camera, 3D geometry estimation
- 1980s: epipolar geometry (stereo)
- …

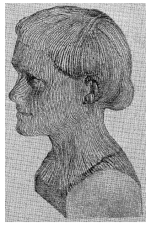
Brief History of Geometric Vision

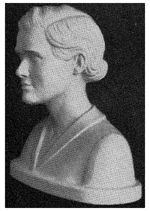
- 1860s: Willème invented photo-scultures





10 E. Morin and E. Rovins, pantographic studio (from *Le Monde illustré*, December 17, 1864)



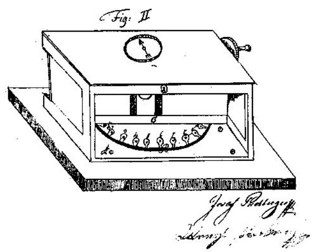


Source: P. Sturm

Brief History of Geometric Vision

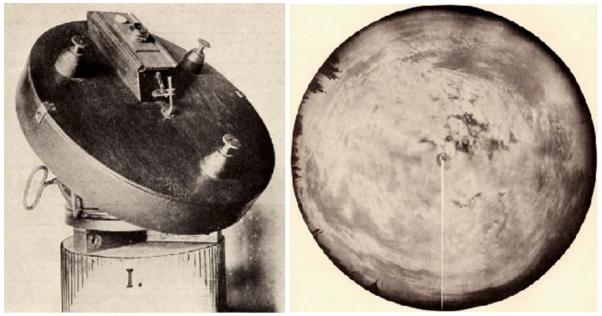
- 1860s: Willème invented photo-scultures
- 1850s: birth of photogrammetry [Laussedat]
- 1840s: panoramic photography

1864



Puchberger 1843

Cylindrograph
Moëssard 1884



“Cloud camera”,
190?

Source: P. Sturm

Brief History of Geometric Vision

- 1860s: Willème invented photo-scultures
- 1850s: birth of photogrammetry [Laussedat]
- 1840s: panoramic photography
- 1822-39: birth of photography [Niépce, Daguerre]
- 1773: general 3-point pose estimation [Lagrange]
- 1715: basic intrinsic calibration (pre-photography!) [Taylor]
- 1700’s: topographic mapping from perspective drawings [Beautemps-Beaupré, Kappeler]

Niépce, “La Table Servie”, 1822

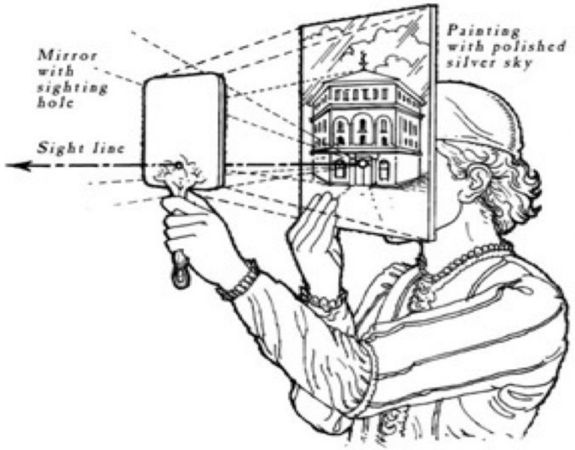
Source: P. Sturm

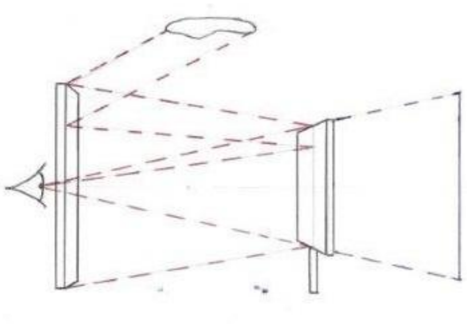
Brief History of Geometric Vision

- 15th century: start of mathematical treatment of 3D, first AR app?

Augmented reality invented by Filippo Brunelleschi (1377-1446)?

Tavoletta prospettica di Brunelleschi





Source: P. Sturm

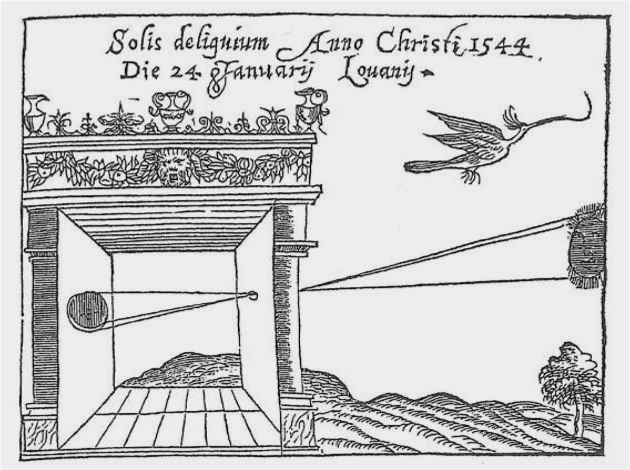
Brief History of Geometric Vision

- 5^{th} century BC: principles of pinhole camera, a.k.a. camera obscura
    - China: 5th century BC
    - Greece: 4th century BC
    - Egypt: 11th century
    - Throughout Europe: from 11th century onwards

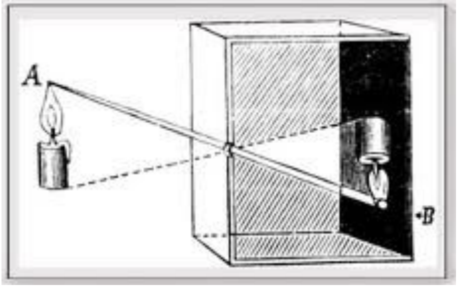
First mention …

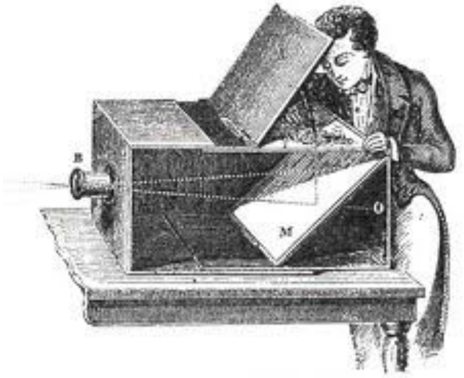
Chinese philosopher Mozi
(470 to 390 BC)

First camera?



Greek philosopher Aristotle
(384 to 322 BC)





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Points

2D points: $\mathbf{x} = (x, y) \in \mathcal{R}^2$ or column vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

3D points: $\mathbf{x} = (x, y, z) \in \mathcal{R}^3$ (often noted $\mathbf{X}$ or $\mathbf{P}$)

Homogeneous coordinates: append a 1

Why? $\bar{\mathbf{x}} = (x, y, 1)$ $\bar{\mathbf{x}} = (x, y, z, 1)$

Everything is easier in Projective Space

- 2D Lines:

  Representation: $l = (a, b, c)$

  Equation: $ax + by + c = 0$

  In homogeneous coordinates: $\bar{x}^T l = 0$

General idea: homogenous coordinates unlock the full power of linear algebra!

Homogeneous coordinates in 2D

2D Projective Space: $\mathcal{P}^2 = \mathcal{R}^3 - (0,0,0)$ (same story in 3D with $\mathcal{P}^3$)

- heterogeneous → homogeneous $\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- homogeneous → heterogeneous $\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$

- points differing only by scale are *equivalent*: $(x, y, w) \sim \lambda\,(x, y, w)$

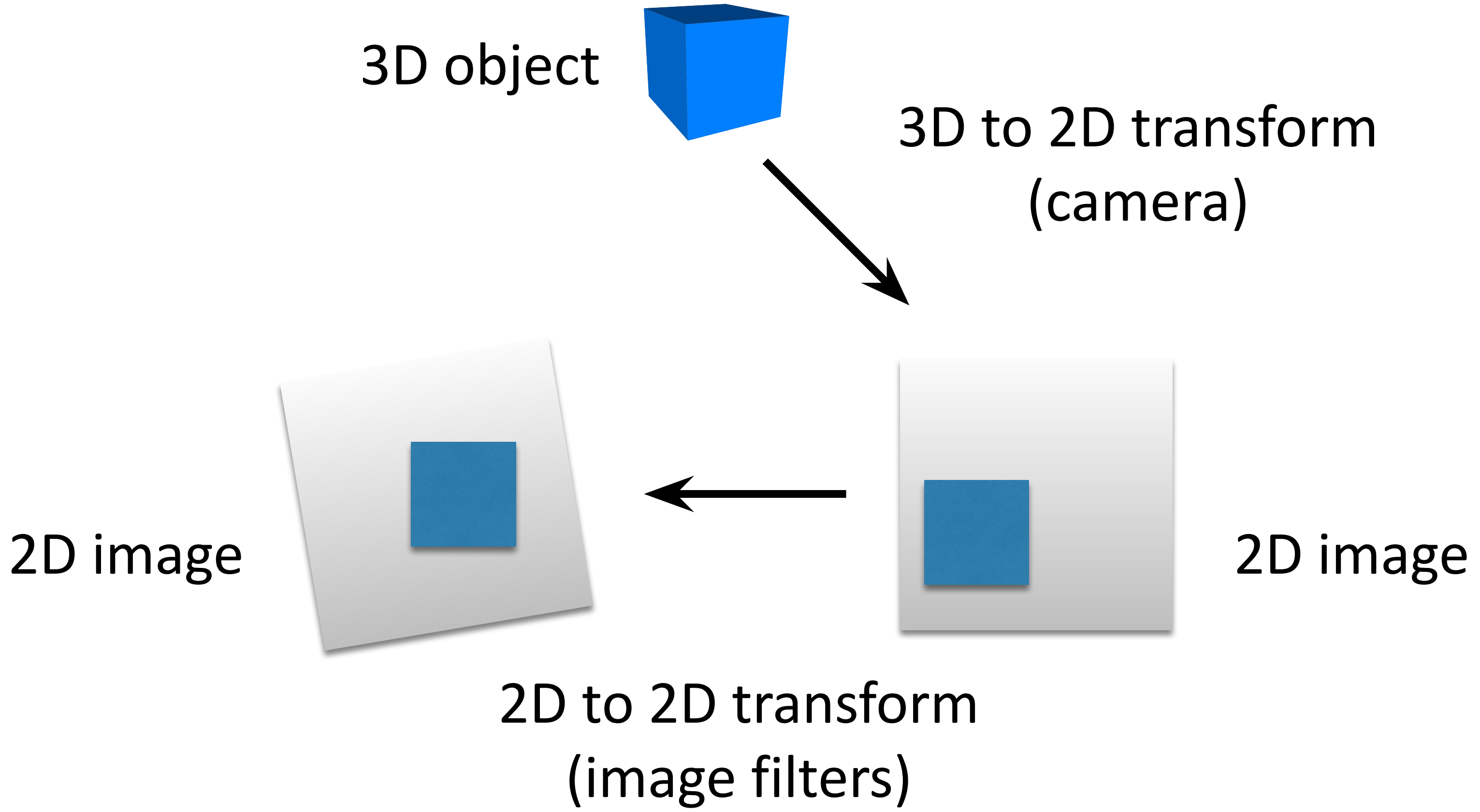
$$\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{w}) = \tilde{w}(x, y, 1) = \tilde{w}\bar{\mathbf{x}}$$

The camera as a coordinate transformation

A camera is a mapping

from: the 3D world

to: a 2D image



Source: K. Kitani

Cameras and objects can move!

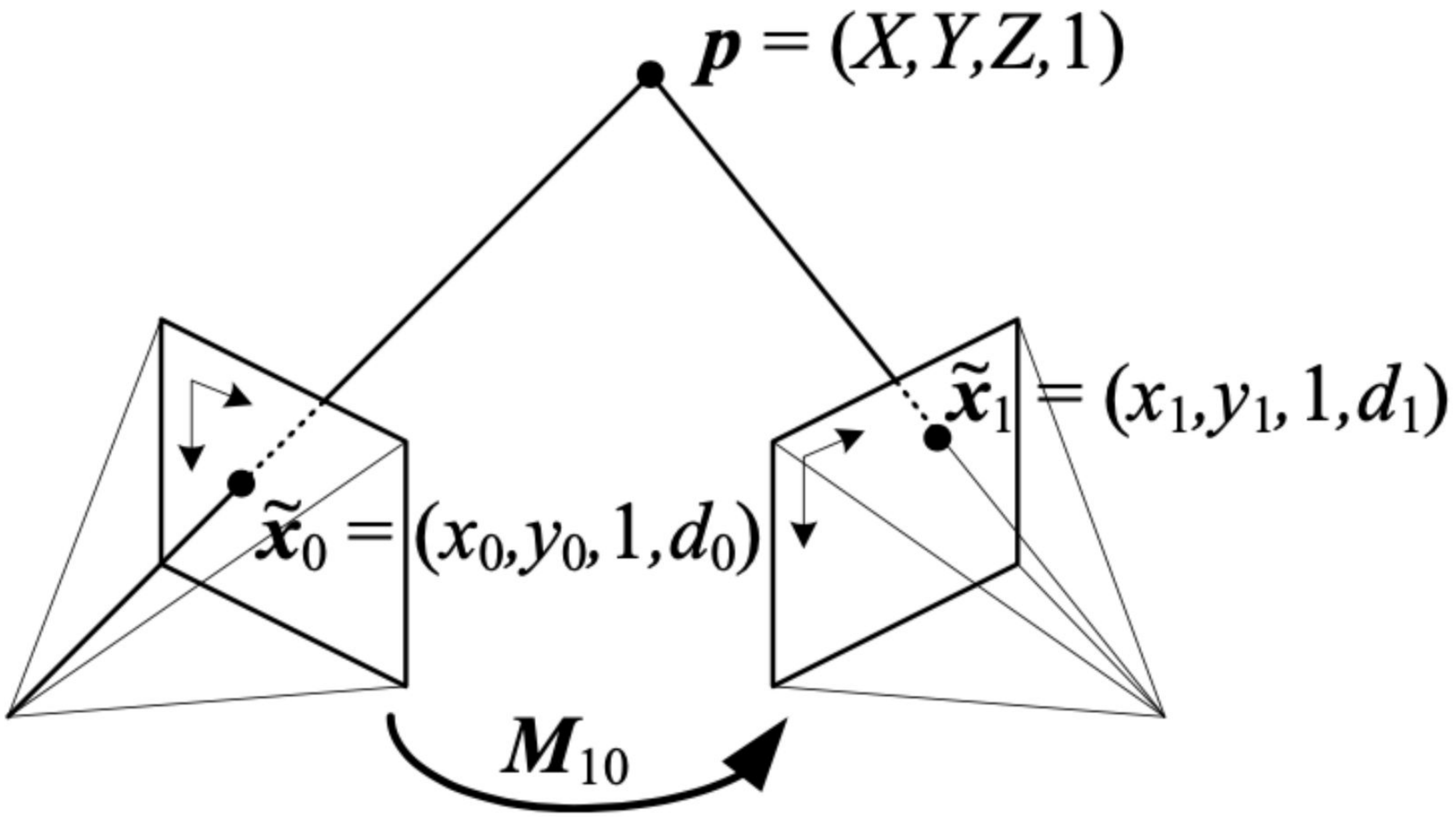


Figure: R. Szeliski

2D Transformations in pixel locations
(not pixel values)

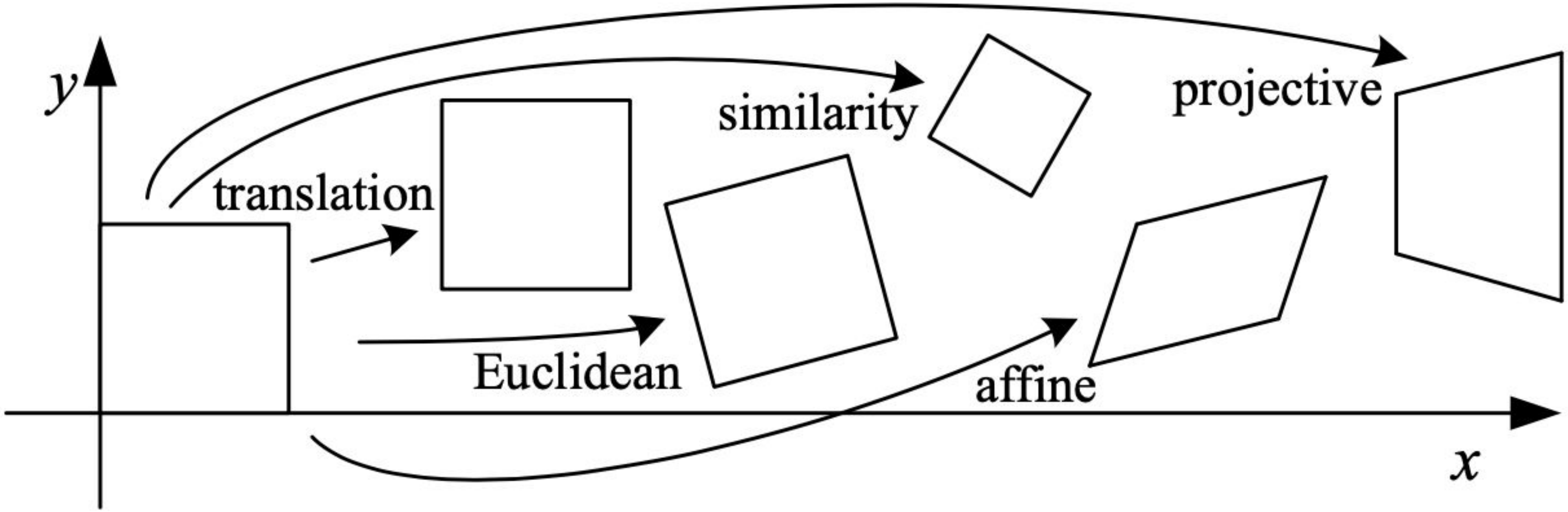
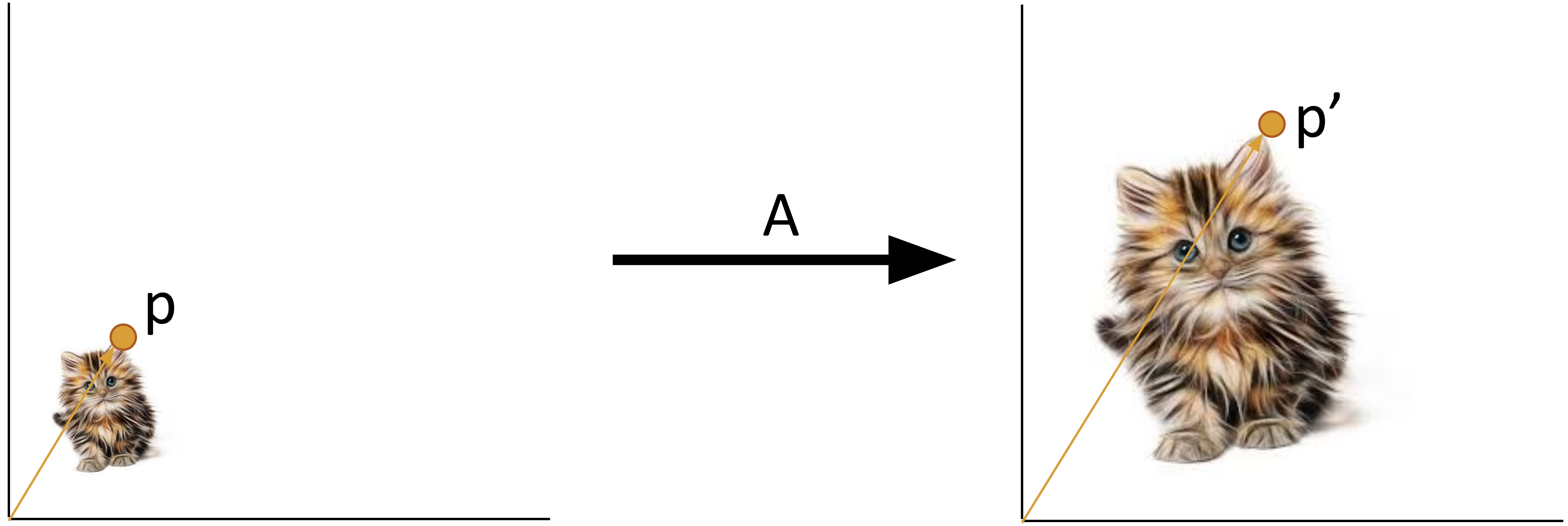


Figure: R. Szeliski

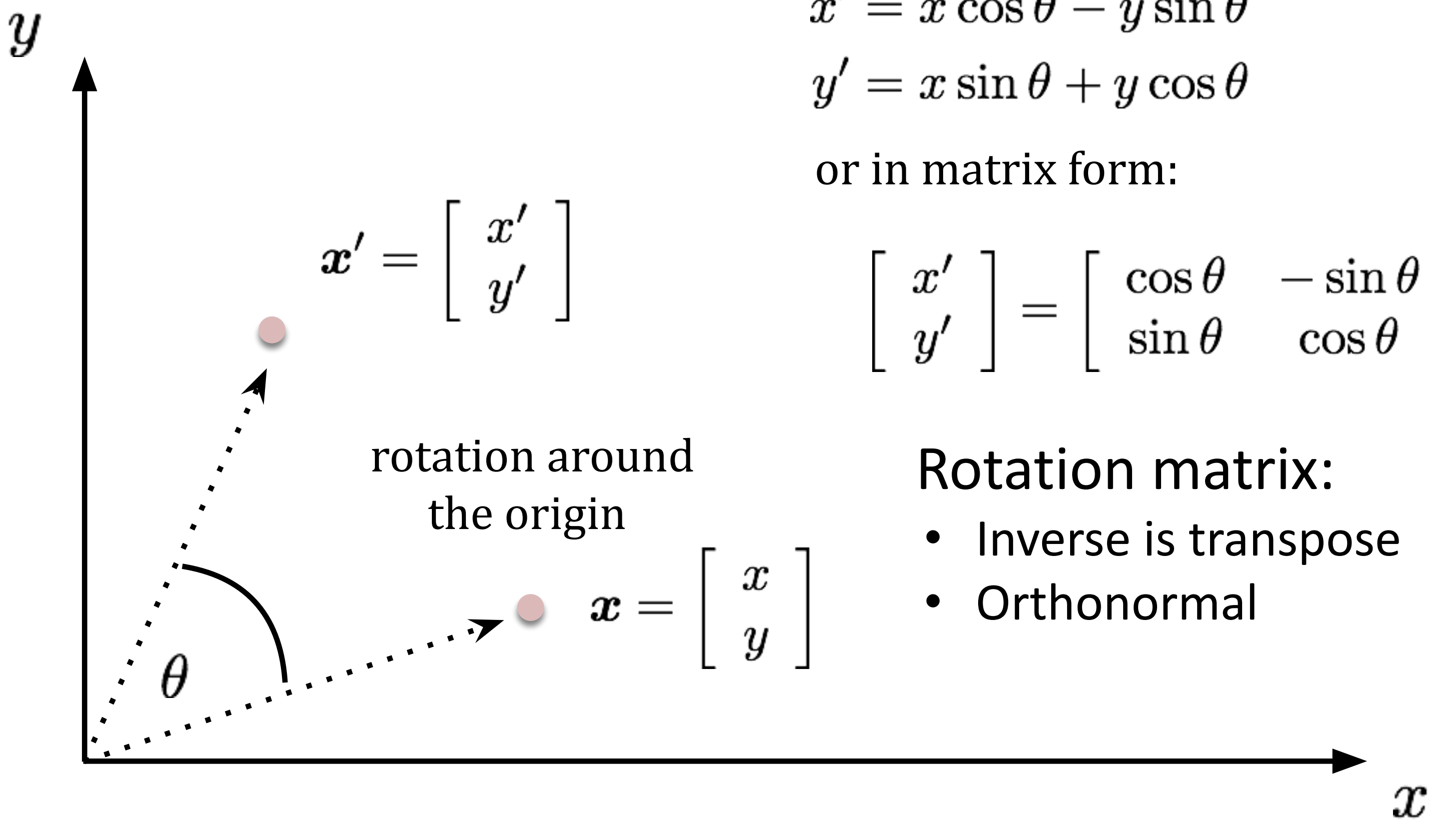
Scaling

$$\underset{A}{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}} \times \underset{p}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{p'}{\begin{bmatrix} s_x x \\ s_y y \end{bmatrix}}$$



Slide: J.C. Niebles

Rotation



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

or in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

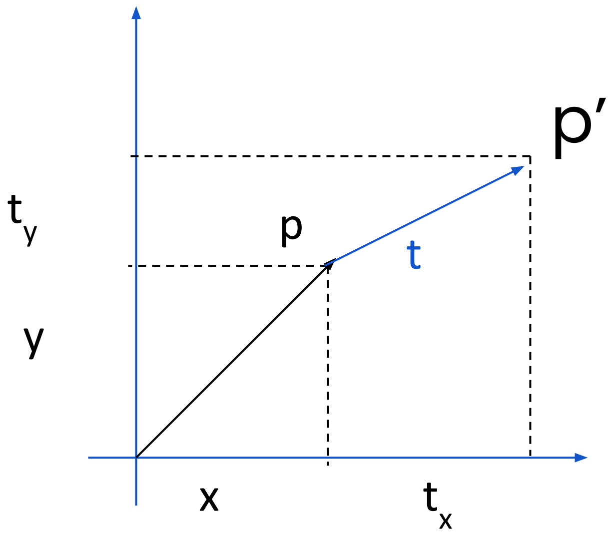
Rotation matrix:

- Inverse is transpose
- Orthonormal

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$
$$\det(\mathbf{R}) = 1$$

Slide: K. Kitani

2D Translation



$$x' = x + t_x$$
$$y' = y + t_y$$

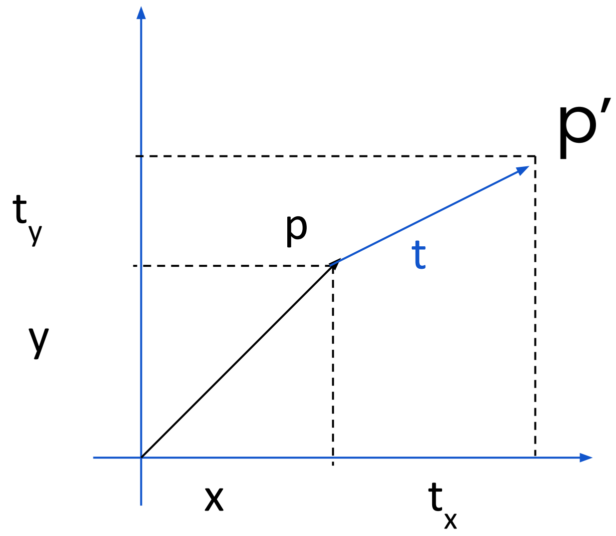
As a matrix?

Slide: JC Niebles

Transformation = Matrix Multiplication

Scale $$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$	Flip across y $$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
Rotate $$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$	Flip across origin $$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
Shear $$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$	Identity $$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D Translation with homogeneous coordinates



$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \rightarrow \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix} p = Tp$$

Slide: JC Niebles

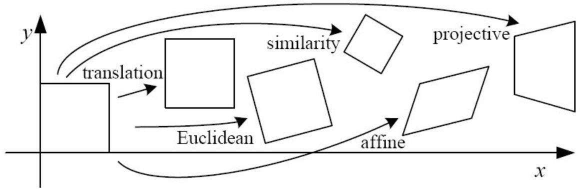
Euclidean transformations: rotation + translation

Euclidean (rigid):
rotation + translation

SE(2): Special Euclidean group
Important in robotics:
describes poses on plane

$$\begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

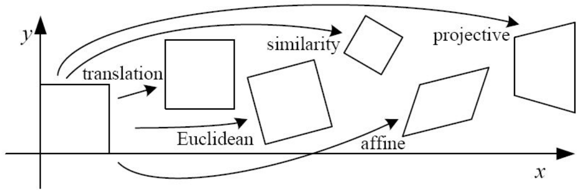
How many degrees of freedom?



Similarity = Euclidean + scaling equally in x and y

Similarity:
Scaling
+ rotation
+ translation

$$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



2D Transformations with homogeneous coordinates

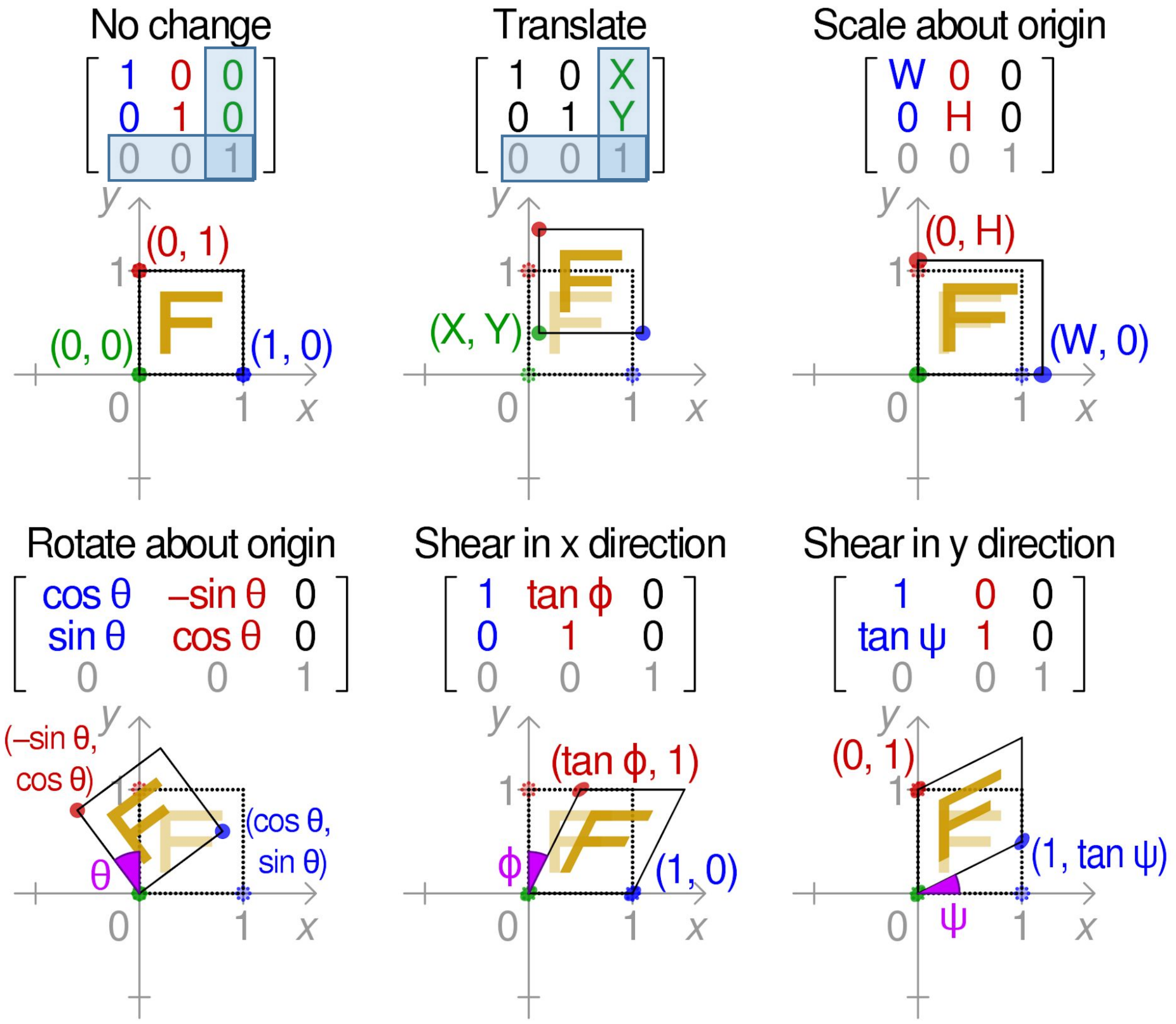


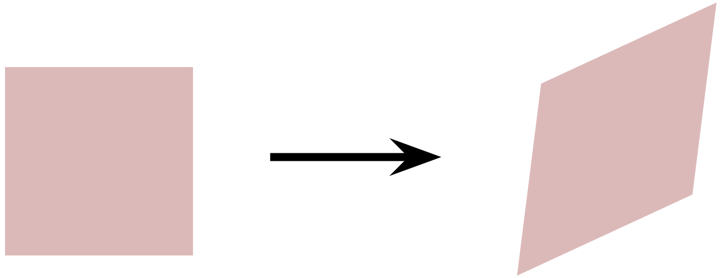
Figure: Wikipedia

Affine transformation = similarity + no restrictions on scaling

Properties of affine transformations:

- arbitrary 6 Degrees Of Freedom
- lines map to lines
- parallel lines map to parallel lines
- ratios are preserved

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$



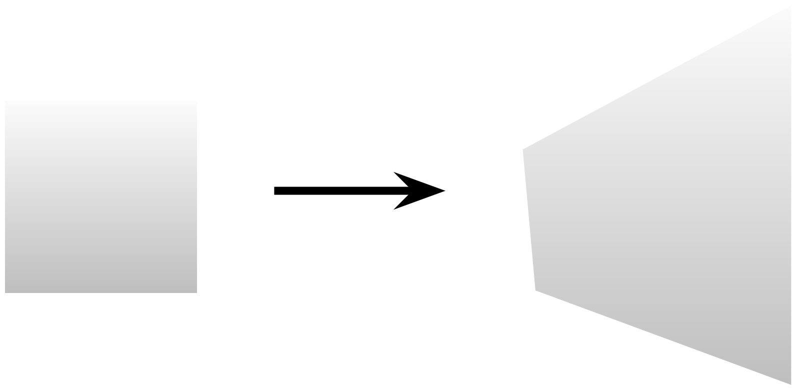
Source: K. Kitani

Projective transformation (homography)

Properties of projective transformations:

- 8 degrees of freedom
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$



Source: K. Kitani

Composing Transformations

- Transformations = Matrices => Composition by Multiplication!

$$p' = R_2 R_1 S p$$

In the example above, the result is equivalent to

$$p' = R_2(R_1(Sp))$$

Equivalent to multiply the matrices into single transformation matrix:

$$p' = (R_2 R_1 S)p$$

Order Matters! Transformations from ***right to left***.

Scaling & Translating != Translating & Scaling

- $$p'' = TSp = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$p''' = STp = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Scaling + Rotation + Translation

p'= (T R S) p

$$p' = TRSp = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} RS & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is the form of the general-purpose transformation matrix

3D Transforms = Matrix Multiplication

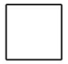
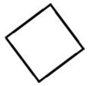
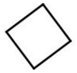
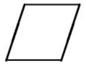
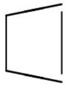
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4\times 4}$	15	straight lines	

Table 2.2 *Hierarchy of 3D coordinate transformations. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 3 $\times$ 4 matrices are extended with a fourth* $[\mathbf{0}^T \;\; 1]$ *row to form a full 4 $\times$ 4 matrix for homogeneous coordinate transformations. The mnemonic icons are drawn in 2D but are meant to suggest transformations occurring in a full 3D cube.*

Figure: R. Szeliski

Today’s agenda

- How biological vision understands geometry
- Brief history of geometric vision
- Geometric transformations
- Pinhole camera
- The Pinhole camera transformation

Reference: Szeliski 2.1, 2.2.3, 7.4

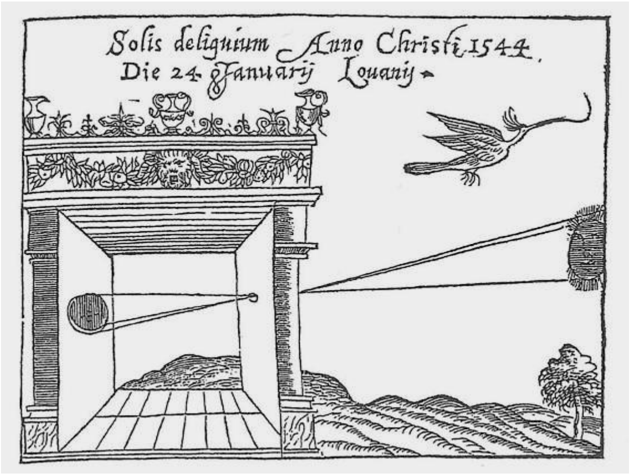
Reminder: Camera Obscura

- 5th century BC: principles of pinhole camera, a.k.a. camera obscura
    - China: 5th century BC
    - Greece: 4th century BC
    - Egypt: 11th century
    - Throughout Europe: from 11th century onwards

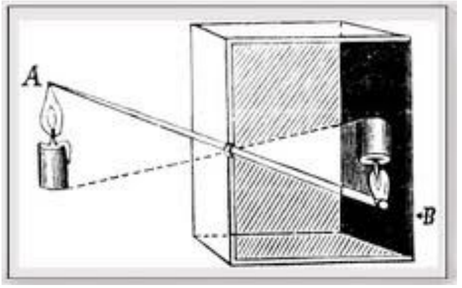
First mention …

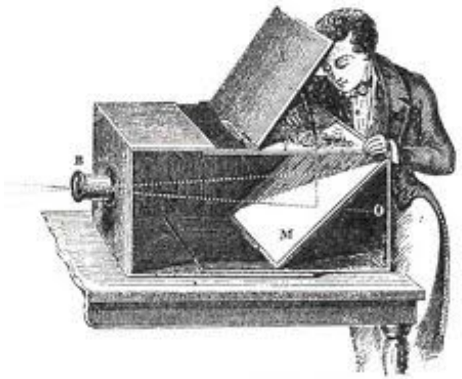
Chinese philosopher Mozi
(470 to 390 BC)

First camera?



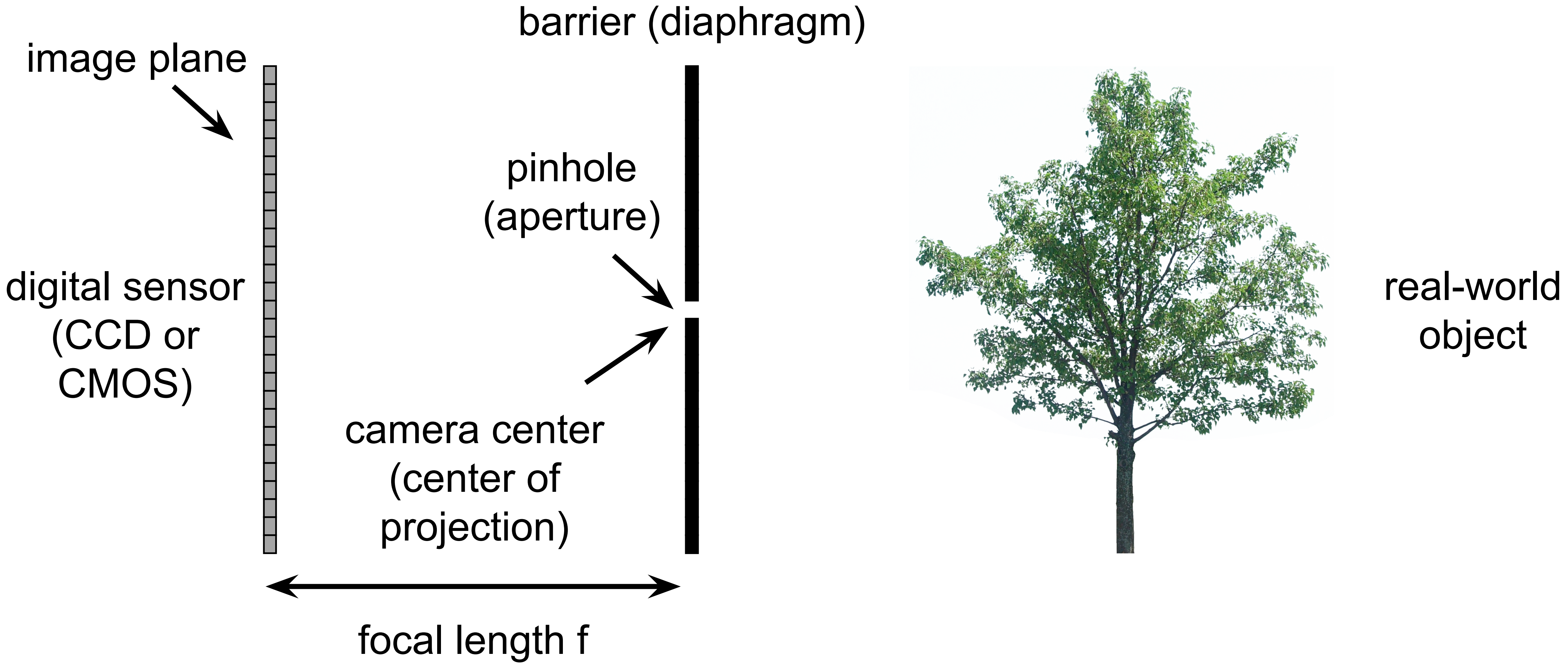
Greek philosopher Aristotle
(384 to 322 BC)



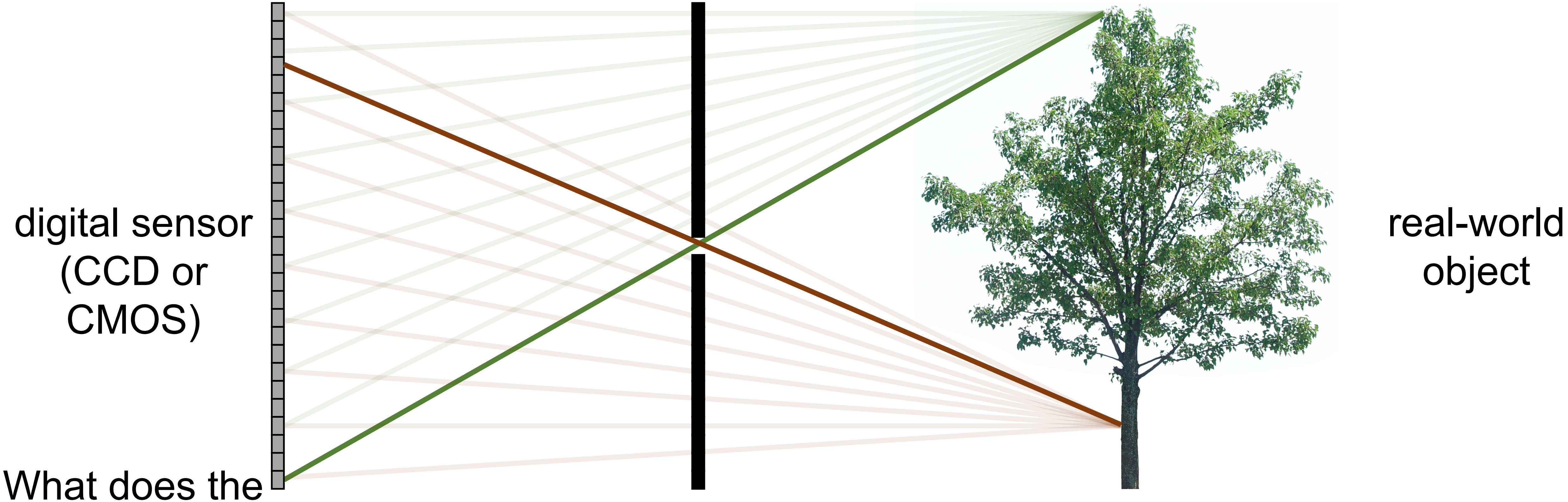


Source: P. Sturm

Pinhole imaging



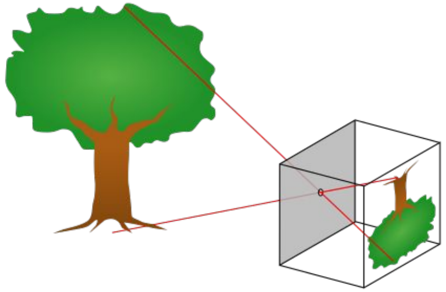
Pinhole imaging

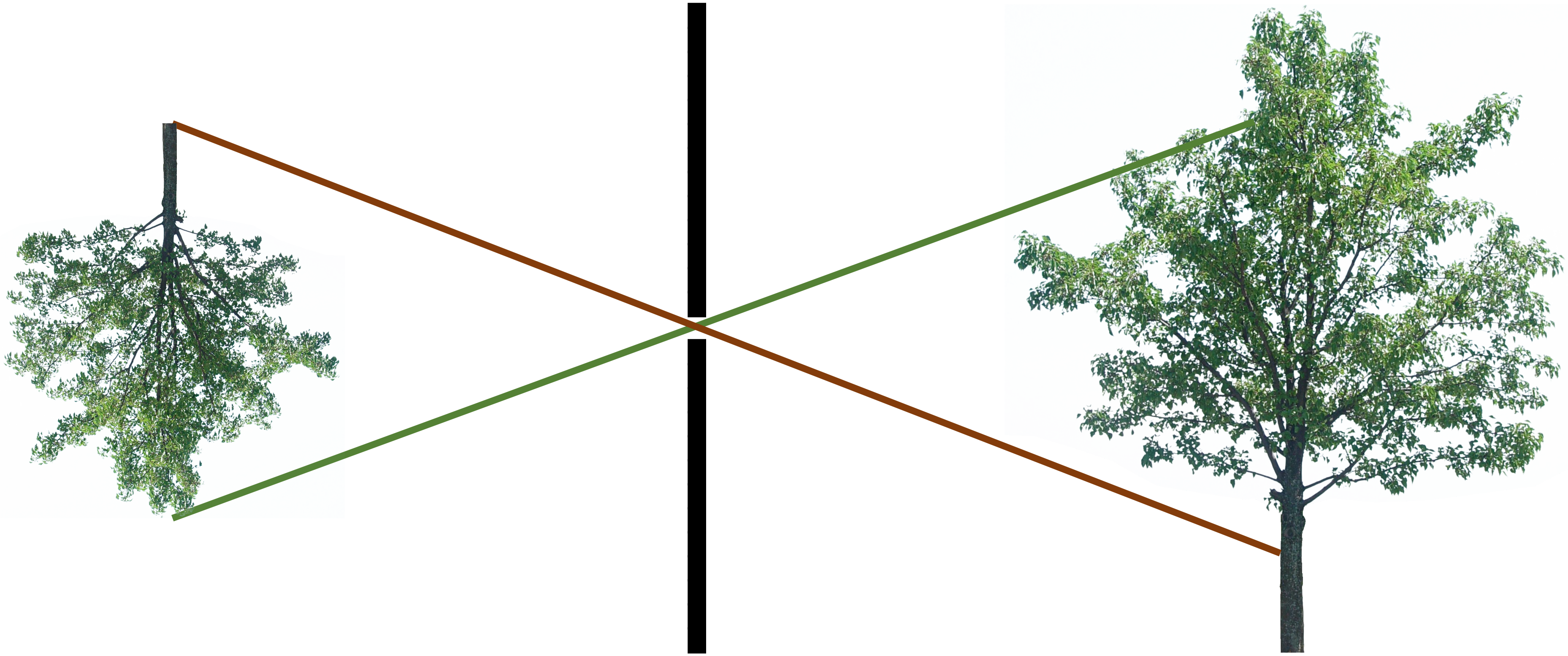


What does the image on the sensor look like?

Each scene point contributes to only one sensor pixel

Pinhole imaging

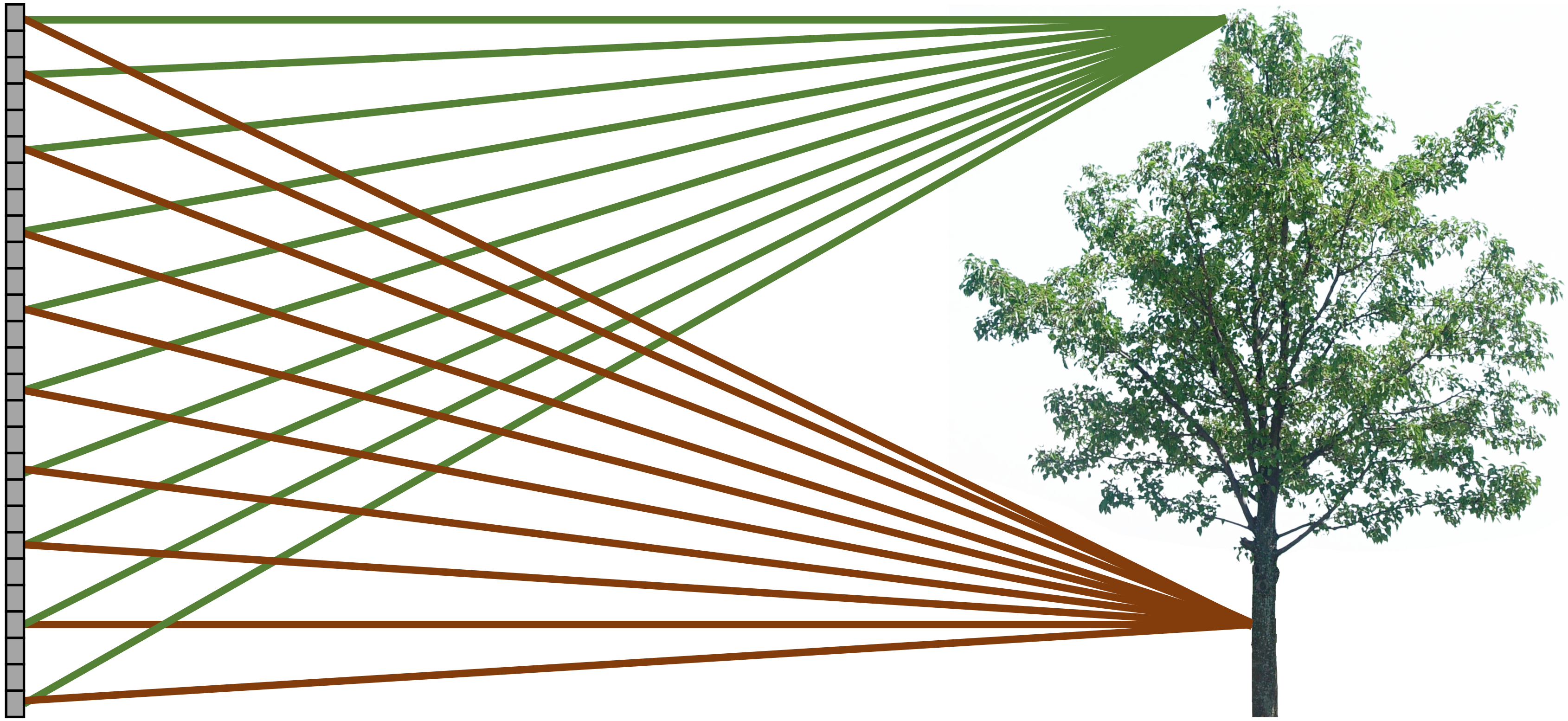




real-world
object

copy of real-world object
(inverted and scaled)

Bare-sensor imaging (without a pinhole camera)



digital sensor (CCD or CMOS)

real-world object

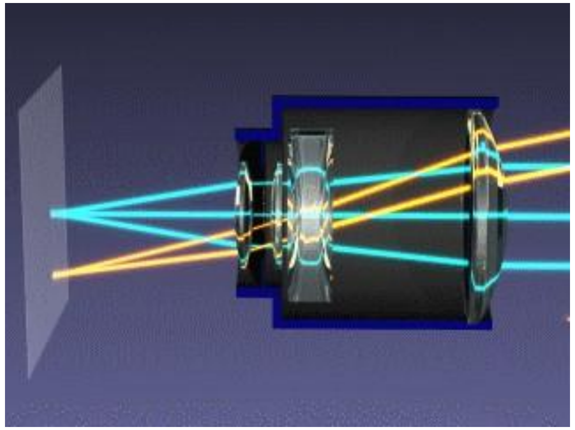
What does the image on the sensor look like?

All scene points contribute to all sensor pixels

Bare-sensor imaging (without a pinhole camera)

All scene points contribute to all sensor pixels

Cameras & Lenses



- Focal length determines the magnification of the image projected onto the image plane.
- Aperture determines the light intensity of that image pixels.

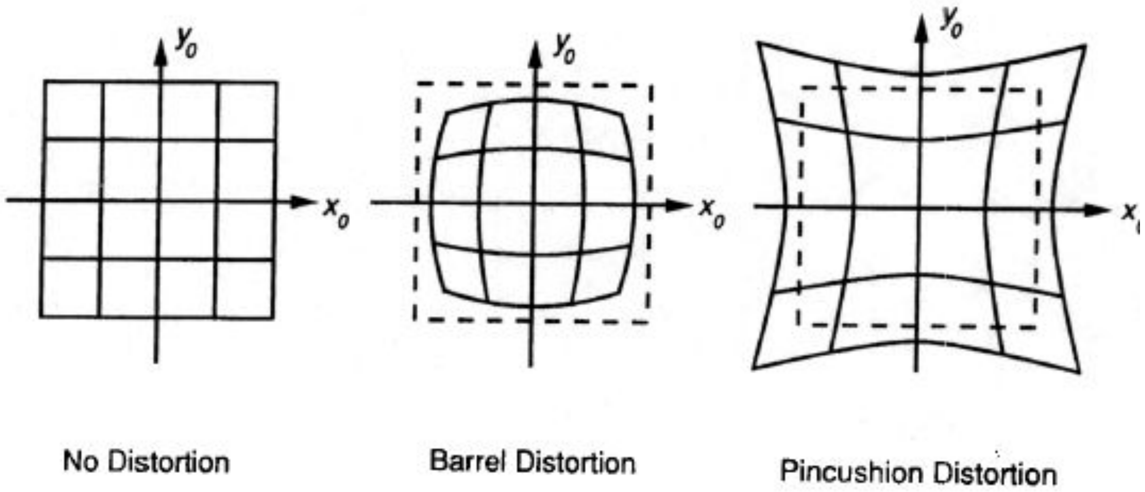
Source wikipedia

What’s going on there?

The buildings look distorted and bending towards each other.

Beyond Pinholes: Radial Distortion

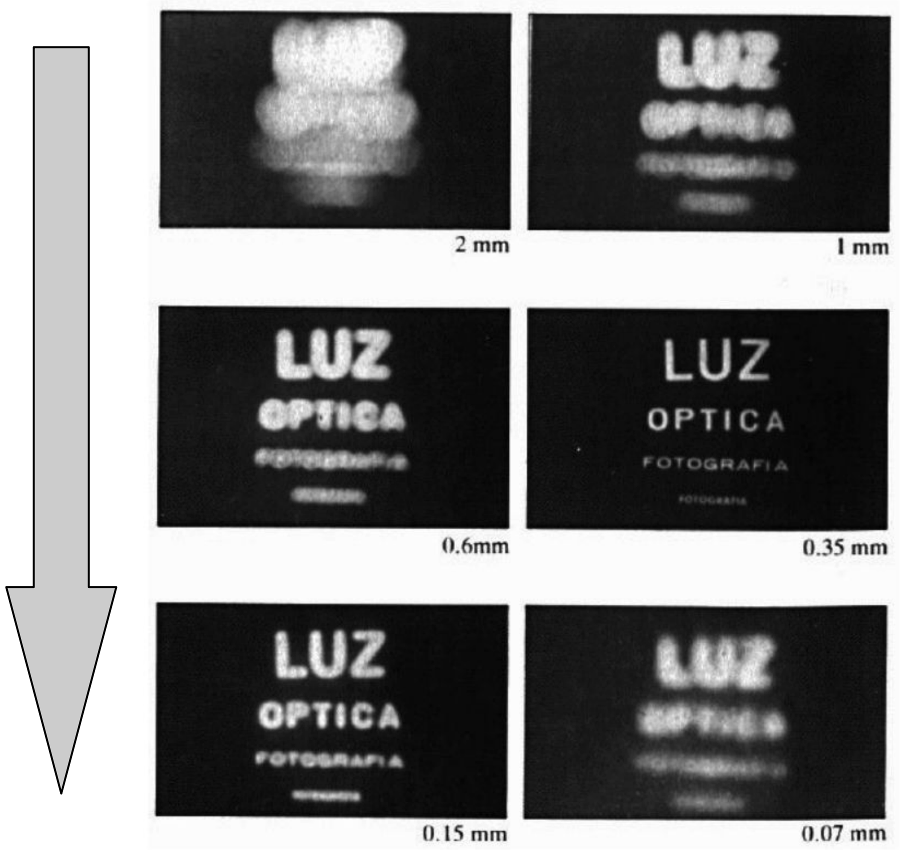
- Common in wide-angle lenses or for special applications (e.g., automotive)
- Creates a projective transformation
- Usually handled through solving for non-linear terms and then correcting image



Corrected Barrel Distortion

Cameras & Lenses

Decreasing aperture size



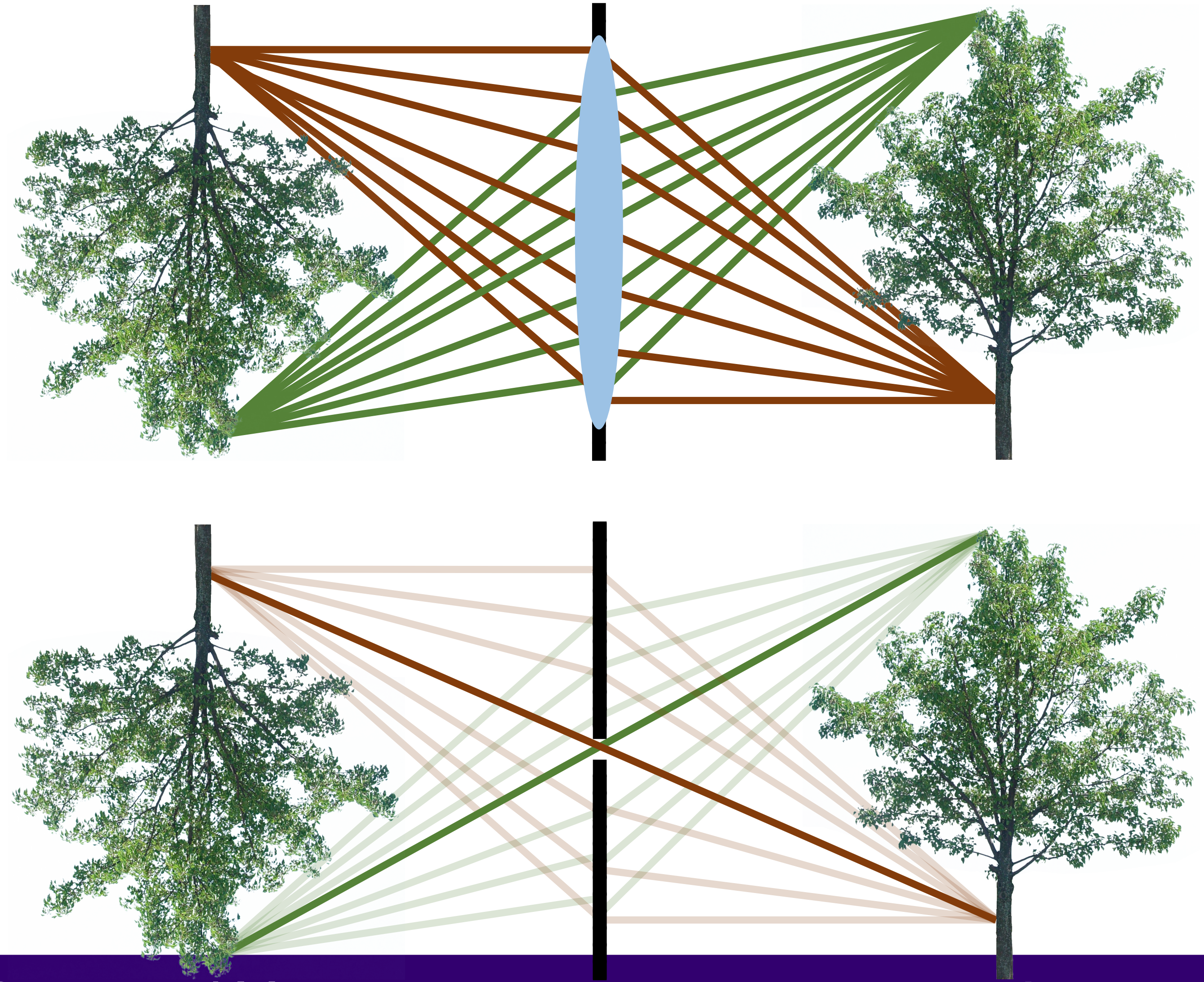
What happens with a smaller aperture?

- Less light passes through
- Less diffraction effect and clearer image

Pinhole is the miniscule aperture, resulting in the least amount of light and clearest image

Slide credit: J. Wu & S. Seitz

Describing both lens and pinhole cameras



For this course, we focus on the pinhole model.

- Similar to thin lens model in Physics: central rays are not deviated.
- Assumes lens camera in focus.
- Useful approximation but ignores important lens distortions.

Today’s agenda

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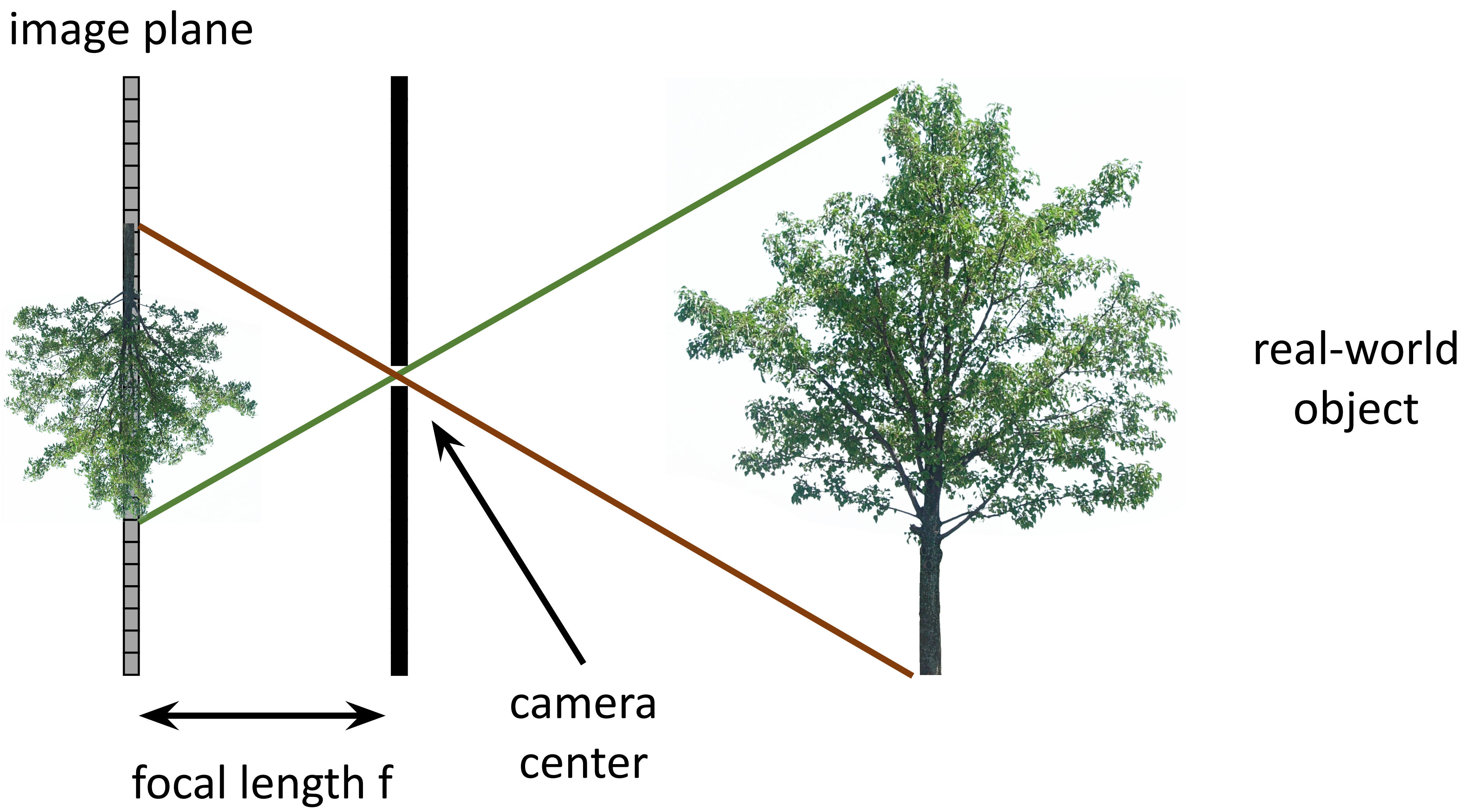
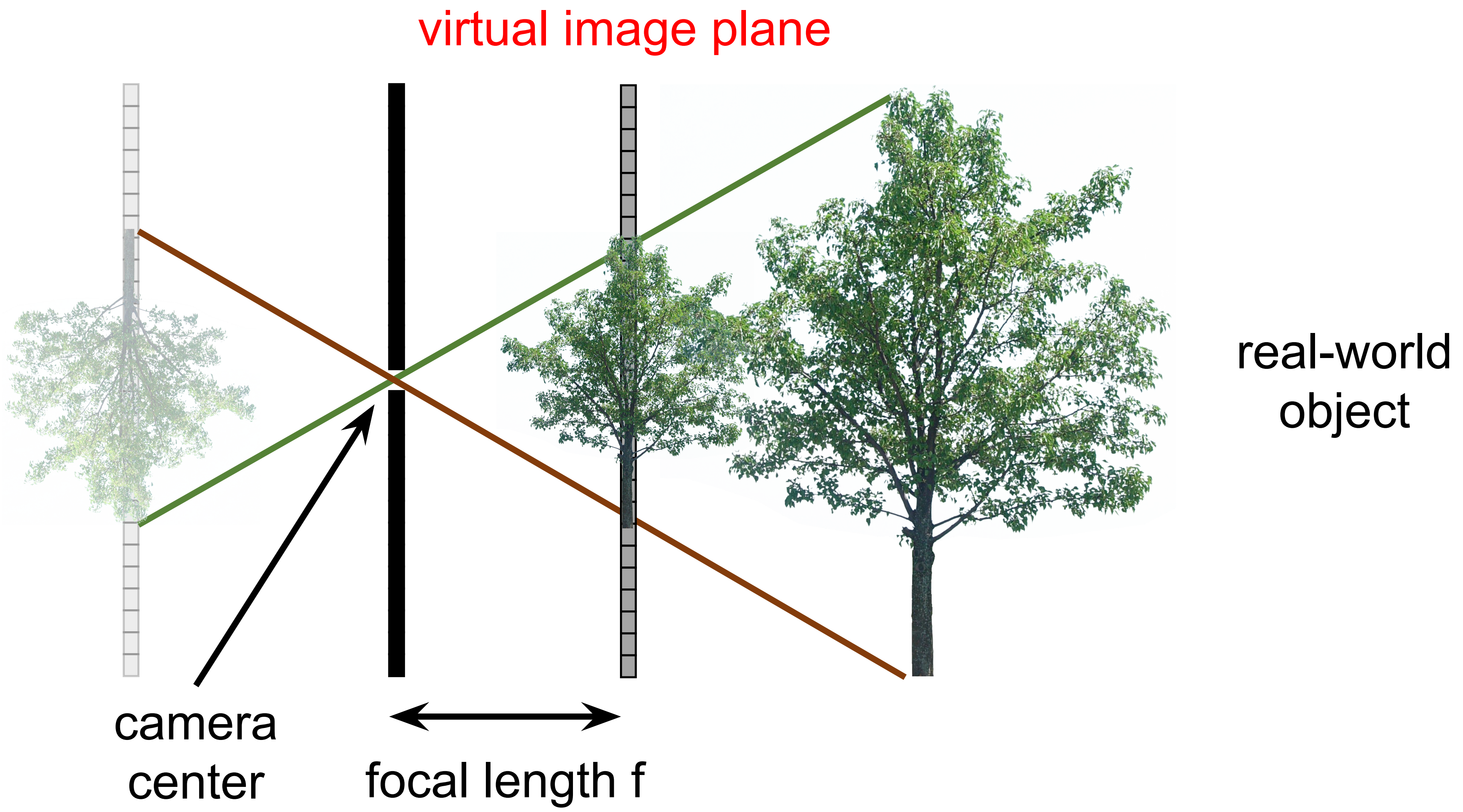
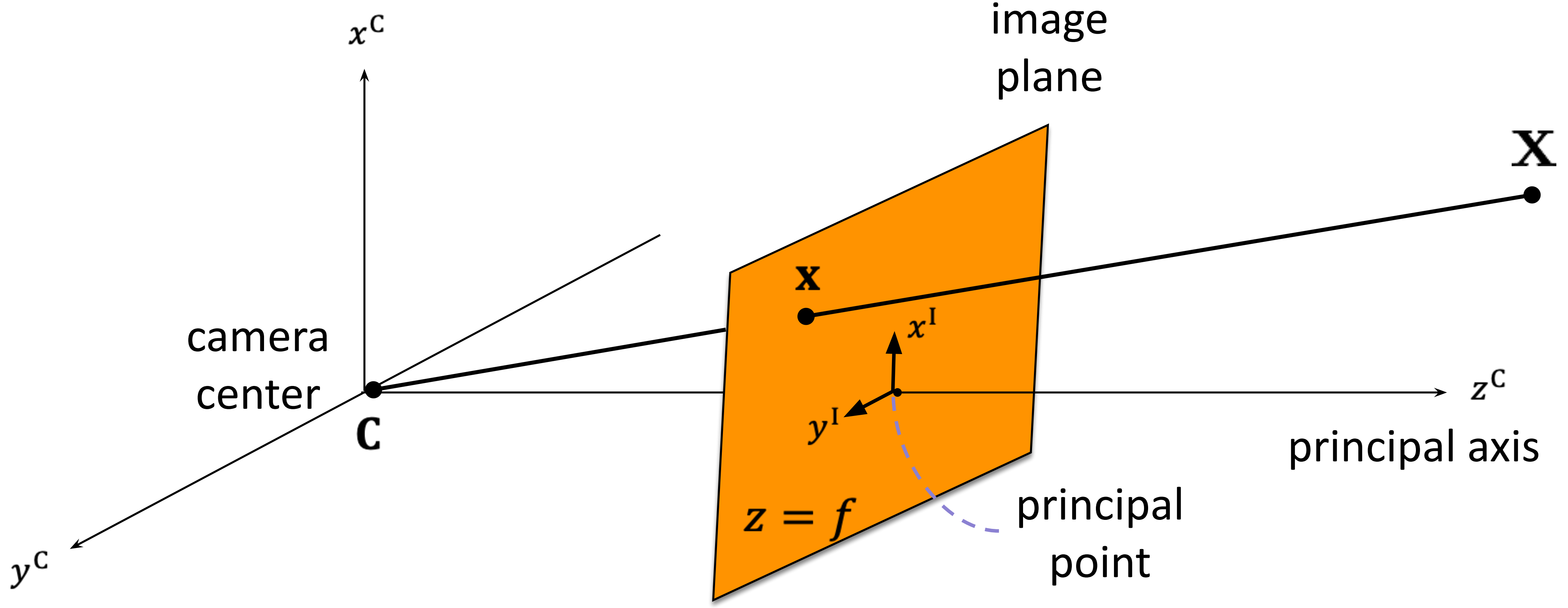
The pinhole camera



The (rearranged) pinhole camera



The (rearranged) pinhole camera



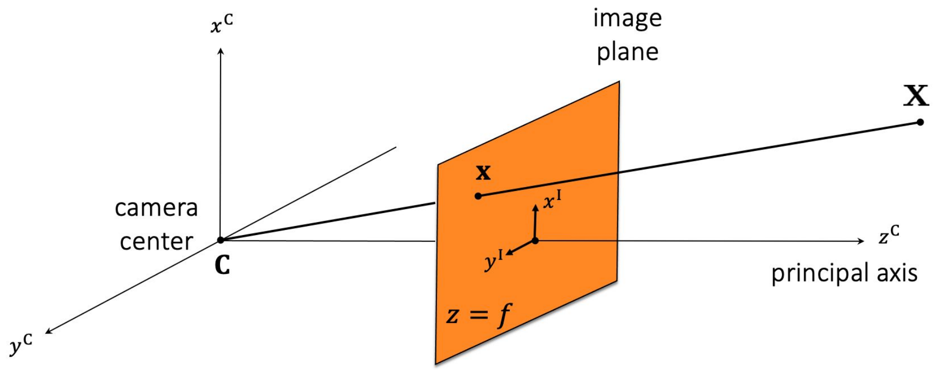
What is the transformation $\mathbf{x} = \mathbf{PX}$?

Pinhole Camera Matrix

Because all transformations are done using homogeneous coordinate system, all transformations are correct up to some scale lambda

$$\text{“}\mathbf{x} = \mathbf{PX}\text{”}$$

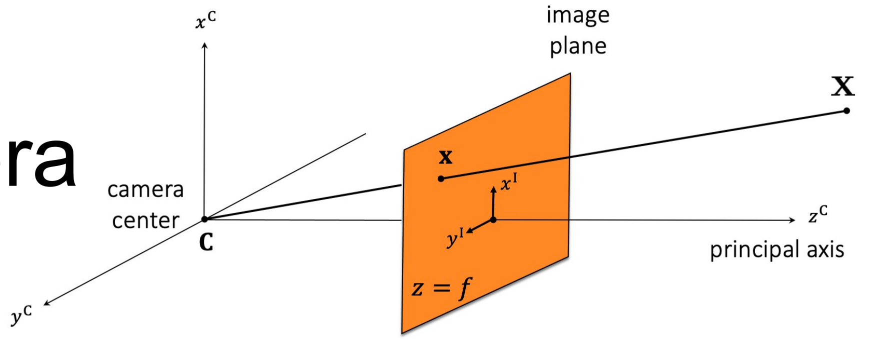
$$\lambda \tilde{\mathbf{x}}^{\mathbf{I}} = \mathbf{P}\tilde{\mathbf{X}}^{\mathbf{C}}$$

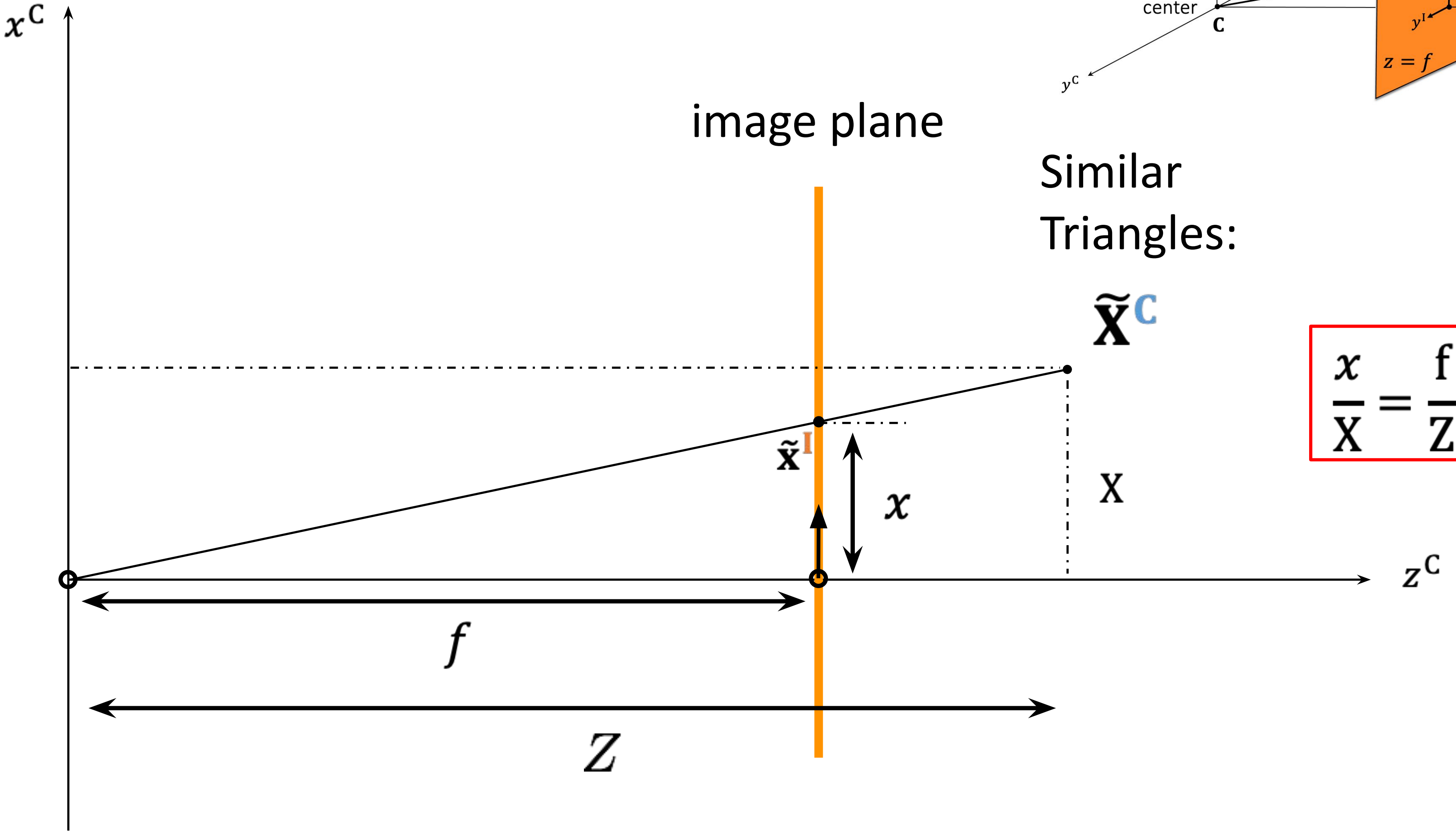


$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \sim \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

image coordinates 3 x 1 | camera matrix 3 x 4 | world (camera) coordinates 4 x 1

2D view of the (rearranged) pinhole camera





Similar Triangles:

$$\boxed{\frac{x}{X} = \frac{f}{Z}}$$

Pinhole Camera Matrix

Transformation from camera coordinates to image coordinates:

$$[X \quad Y \quad Z]^\top \mapsto [fX/Z \quad fY/Z]^\top$$

General camera model *in homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \sim \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Pinhole camera has a much simpler projection matrix (assume only scaling):

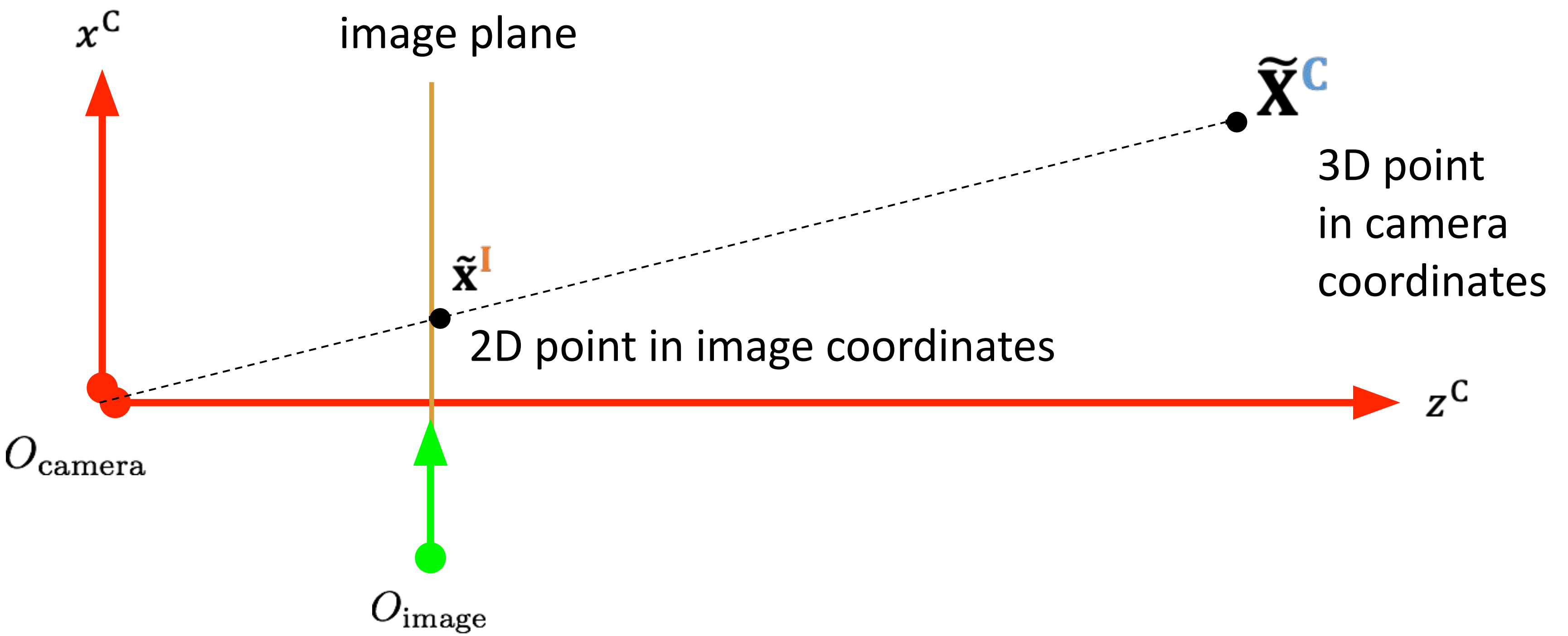
$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} fX \\ fY \\ Z \end{bmatrix} \to \begin{bmatrix} fX/Z \\ fY/Z \end{bmatrix}$$

Reminder: conversion from homogeneous coordinates

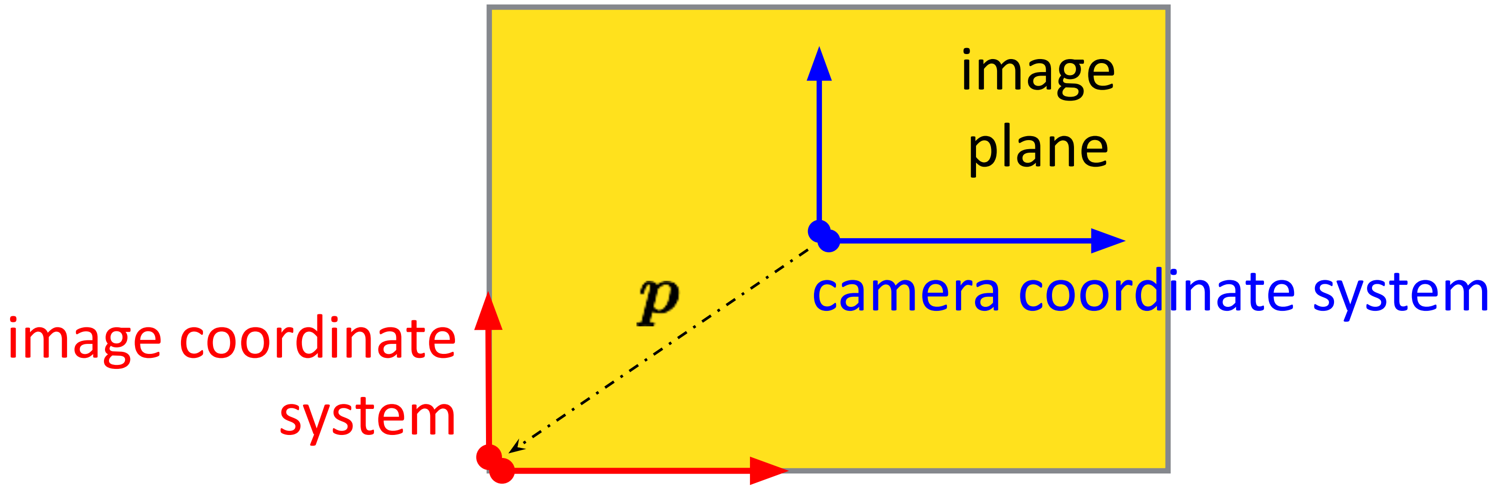
Generalizing the camera matrix

In general, the camera and image have *different* coordinate systems.



Generalizing the camera matrix

In particular, the camera origin and image origin may be different:

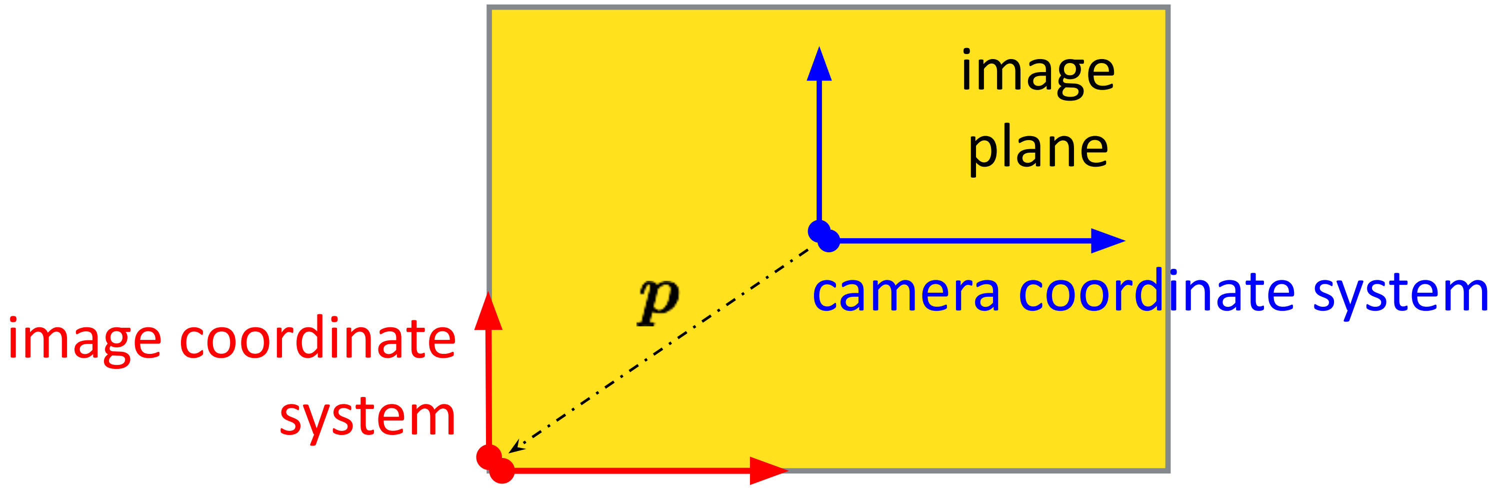


Q. How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Generalizing the camera matrix

In particular, the camera origin and image origin may be different:



Q. How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Translate the camera origin to image origin

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \left[\begin{array}{ccc} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\mathbf{P} = \left[\begin{array}{cccc} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(homogeneous) transformation from 2D to 2D, accounting for focal length *f* and origin translation

(homogeneous) perspective projection from 3D to 2D, assuming image plane at z = 1 and shared camera/image origin

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(homogeneous) transformation from 2D to 2D, accounting for focal length *f* and origin translation

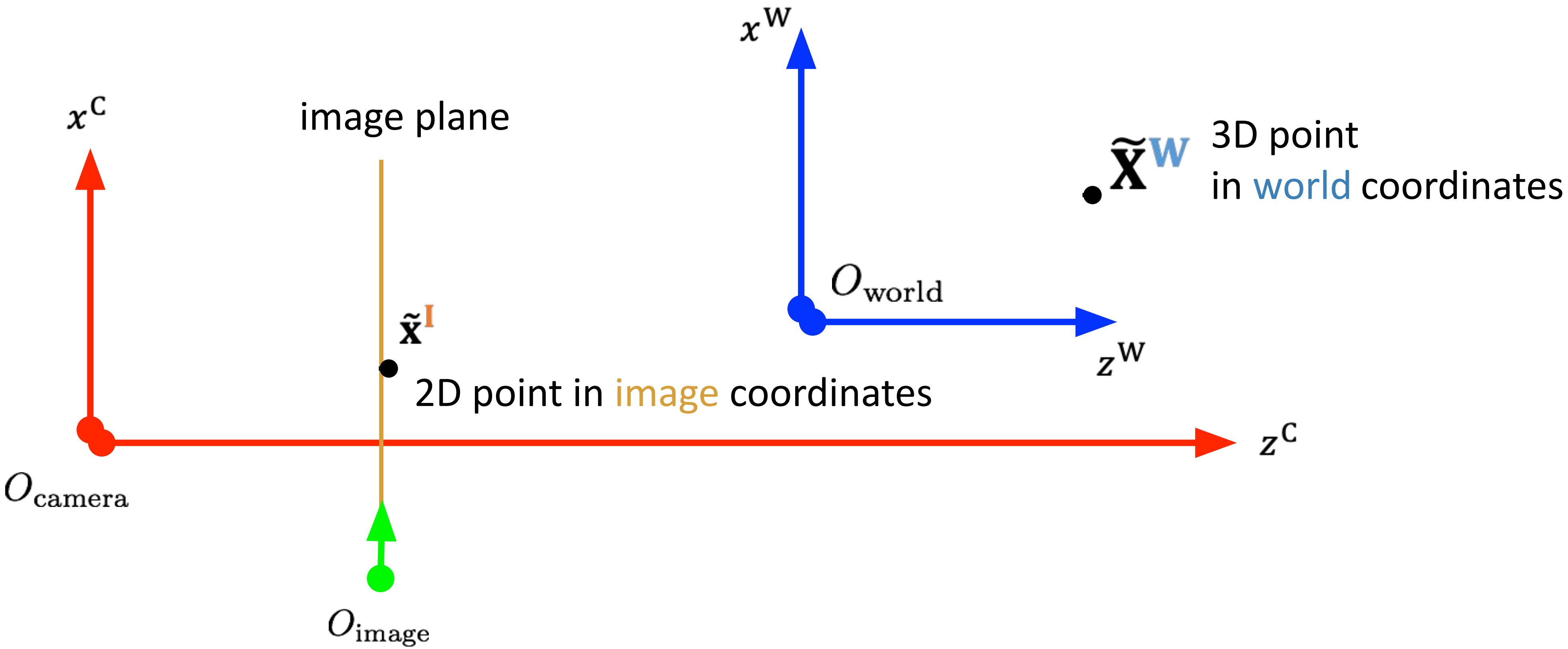
(homogeneous) perspective projection from 3D to 2D, assuming image plane at z = 1 and shared camera/image origin

Also written as: $\mathbf{P} = \mathbf{K}[\mathbf{I}|\mathbf{0}]$ where $\mathbf{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$

K is called the camera intrinsics

Generalizing the camera matrix

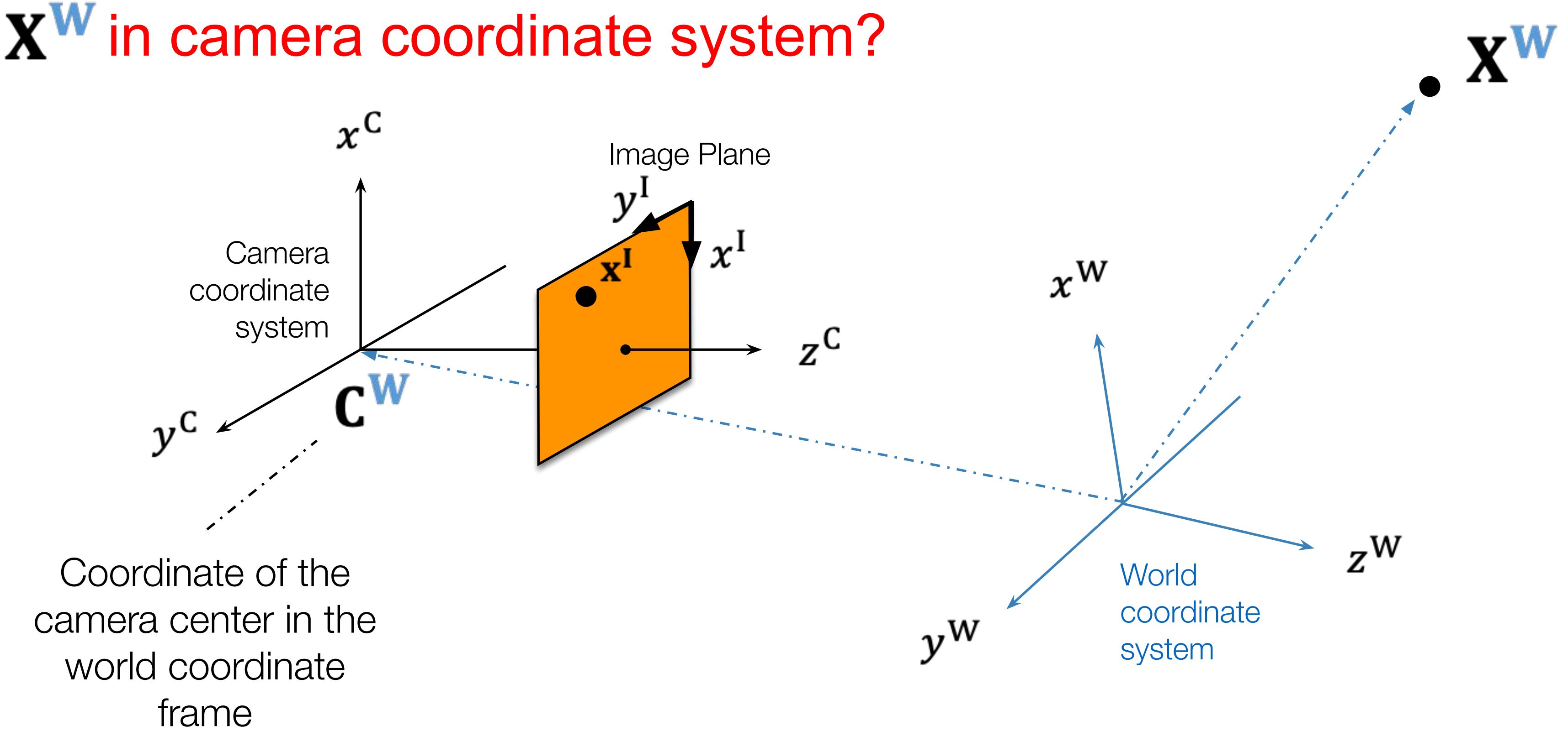
In general, there are 3 different coordinate systems (camera moves in the world).



World-to-camera coordinate transformation

Let's assume camera is at location $\mathbf{C}^{\mathrm{W}}$ in world coordinate system

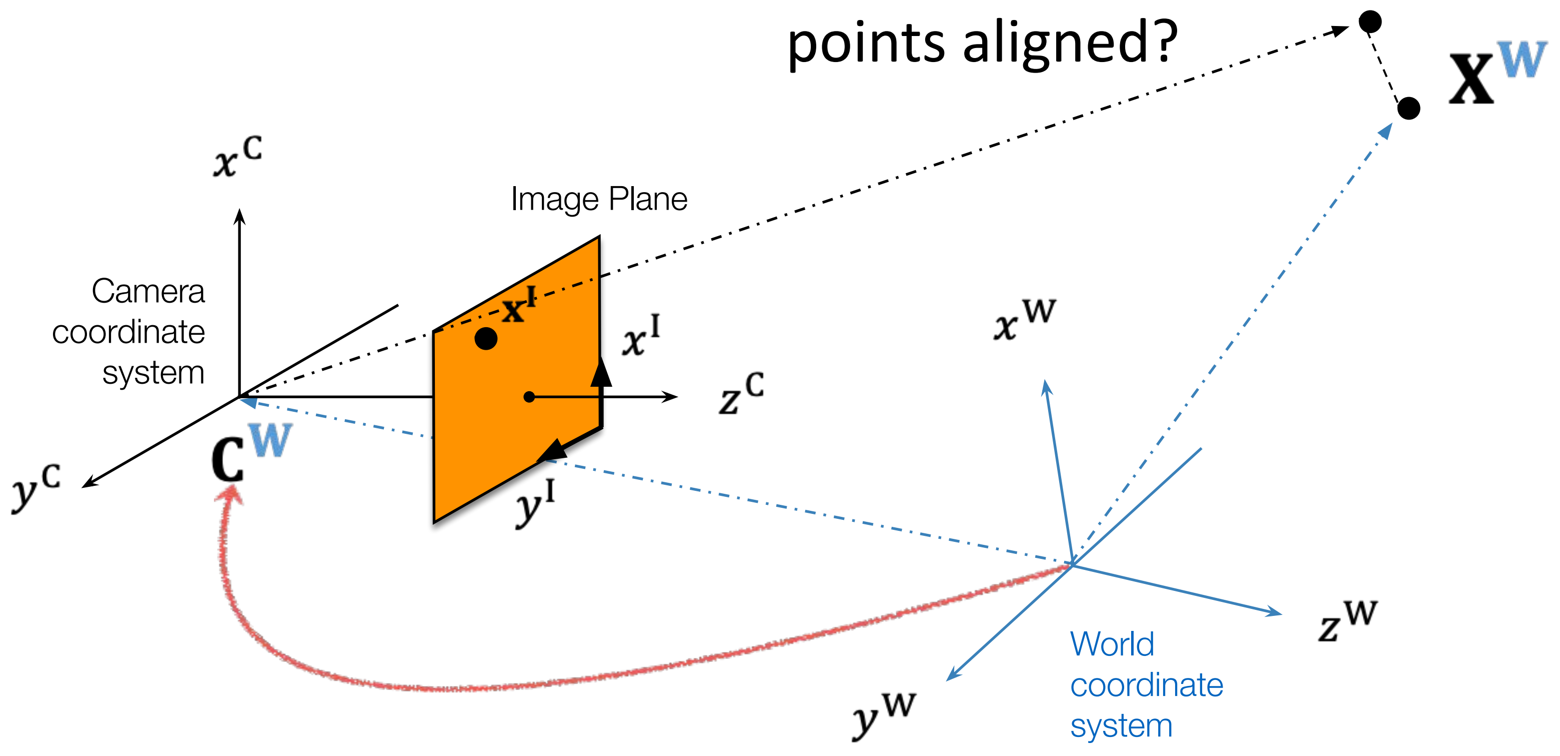
Q. What is $\mathbf{X}^{\mathrm{W}}$ in camera coordinate system?



Note: heterogeneous coordinates for now

World-to-camera coordinate transformation

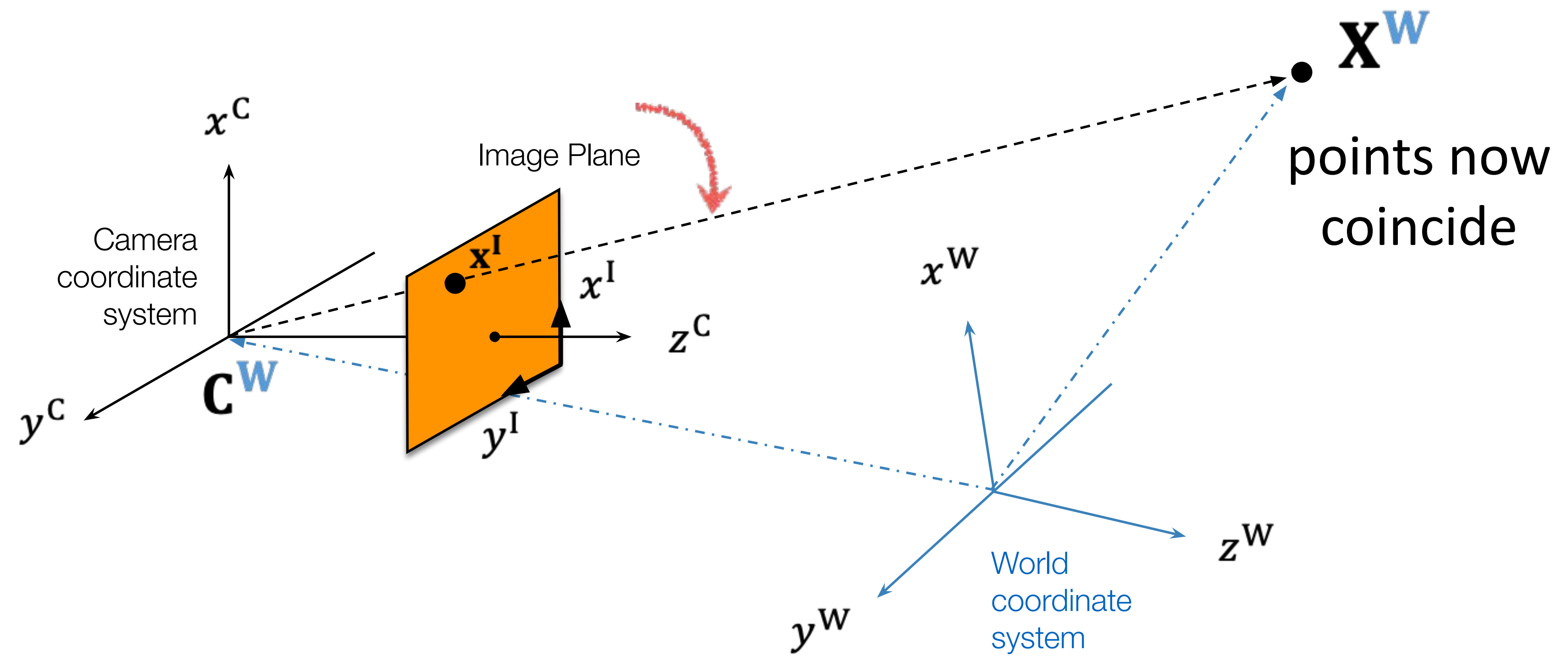
Why aren't the points aligned?



$$\mathbf{X}^{\mathrm{W}} - \mathbf{C}^{\mathrm{W}}$$

translate

World-to-camera coordinate transformation



$$\mathbf{R}\,(\mathbf{X}^{\mathrm{W}} - \mathbf{C}^{\mathrm{W}})$$

rotate translate

Coordinate system transformation

In *heterogeneous* coordinates, we have:

$$\mathbf{X}^{\mathbf{C}} = \mathbf{R}\left(\mathbf{X}^{\mathbf{W}} - \mathbf{C}^{\mathbf{W}}\right)$$

Q. How do we write this transformation in homogeneous coordinates?

Coordinate system transformation

In *heterogeneous* coordinates, we have:

$$\mathbf{X}^{\mathbf{C}} = \mathbf{R}\,(\mathbf{X}^{\mathbf{W}} - \mathbf{C}^{\mathbf{W}})$$

Q. How do we write this transformation in homogeneous coordinates?

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

or

$$\widetilde{\mathbf{X}}^{\mathbf{C}} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\mathbf{C}^{\mathbf{W}} \\ \mathbf{0} & 1 \end{bmatrix} \widetilde{\mathbf{X}}^{\mathbf{W}}$$

Let's update our camera transformation

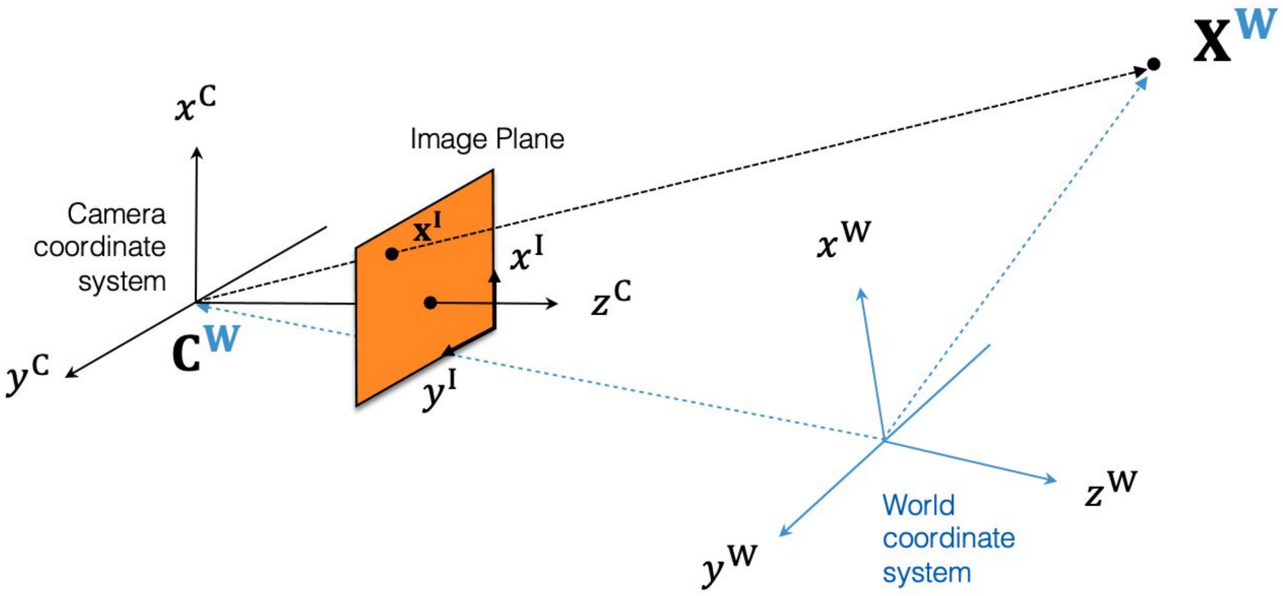
The previous camera transformation we calculated is for homogeneous 3D coordinates in camera coordinate system:

(omitting ~ for simplicity: everything in homogeneous coordinates)

$$\mathbf{x}^{\mathbf{I}} \sim \mathbf{K}[\mathbf{I}|\mathbf{0}]\mathbf{X}^{\mathbf{C}}$$

We also just derived:

$$\mathbf{X}^{\mathbf{C}} = \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X}^{\mathbf{W}}$$



Putting it all together

We can write everything into a single projection: $\mathbf{x}^{\mathbf{I}} \sim \mathbf{K}[\mathbf{I}|\mathbf{0}] \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X}^{\mathbf{W}} = \mathbf{P}\mathbf{X}^{\mathbf{W}}$

The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{I} \quad | \quad \mathbf{0}] \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix}$$

intrinsic parameters (3 x 3): correspond to camera internals (image-to-image transformation)

perspective projection (3 x 4): maps 3D to 2D points (camera-to-image transformation)

extrinsic parameters (4 x 4): correspond to camera externals (world-to-camera transformation)

Putting it all together

We can write everything into a single projection: $\mathbf{x}^{\mathbf{I}} \sim \mathbf{P}\mathbf{X}^{\mathbf{W}}$

The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \;\middle|\; \overbrace{-\mathbf{RC}}^{\mathbf{t}} \right]$$

intrinsic parameters (3 x 3):
correspond to camera internals
(sensor not at f = 1 and origin shift)

extrinsic parameters (3 x 4):
correspond to camera externals
(world-to-image transformation)

General pinhole camera matrix

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$ where $$\mathbf{t} = -\mathbf{RC}$$

General pinhole camera matrix

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$ where $$\mathbf{t} = -\mathbf{RC}$$

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc:c} r_1 & r_2 & r_3 & t_1 \\ r_4 & r_5 & r_6 & t_2 \\ r_7 & r_8 & r_9 & t_3 \end{array}\right]$$

intrinsic parameters

extrinsic parameters

$$\mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

3D rotation

3D translation

More general camera matrices

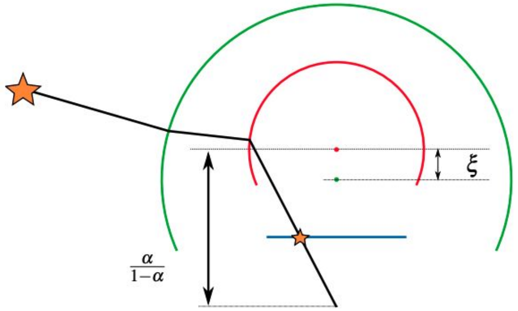
Non-square pixels, sensor may be skewed
(causing focal length to be different along x and y).

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \;\vdots\; -\mathbf{RC} \right]$$

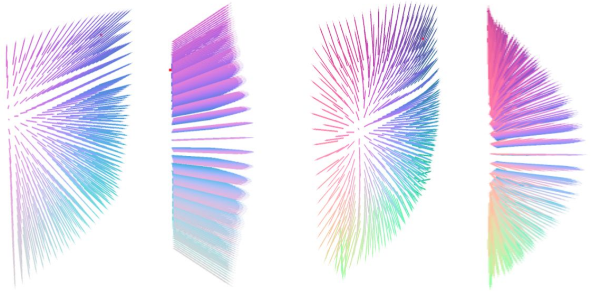
Q. How many degrees of freedom?

Camera Models: Still an Active Area

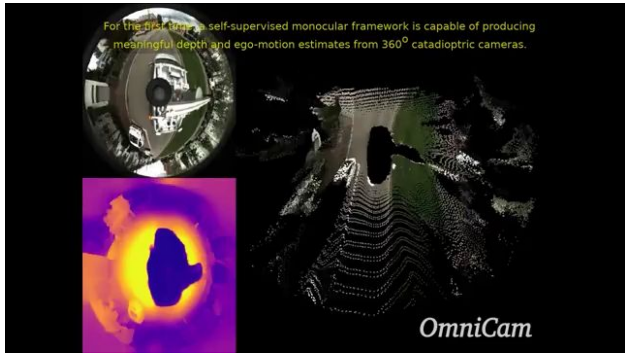
Is everybody only using a 2400 years old model?



- More complex cameras: pinhole + distortion, fisheye catadioptric, dashcams, underwater…
- The Double Sphere Camera Model, Usenko *et al* ECCV 2018 (commonly used in robotics, like in our ICRA’22 paper)
- Learning Camera Models Neural Ray Surfaces, Vasiljevic *et al,* 3DV 2020



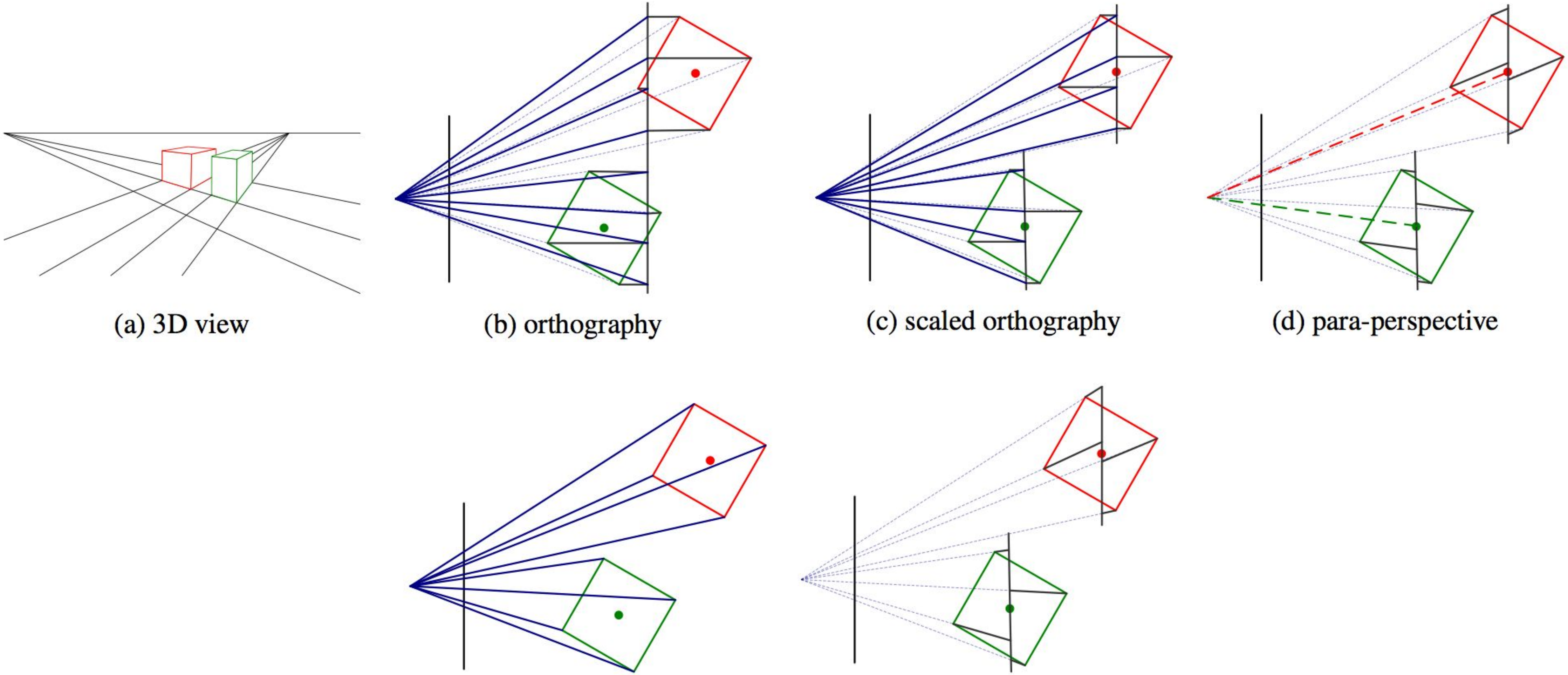
(a) Pinhole (KITTI) (b) Catadioptric (OmniCam)



Next time

Camera calibration

Many other types of cameras


(a) 3D view (b) orthography (c) scaled orthography (d) para-perspective

(e) perspective (f) object-centered

Figure: R. Szeliski