Lecture 14

Recognition and kNN

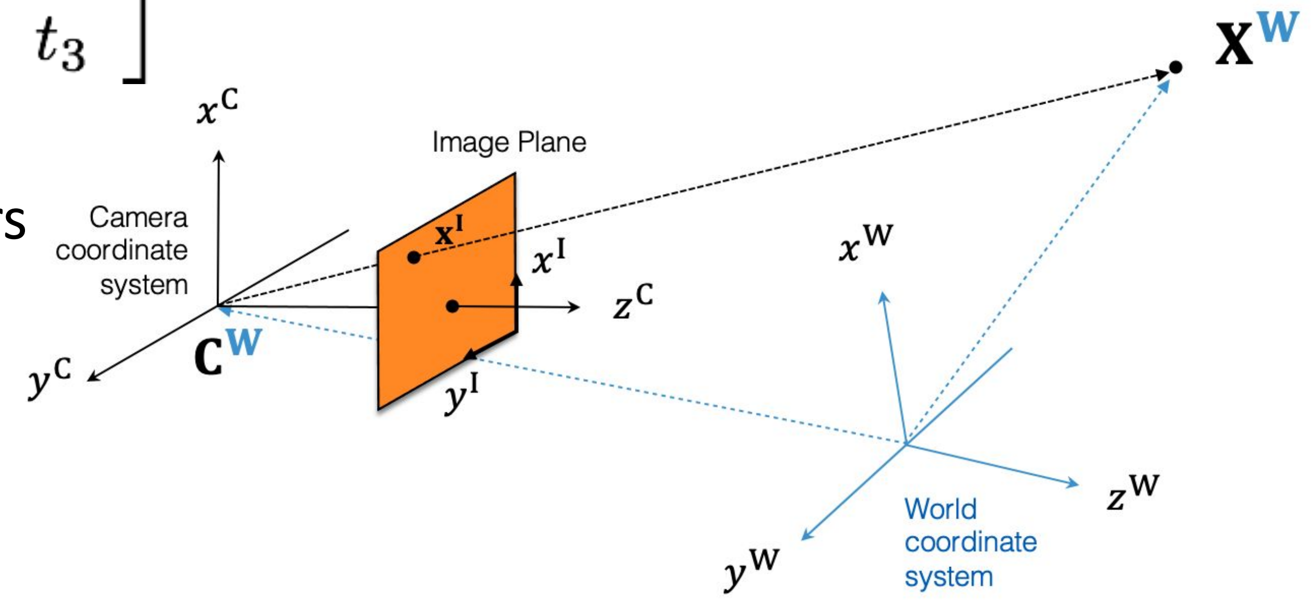
So far: General pinhole camera matrix

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}] \qquad \text{where} \qquad \mathbf{t} = -\mathbf{RC}$$

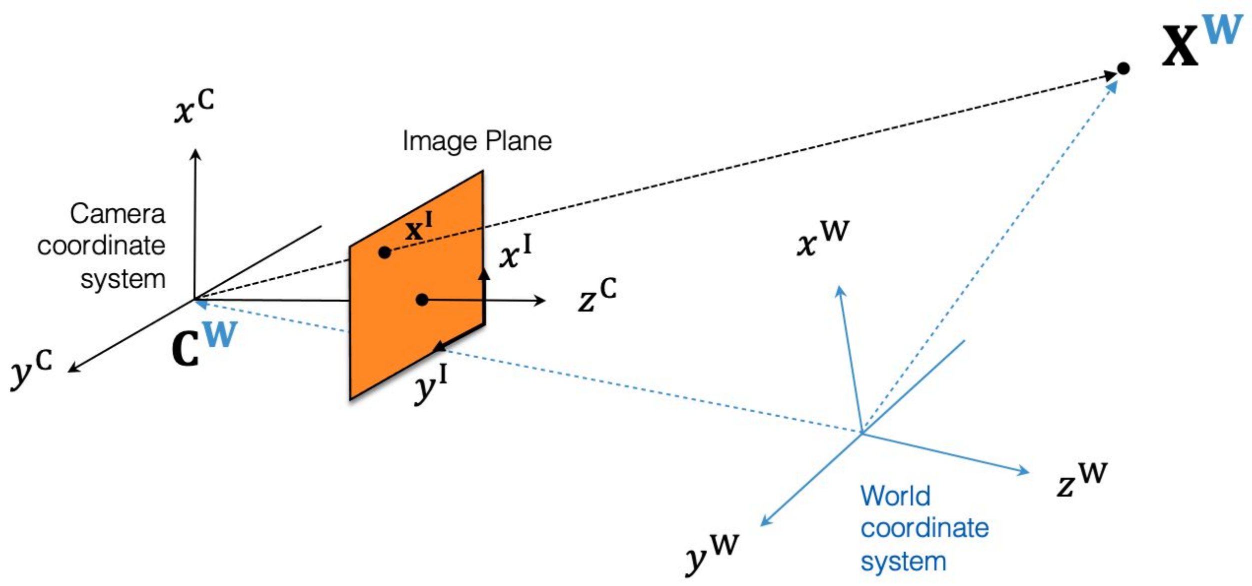
$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 & t_1 \\ r_4 & r_5 & r_6 & t_2 \\ r_7 & r_8 & r_9 & t_3 \end{bmatrix}$$

intrinsic parameters

extrinsic parameters



So far: The Pinhole Camera Model



$$\tilde{\mathbf{x}}^{\mathbf{I}} \sim \mathbf{P}\tilde{\mathbf{X}}^{\mathbf{W}} \qquad \mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{I} \mid \mathbf{0}] \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

intrinsic parameters **K** (3 x 3): correspond to camera internals (image-to-image transformation)

perspective projection (3 x 4): maps 3D to 2D points (camera-to-image transformation)

extrinsic parameters (4 x 4): correspond to camera externals (world-to-camera transformation)

So far: Solving for camera matrix via total least squares

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} ||\mathbf{A}\boldsymbol{x}||^2 \text{ subject to } ||\boldsymbol{x}||^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{X}_1^T & 0 & -x_1'\boldsymbol{X}_1^T \\ \mathbf{0} & \boldsymbol{X}_1^T & -y_1'\boldsymbol{X}_1^T \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^T & 0 & -x_N'\boldsymbol{X}_N^T \\ \mathbf{0} & \boldsymbol{X}_N^T & -y_N'\boldsymbol{X}_N^T \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

Equivalently, solution ***x*** is the Eigenvector corresponding to the smallest Eigenvalue of **A**

So far: Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{c|c} \bar{\mathbf{P}} & \begin{array}{c} p_4 \\ p_8 \\ p_{12} \end{array} \end{array} \right] \sim \mathbf{K}\,[\mathbf{R} \mid -\mathbf{RC}]$$

$\bar{\mathbf{P}}^\top \bar{\mathbf{P}} \sim \mathbf{K}^\top \mathbf{K}$ with $\mathbf{K}$ upper triangular p.d.

Obtain $\mathbf{K}$ by Cholesky decomposition of $\bar{\mathbf{P}}^\top \bar{\mathbf{P}} = \mathbf{L}\mathbf{L}^\top$

$$\mathbf{K} \sim \mathbf{L}^\top$$

$|\mathbf{R}| = 1 \Rightarrow \lambda = |\mathbf{K}^{-1}\bar{\mathbf{P}}|^{-1/3}$

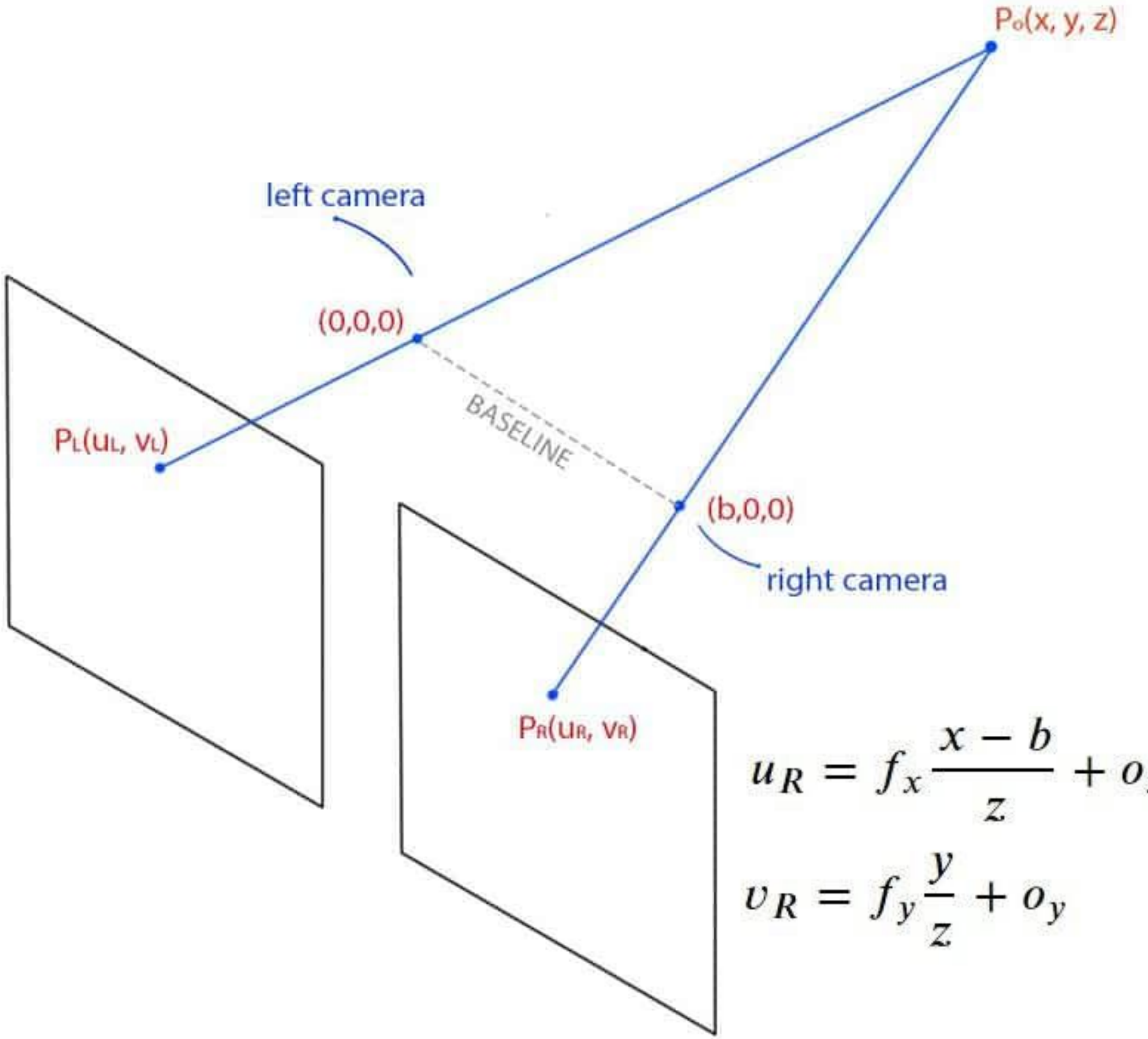
Once $\mathbf{K}$ is known, we can compute $\mathbf{R} = \lambda \mathbf{K}^{-1}\bar{\mathbf{P}}$

Finally, easy to know the camera center: $\mathbf{C} = -\lambda^{-1}\,\mathbf{R}^\top \mathbf{K}^{-1} [p_4 \;\; p_8 \;\; p_{12}]^\top$

So far: Estimating depth

$$u_L = f_x \frac{x}{z} + o_x$$

$$v_L = f_y \frac{y}{z} + o_y$$



$$u_R = f_x \frac{x - b}{z} + o_x$$

$$v_R = f_y \frac{y}{z} + o_y$$

$$\boxed{z = \frac{b f_x}{u_L - u_R}}$$

Today's agenda

- Introduction to recognition
- A object recognition pipeline
- Choosing the right features
- A training algorithm: KNN
- Testing an algorithm
- Challenges with kNN
- Dimensionality reduction
- Principal Component Analysis (PCA)

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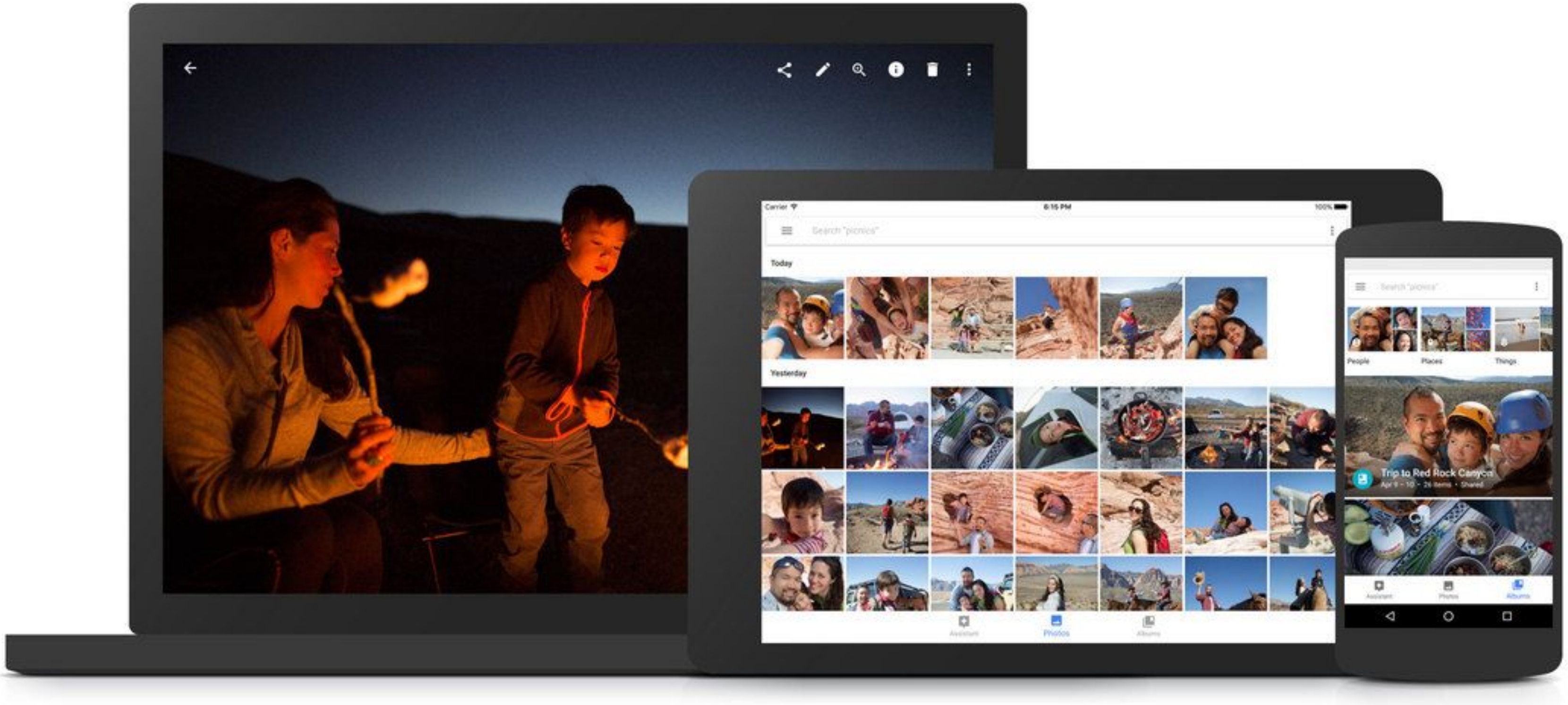
What do we mean by recognition?

Classification: Does this image contain a building? [yes/no]

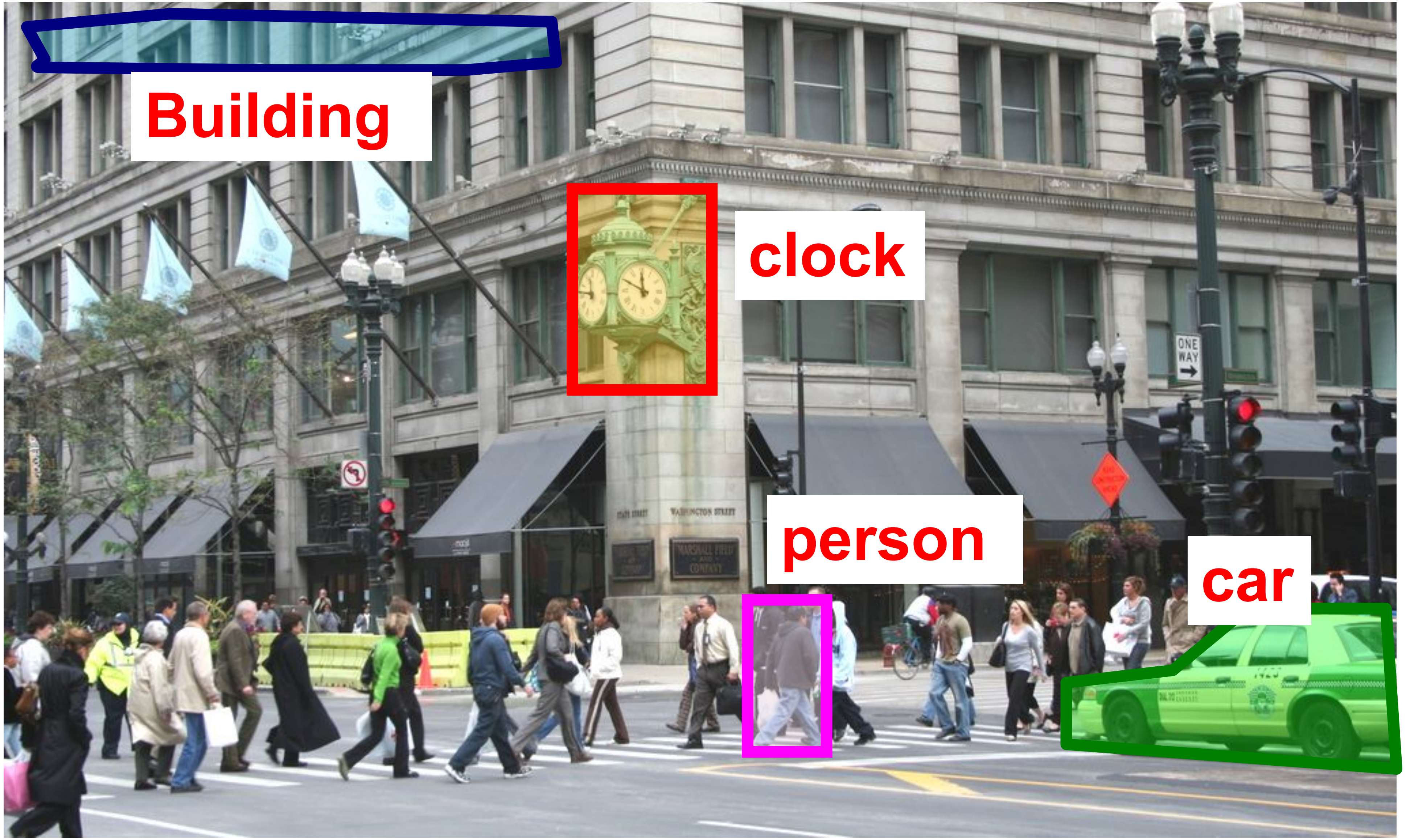
Classification: Is this an beach?

Applications: Image Search & Organizing photo collections



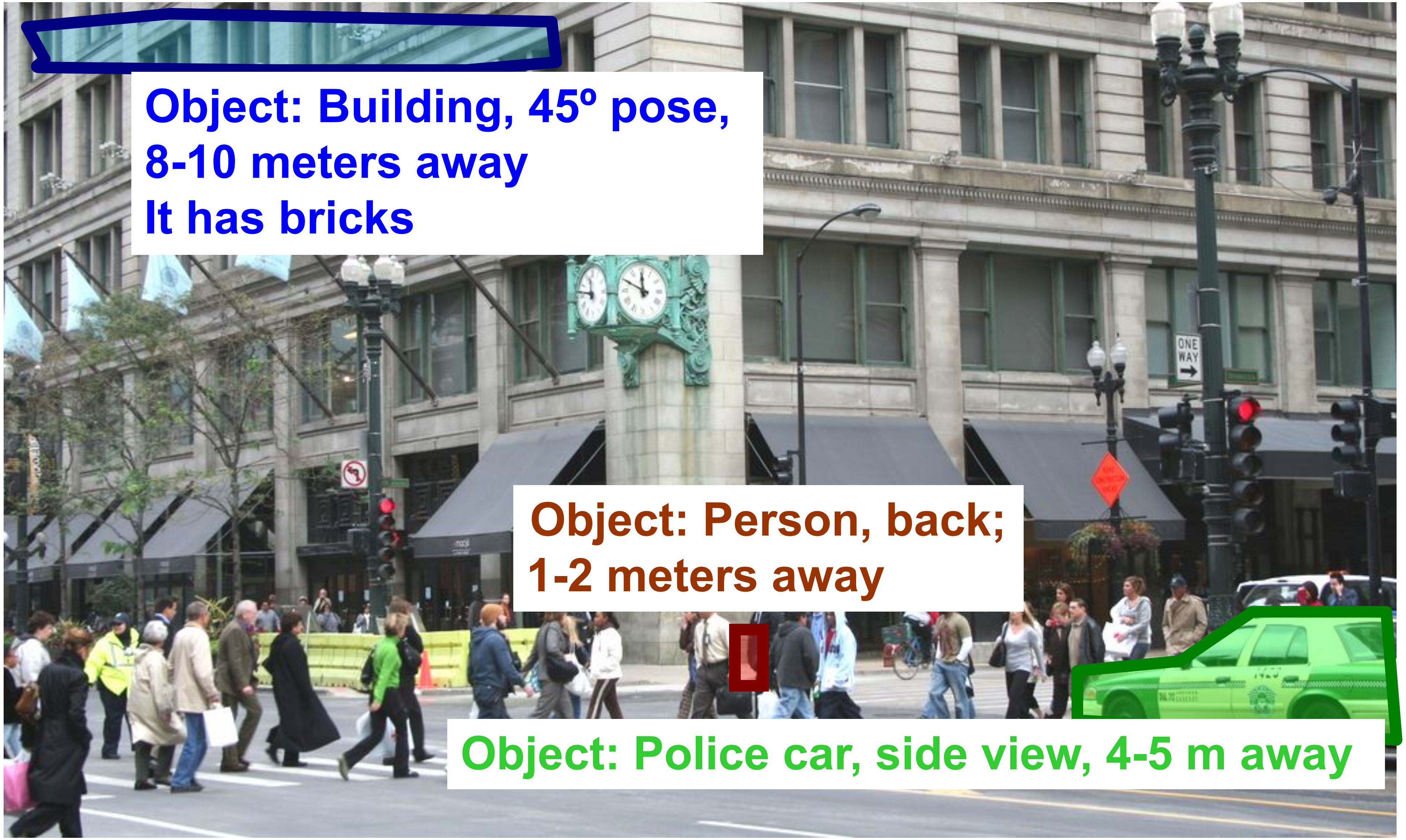
Detection: Does this image contain a car? [where?]

Detection: Which object does this image contain? [where?]



Detection: Accurate localization (segmentation)

Detection: Estimating object semantic & geometric attributes



Applications of computer vision

Computational photography

Assistive technologies

Surveillance

Security

Assistive driving

Levels of recognition: Category-level vs instance-level

Does this image contain the Chicago Macy's building?

Categorization vs Single instance recognition

We have seen a form of single instance categorization already: **Where is the crunchy nut?**

Applications of computer vision

Recognizing landmarks in mobile devices

Activity recognition: What are these people doing?

Visual Recognition

- Design algorithms that can:
  - Classify images or videos
  - Detect and localize objects
  - Estimate semantic and geometrical attributes
  - Classify human activities and events

Why is this challenging?

How many object categories are there?

Challenges: viewpoint variation

Michelangelo 1475-1564

Challenges: illumination

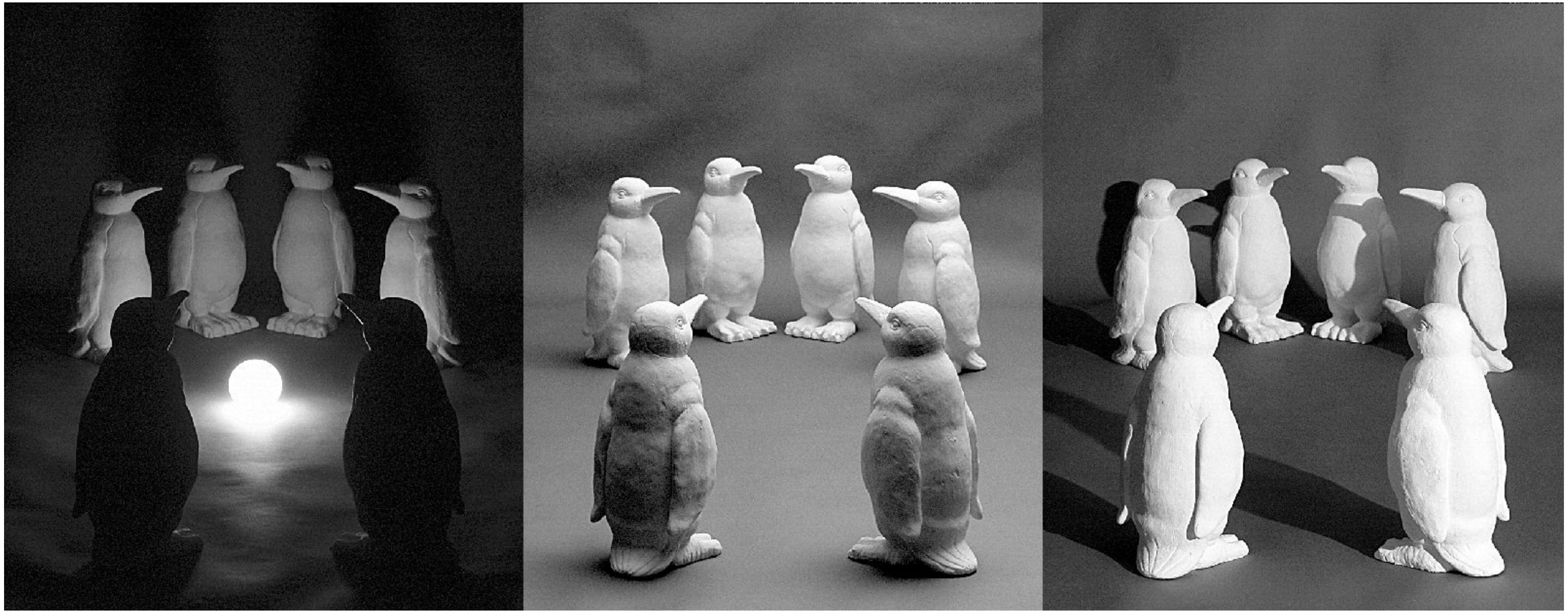
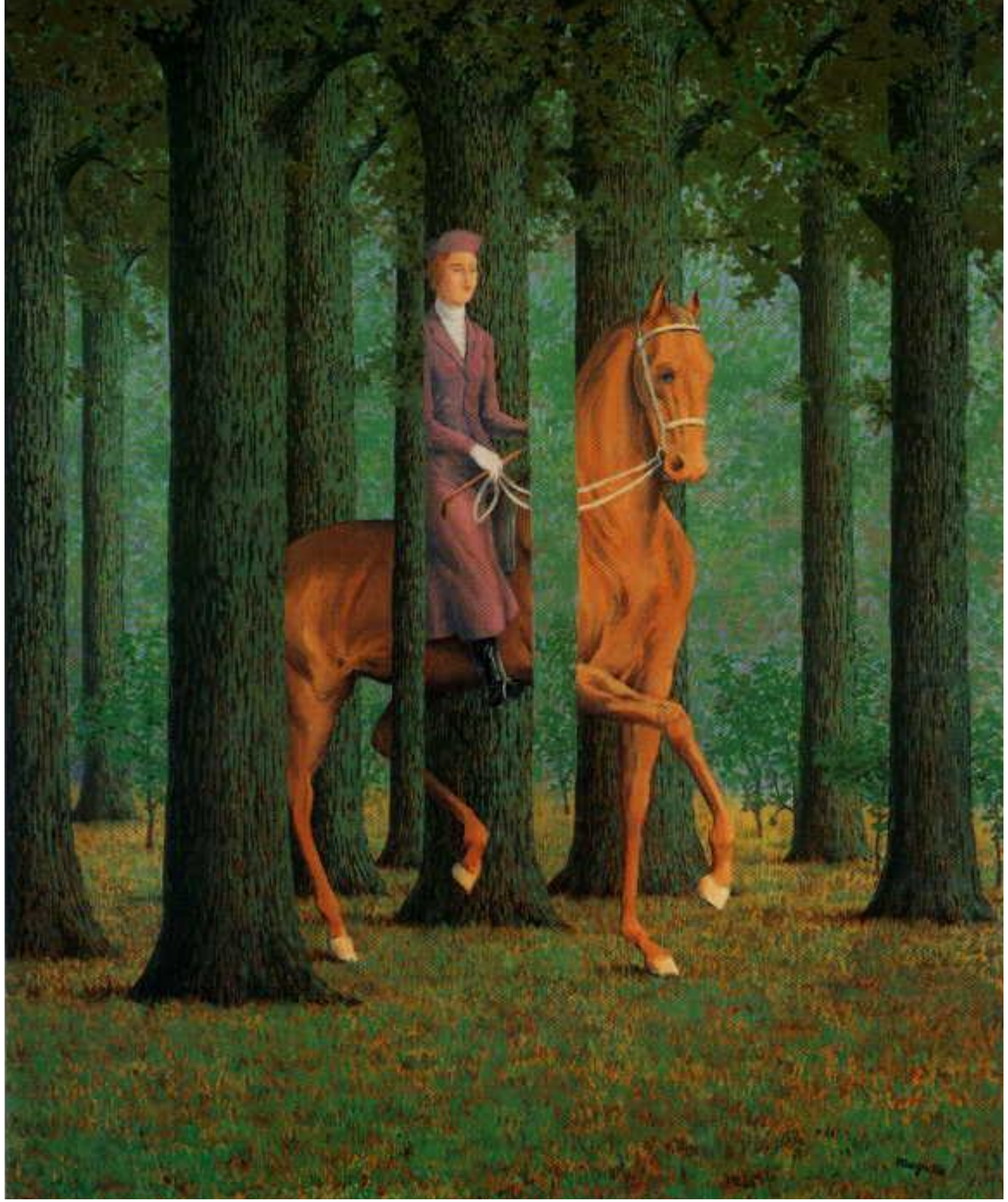


image credit: J. Koenderink

Challenges: scale

Challenges: deformation

Challenges: occlusion



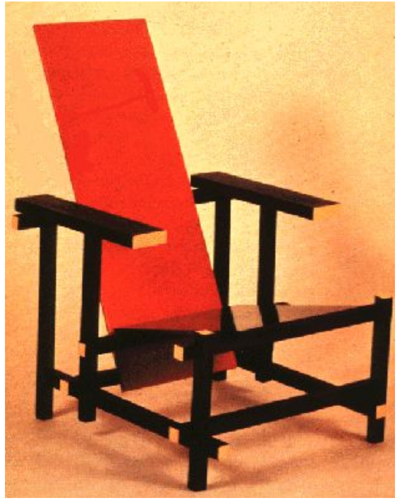
Magritte, 1957

Art Segway - Magritte

Challenges: background clutter

Kilmeny Niland. 1995

Challenges: intra-class variation



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Object recognition:
a classification framework

- Apply a prediction function to a feature representation of the image to get the desired output:

f() = “apple”

f() = “tomato”

f() = “cow”

A simple pipeline - Training

Training Images

Image Features

A simple pipeline - Training

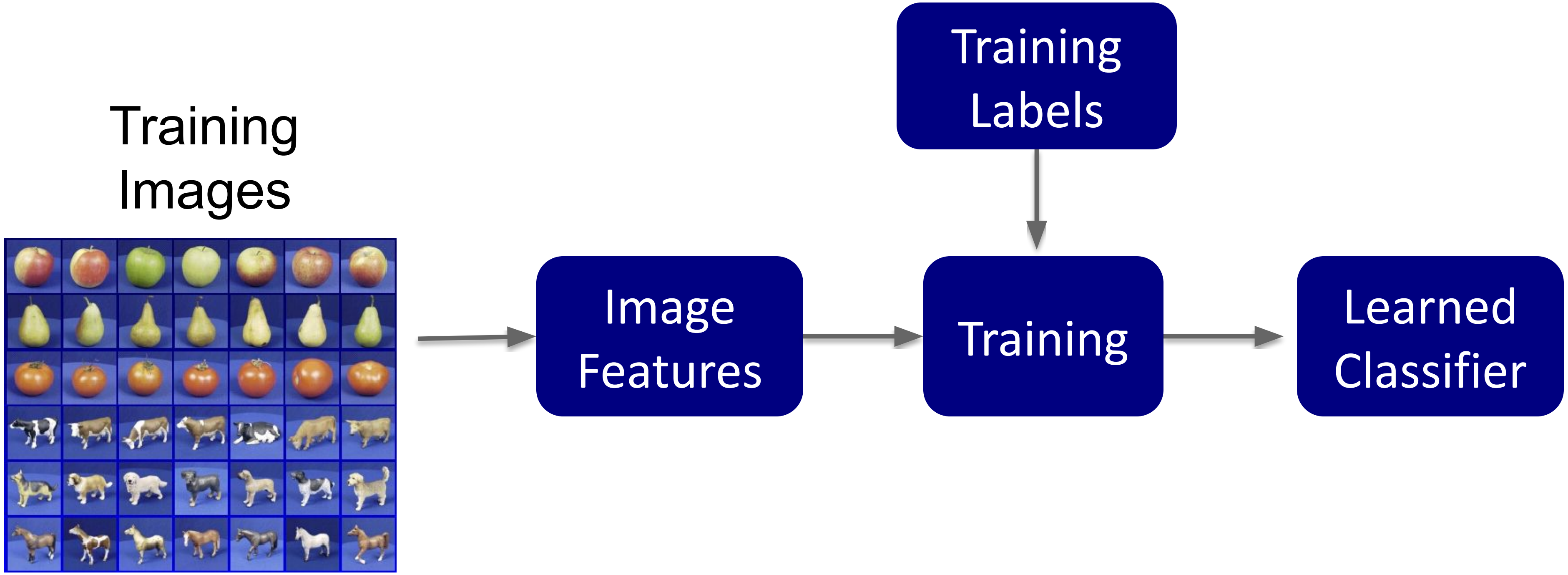
Training Images

Image Features

Training Labels

Training

A simple pipeline - Training



A simple pipeline - Training

Training Images

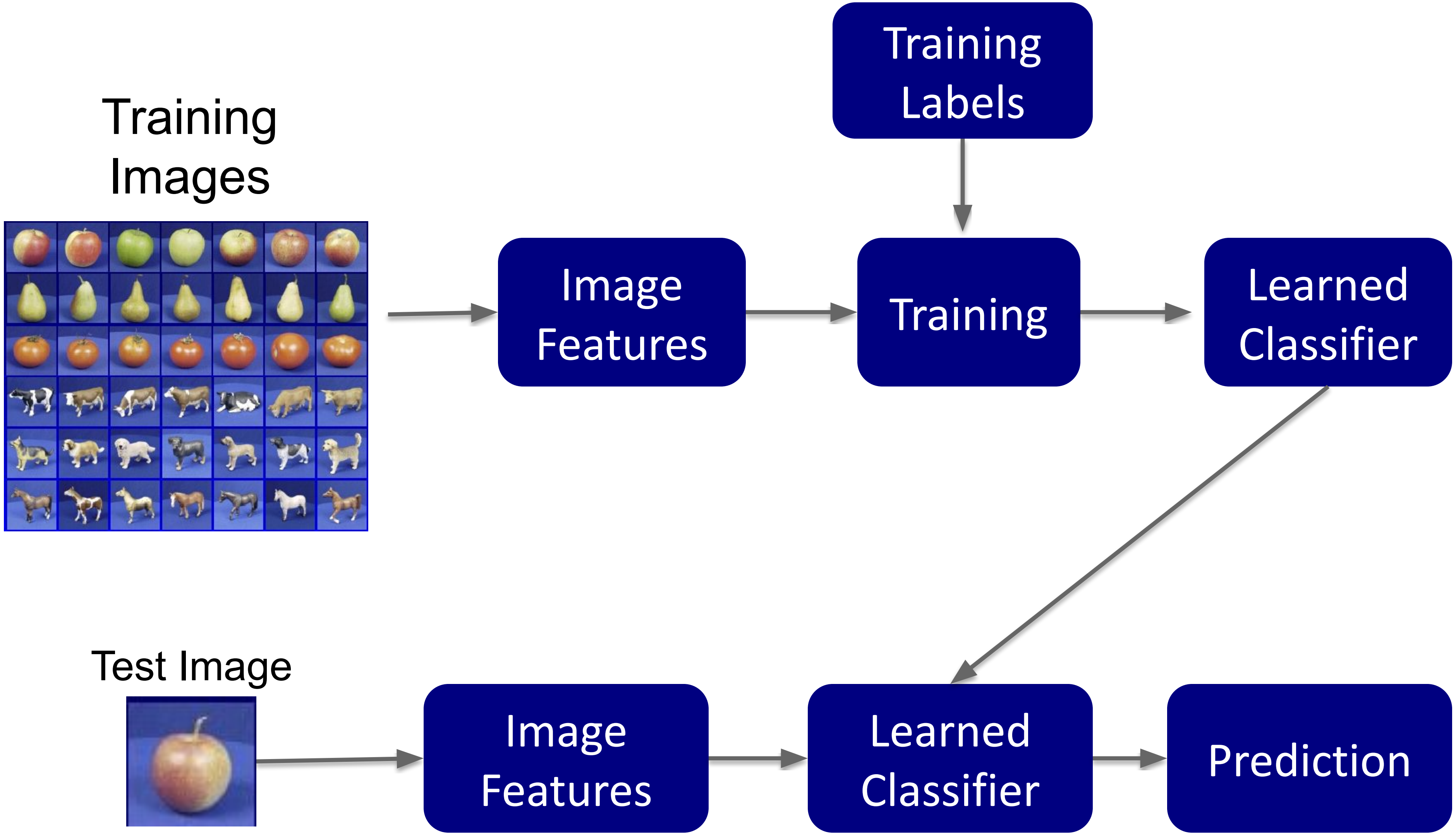
→ Image Features → Training → Learned Classifier

Training Labels → Training

Test Image

→ Image Features

A simple pipeline - Training



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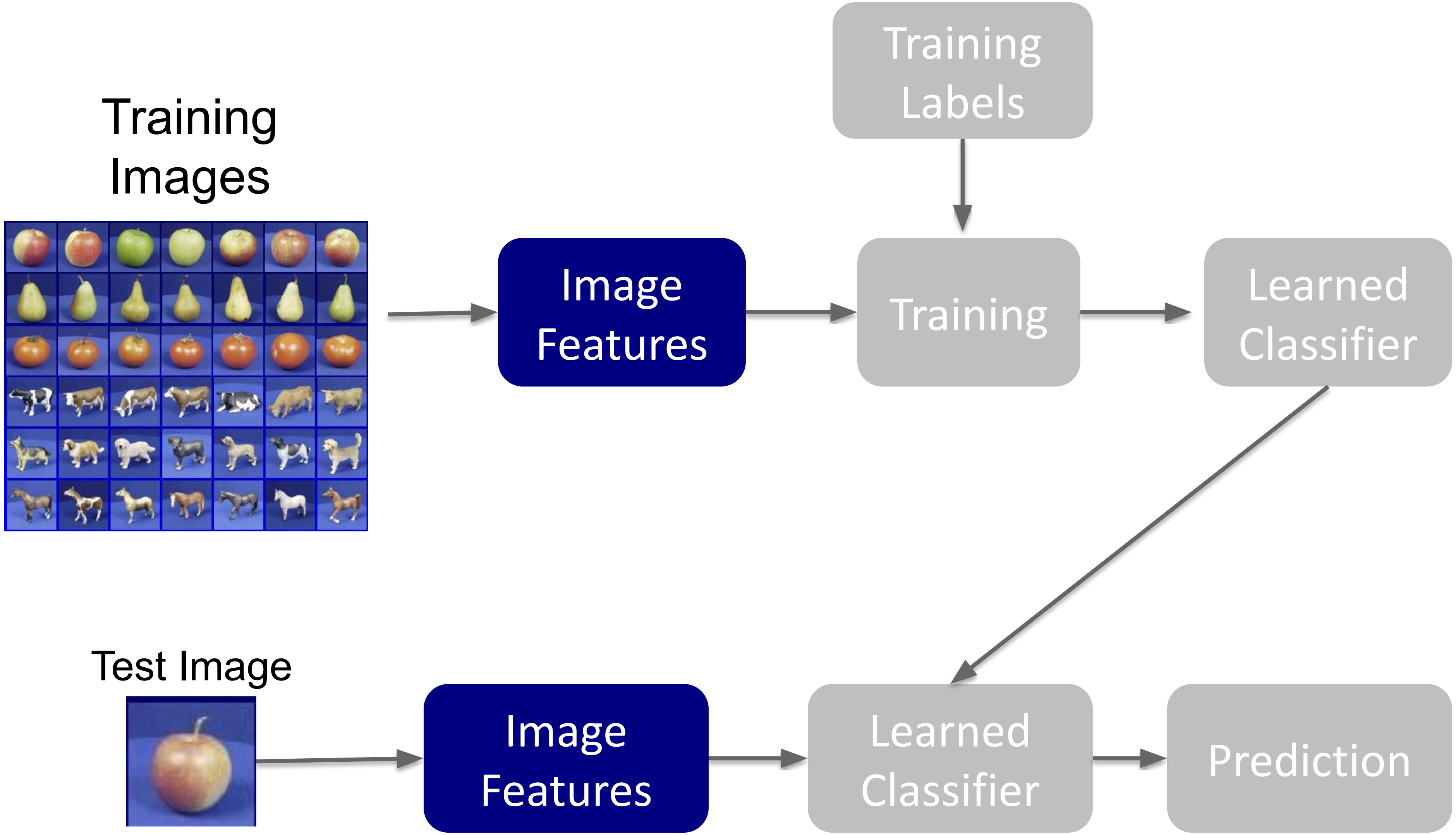
A simple pipeline - Training



Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	?						

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	?					

Choices of features

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RGB-histogram	✅	❌	?	?			

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	?		

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	?	?

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	?	?	?	?			

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	✅	❌	❌	❌	?	?	?

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	✅	❌	❌	❌	❌	✅	✅

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	✅	❌	❌	❌	❌	✅	✅
SIFT	?	?	?	?			

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	✅	❌	❌	❌	❌	✅	✅
SIFT	✅	✅	✅	✅	?	?	?

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	✅	❌	❌	❌	❌	✅	✅
SIFT	✅	✅	✅	✅	❌	✅	✅
Deep learning							

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	✅	❌	❌	❌	❌	✅	✅
SIFT	✅	✅	✅	✅	❌	✅	✅
Deep learning	usually	usually	usually	sometimes			

Choices of features

	Invariances						
	Translation	**Scale**	**Rotation** (relative to camera plane)	**Rotation** (unconstrained)	**Occlusion**	**Illumination changes**	**Gaussian Noise**
RGB-histogram	✅	❌	✅	❌	❌	❌	❌
HoG	✅	❌	❌	❌	❌	✅	✅
SIFT	✅	✅	✅	✅	❌	✅	✅
Deep learning	usually	usually	usually	sometimes	❌	✅	✅

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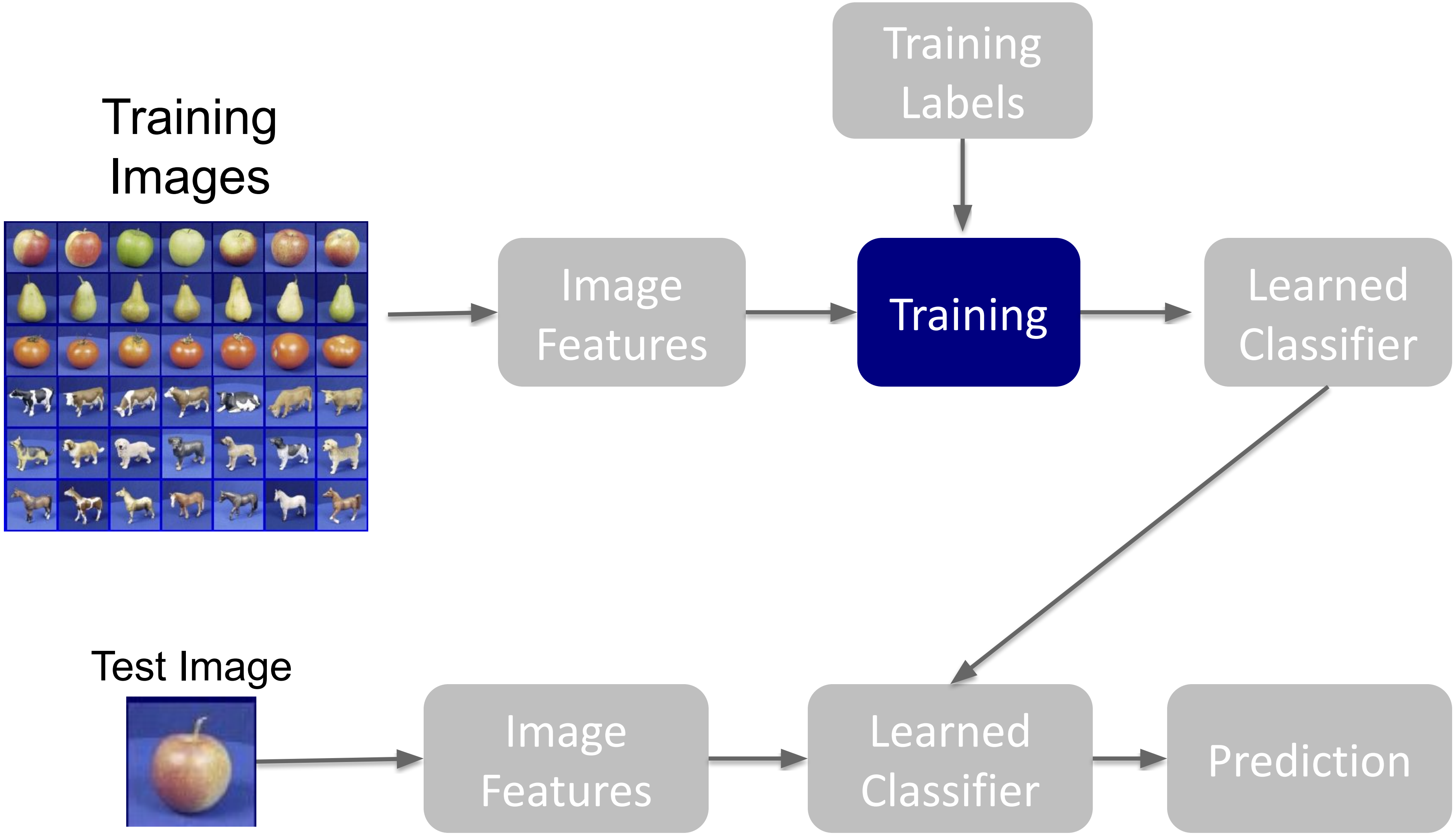
A simple pipeline - Training



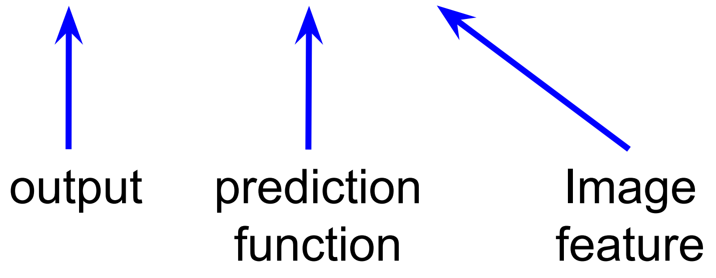
Many classifiers to choose from

- **K-nearest neighbor**
- SVM
- Neural networks
- Naïve Bayes
- Bayesian network
- Logistic regression
- Randomized Forests
- Boosted Decision Trees
- RBMs
- Etc.

Which is the best one?

Learning a classifier to map inputs to outputs

$$y = f(\mathbf{x})$$



- **Training:** given a *training set* of labeled examples $\{(\mathbf{x}_1,y_1), \ldots, (\mathbf{x}_N,y_N)\}$, estimate the prediction function f by minimizing the prediction error on the training set
- **Testing:** apply f to a never before seen *test example* $\mathbf{x}$ and output the predicted value $y = f(\mathbf{x})$

Slide credit: L. Lazebnik

An example training dataset

Apples

Pear

Tomatos

Cow

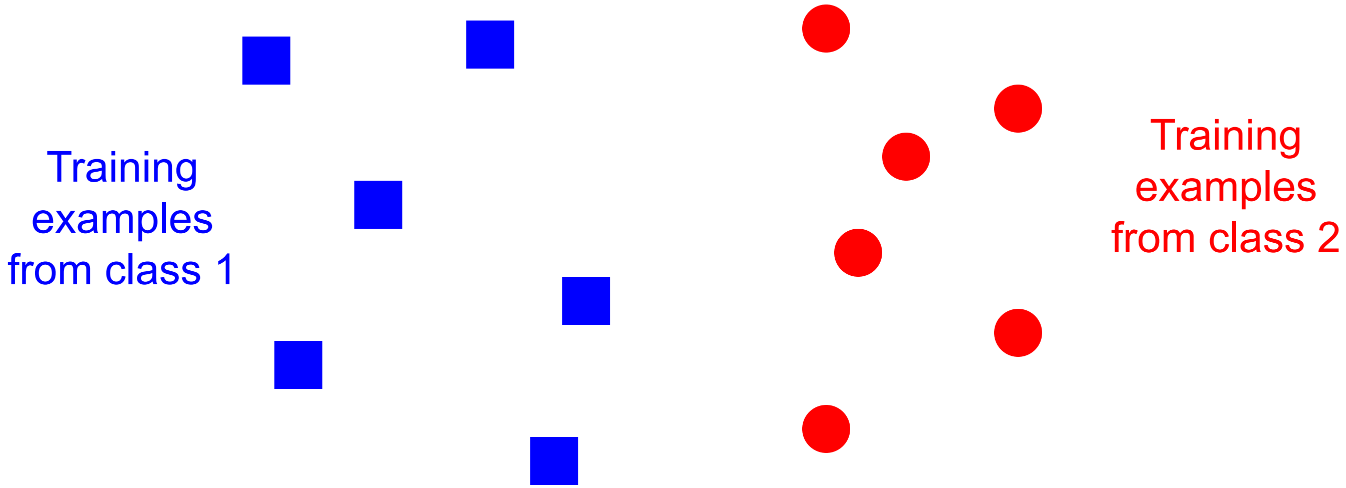
Dog

Horse

Training set (labels known)

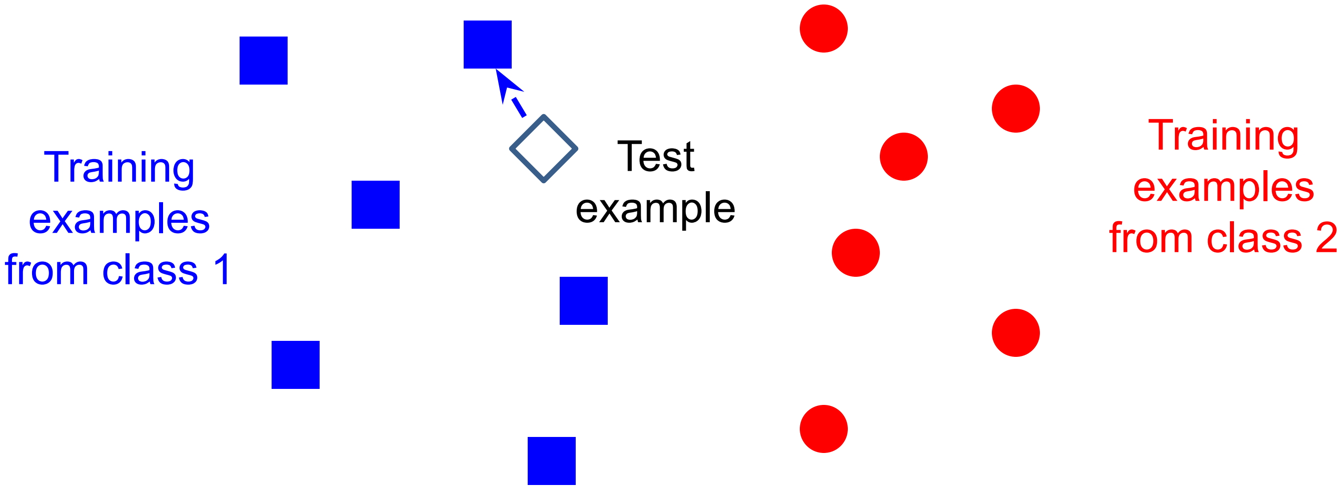
For kNN classifier, training simply means to store all training data.

A stored training set



Slide credit: L. Lazebnik

During testing, we assign the label of the nearest neighbot in feature space



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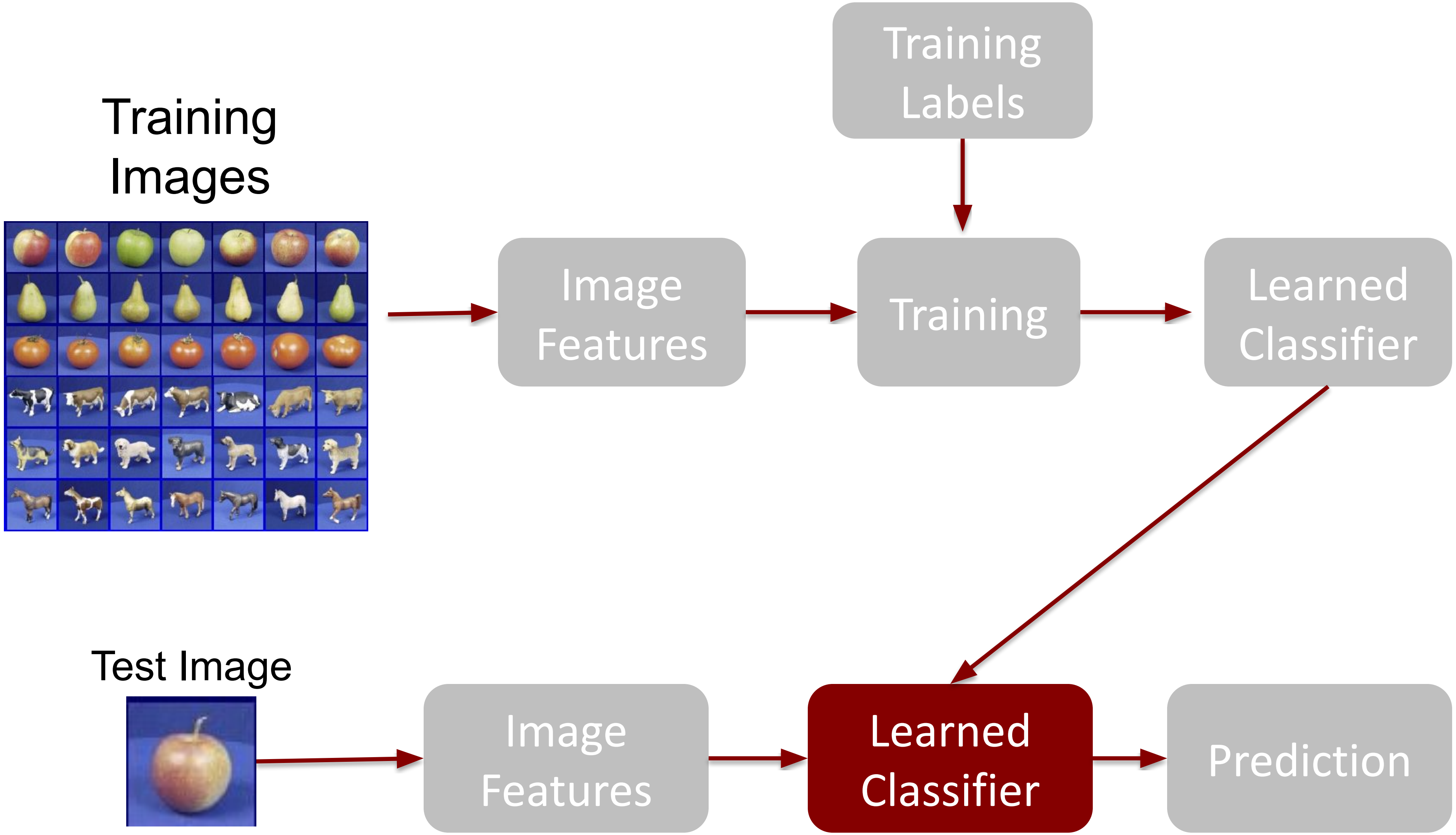


Generalization

Training set (labels known)

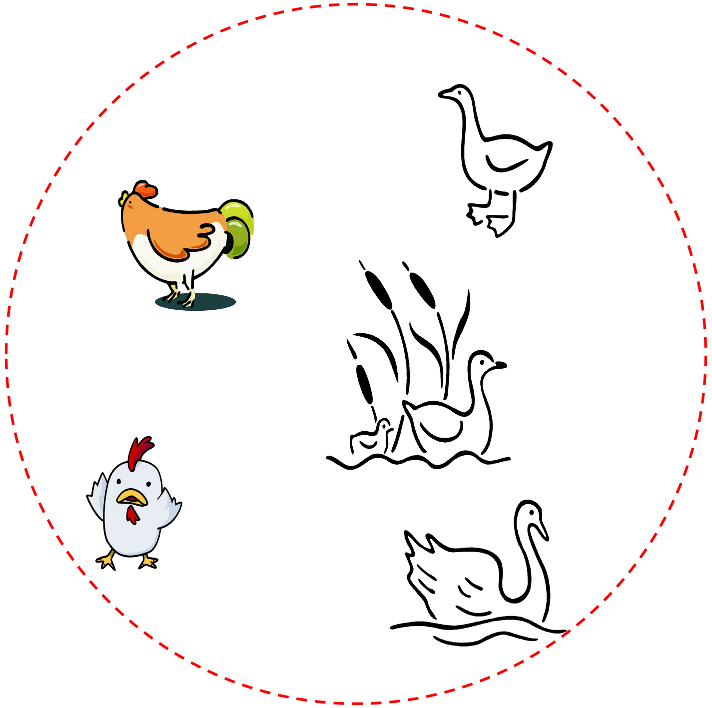
Test set (labels unknown)

- How well does a learned model generalize from the data it was trained on to a new test set?

Slide credit: L. Lazebnik

Intuition for Nearest Neighbor Classifier

Given a training dataset, simply store each image's features and their corresponding label.

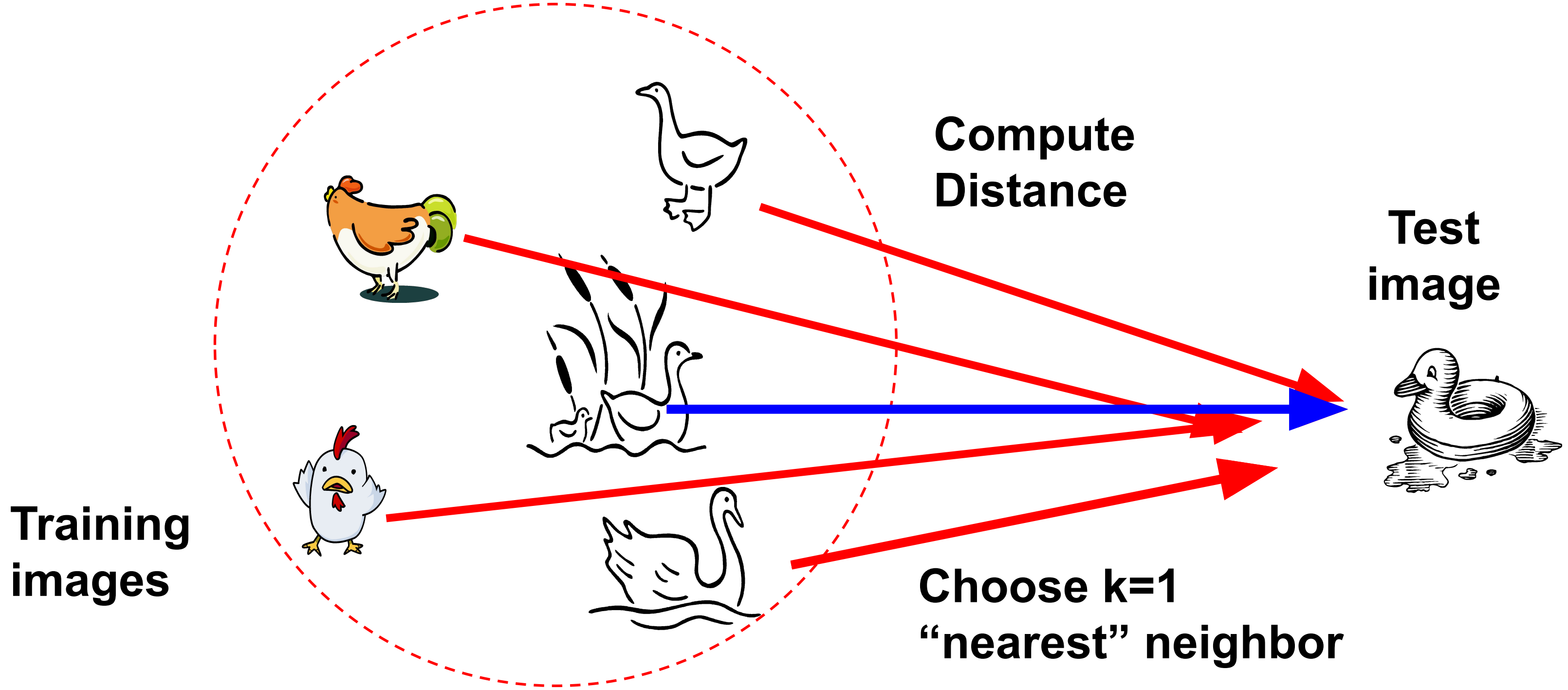


Training images

Source: N. Goyal

Intuition for Nearest Neighbor Classifier

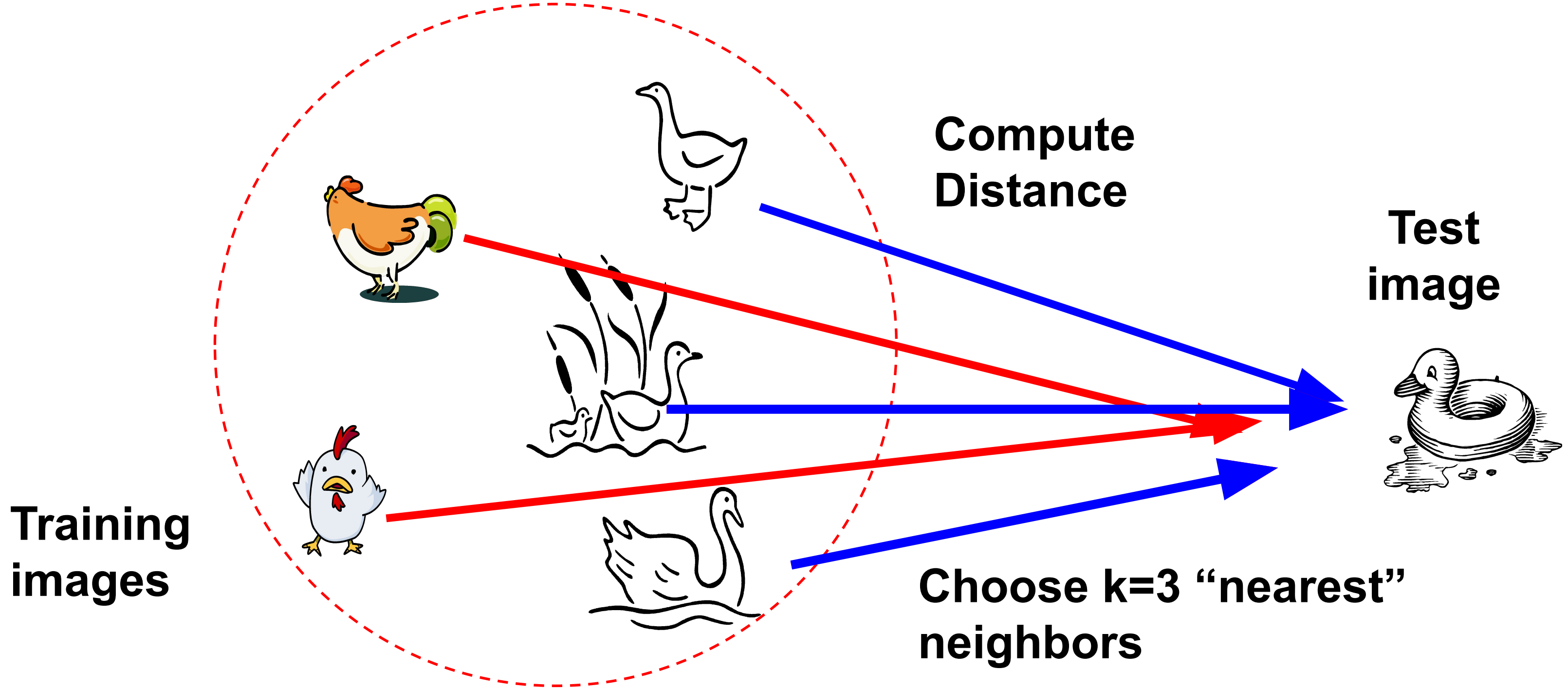
Given a training dataset, simply store each image's features and their corresponding label.



Source: N. Goyal

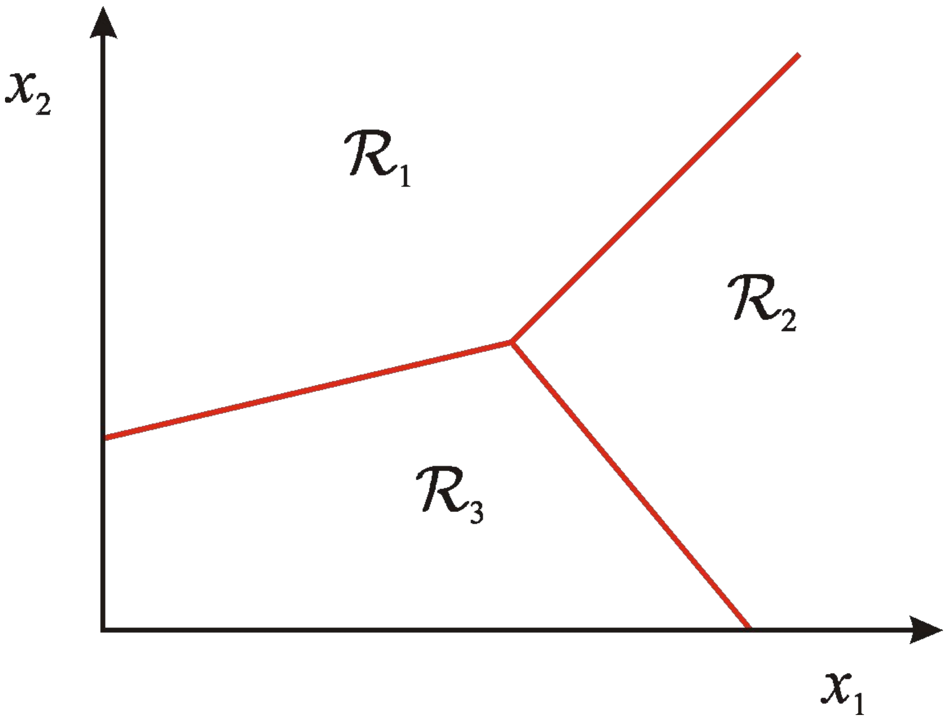
Nearest Neighbor Classifier

- Assign label of majority of K=3 nearest neighbors



Classification

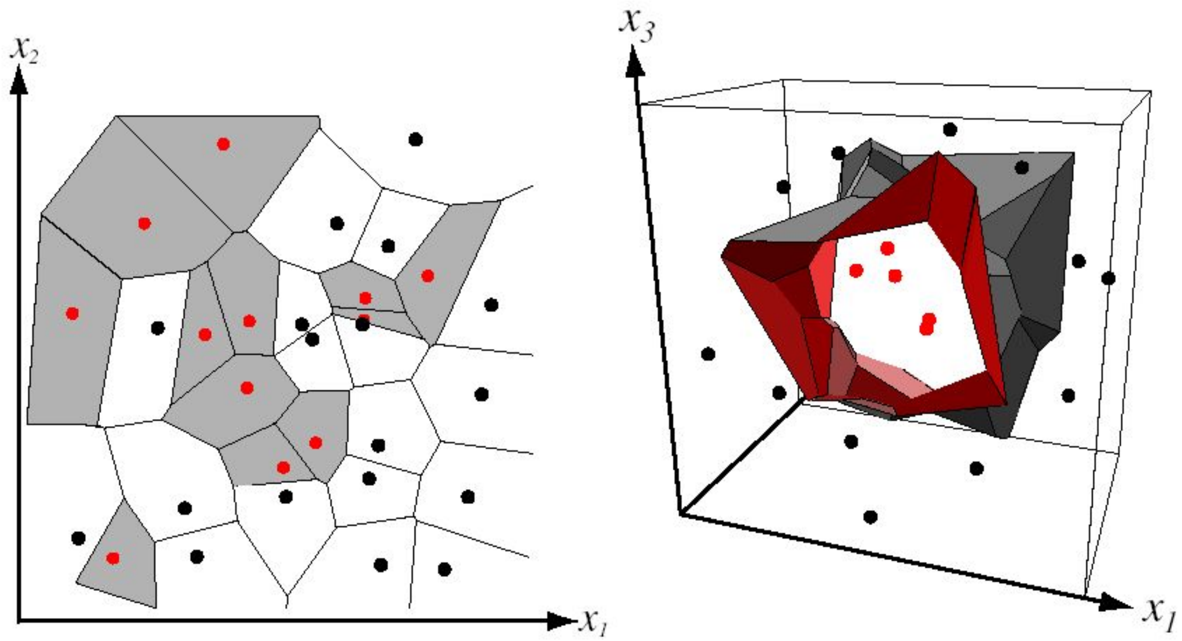
- Assign input vector to one of many classes (categories)
- **Geometric interpretation** of classifiers: A classifier divides input space into *decision regions* separated by *decision boundaries*



Slide credit: L. Lazebnik

Nearest Neighbor Classifier

- Assign label of nearest training data point to each test data point



from Duda *et al.*

Partitioning of feature space
for two-category 2D and 3D data

Source: D. Lowe

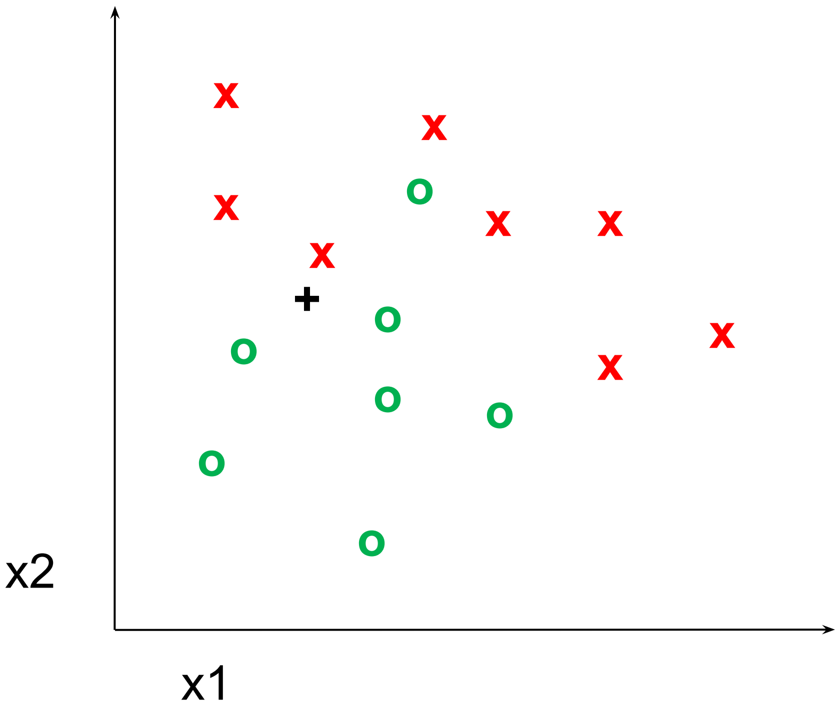
How do we find the nearest neighbors in feature space?

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where X^n and X^m are the n-th and m-th data points



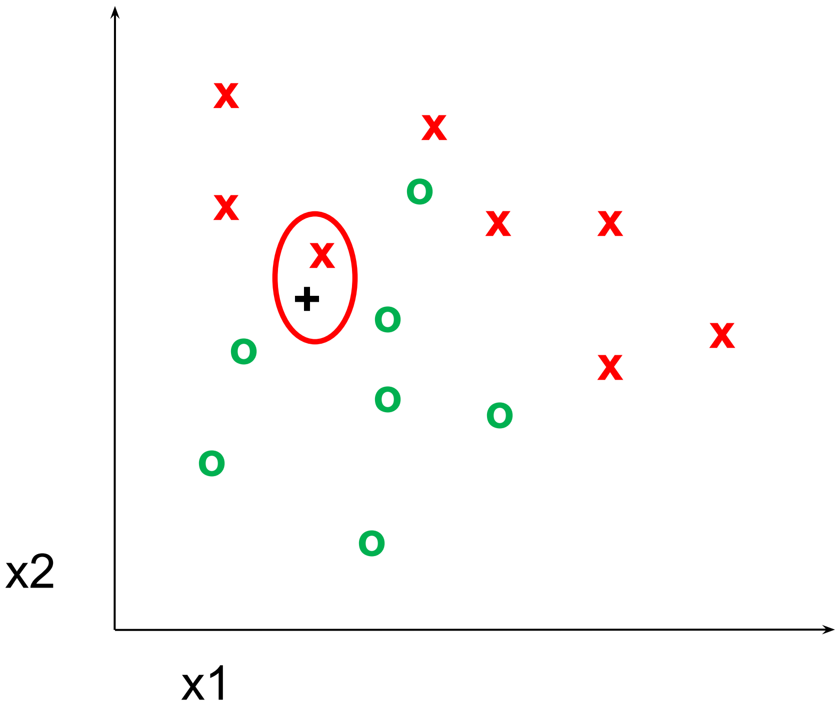
1-nearest neighbor

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where X^n and X^m are the n-th and m-th data points



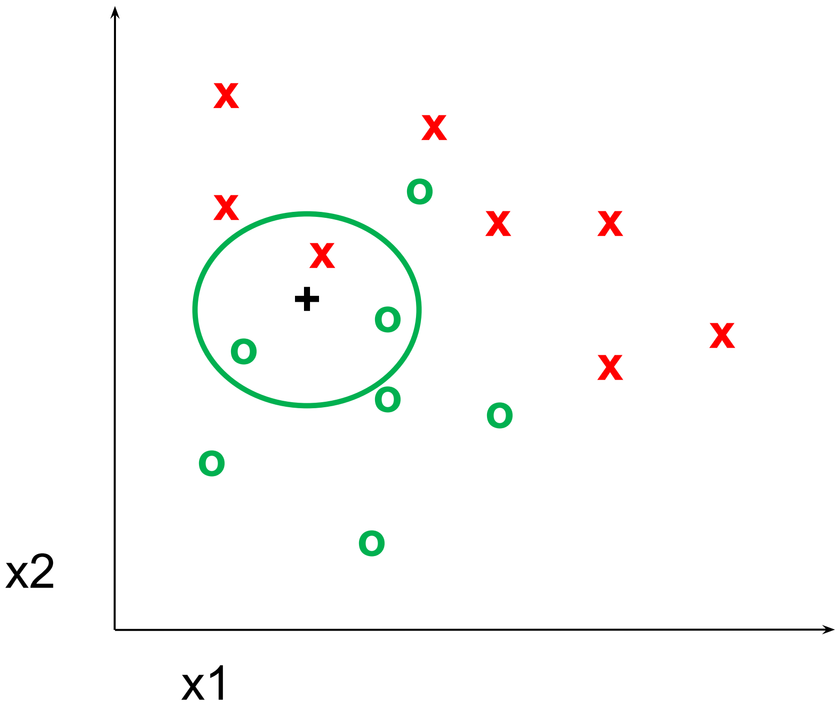
3-nearest neighbor

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where X^n and X^m are the n-th and m-th data points



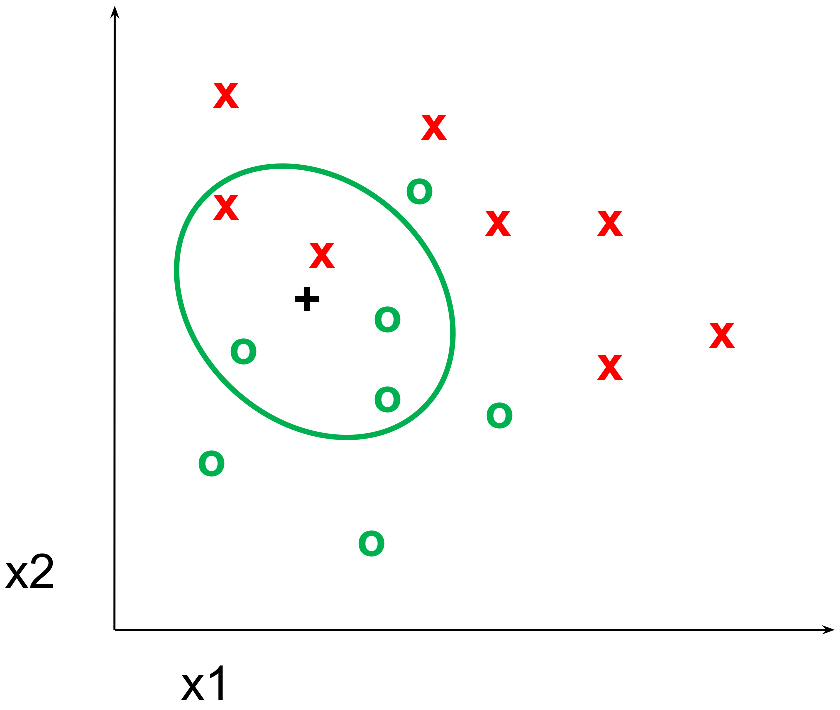
5-nearest neighbor

Distance measure (same as the ones from segmentation)

Euclidean:

$$Dist(X^n, X^m) = \sqrt{\sum_{i=1}^{D} (X_i^n - X_i^m)^2}$$

Where X^n and X^m are the n-th and m-th data points



Choosing the right features is important but dataset-dependent

	Color	$D_x D_y$	Mag-Lap	PCA Masks	PCA Gray	Cont. Greedy	Cont. DynProg	Avg.
apple	57.56%	**85.37%**	80.24%	78.78%	**88.29%**	77.07%	76.34%	77.66%
pear	66.10%	90.00%	85.37%	**99.51%**	**99.76%**	90.73%	91.71%	89.03%
tomato	**98.54%**	94.63%	**97.07%**	67.80%	76.59%	70.73%	70.24%	82.23%
cow	86.59%	82.68%	**94.39%**	75.12%	62.44%	86.83%	86.34%	82.06%
dog	34.63%	62.44%	74.39%	72.20%	66.34%	**81.95%**	**82.93%**	67.84%
horse	32.68%	58.78%	70.98%	77.80%	77.32%	**84.63%**	**84.63%**	69.55%
cup	79.76%	66.10%	77.80%	**96.10%**	**96.10%**	**99.76%**	**99.02%**	87.81%
car	62.93%	**98.29%**	77.56%	**100.0%**	**97.07%**	**99.51%**	**100.0%**	90.77%
total	64.85%	79.79%	82.23%	83.41%	82.99%	86.40%	86.40%	80.87%

Dataset: ETH-80, by B. Leibe, 2003

Results

Category	Primary feature(s)	Secondary feature(s)
apple	PCA Gray	Texture $D_x D_y$
pear	PCA Gray / Masks	
tomato	Color	Texture Mag-Lap
cow	Texture Mag-Lap	Contour / Color
dog	Contour	
horse	Contour	
cup	Contour	PCA Gray / Masks
car	PCA Masks / Contour	Texture $D_x D_y$

Dataset: ETH-80, by B. Leibe, 2003

K-NN: a very useful algorithm

- Simple, a good one to try first
- Very flexible decision boundaries
- With infinite examples, 1-NN has a strong theoretical guarantee (out of scope for this class)

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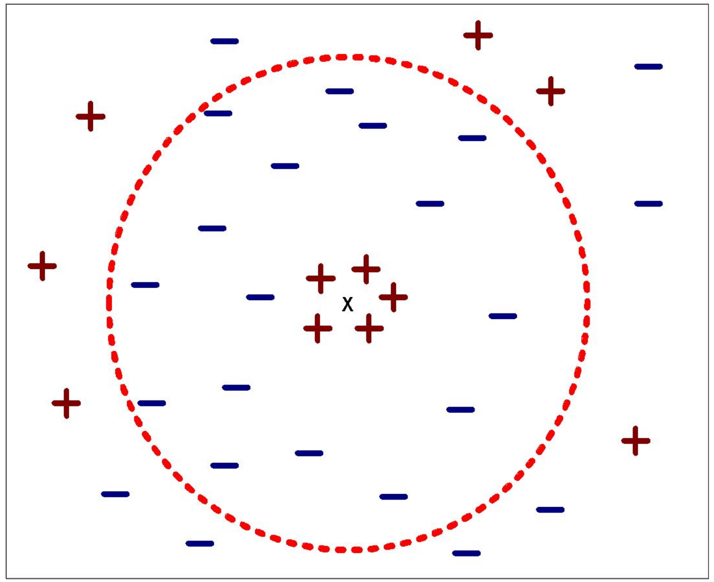
K-NN: issues to keep in mind

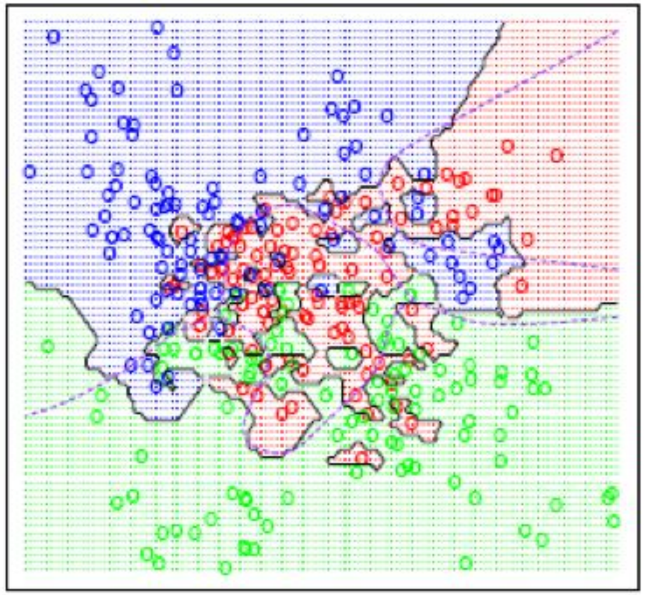
- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes



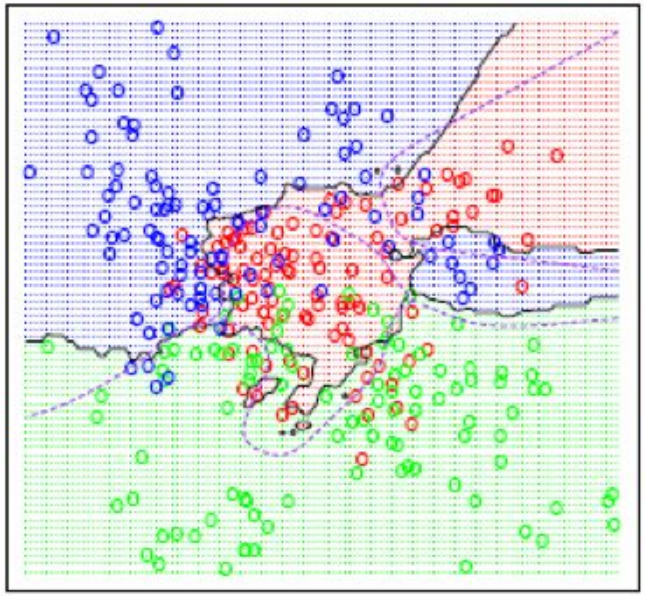
K-NN: issues to keep in mind

- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes

K=1

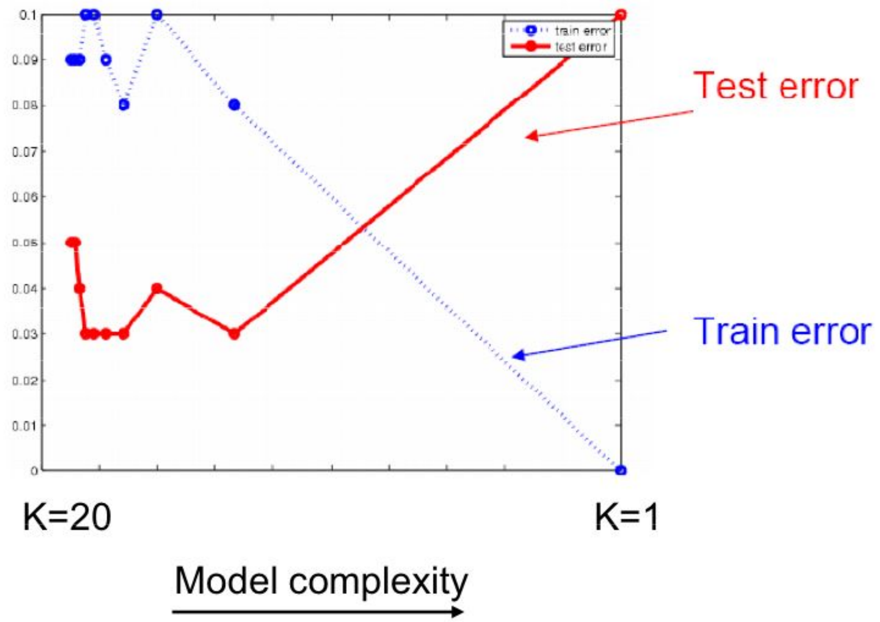


K=15

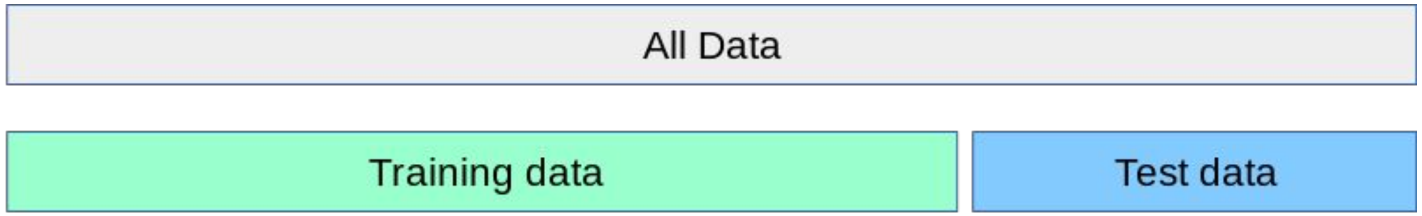


K-NN: issues to keep in mind

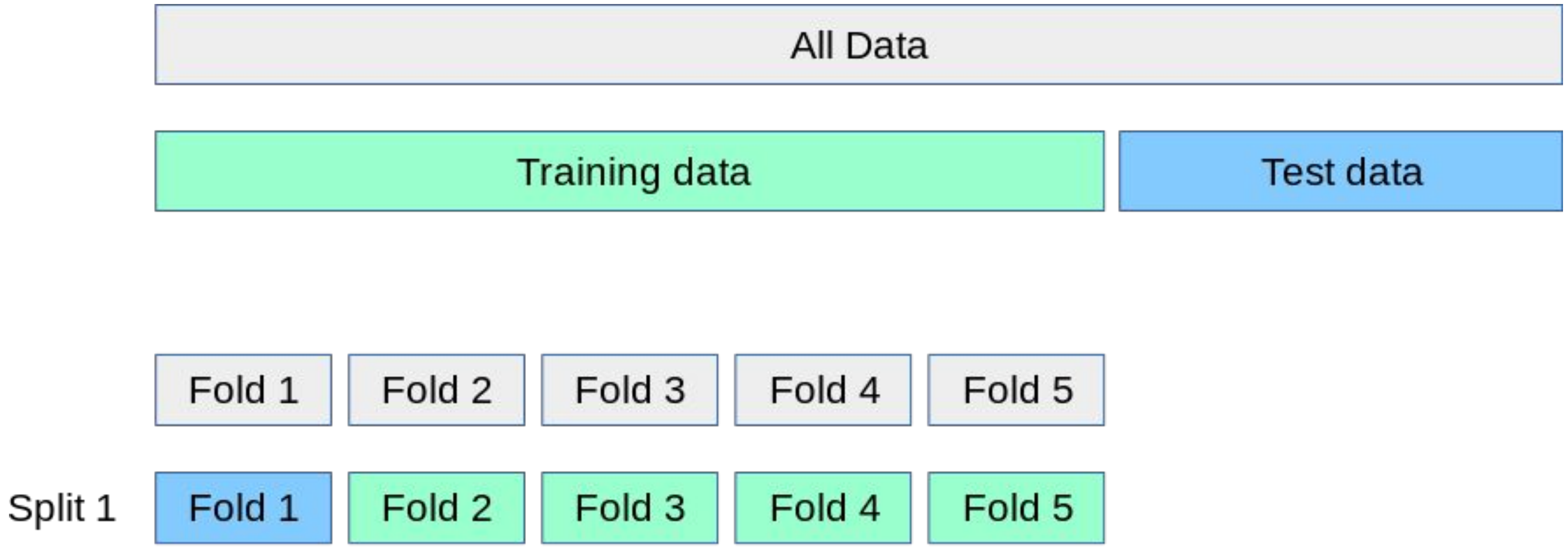
- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes
  - Solution: Cross validate



Cross validation

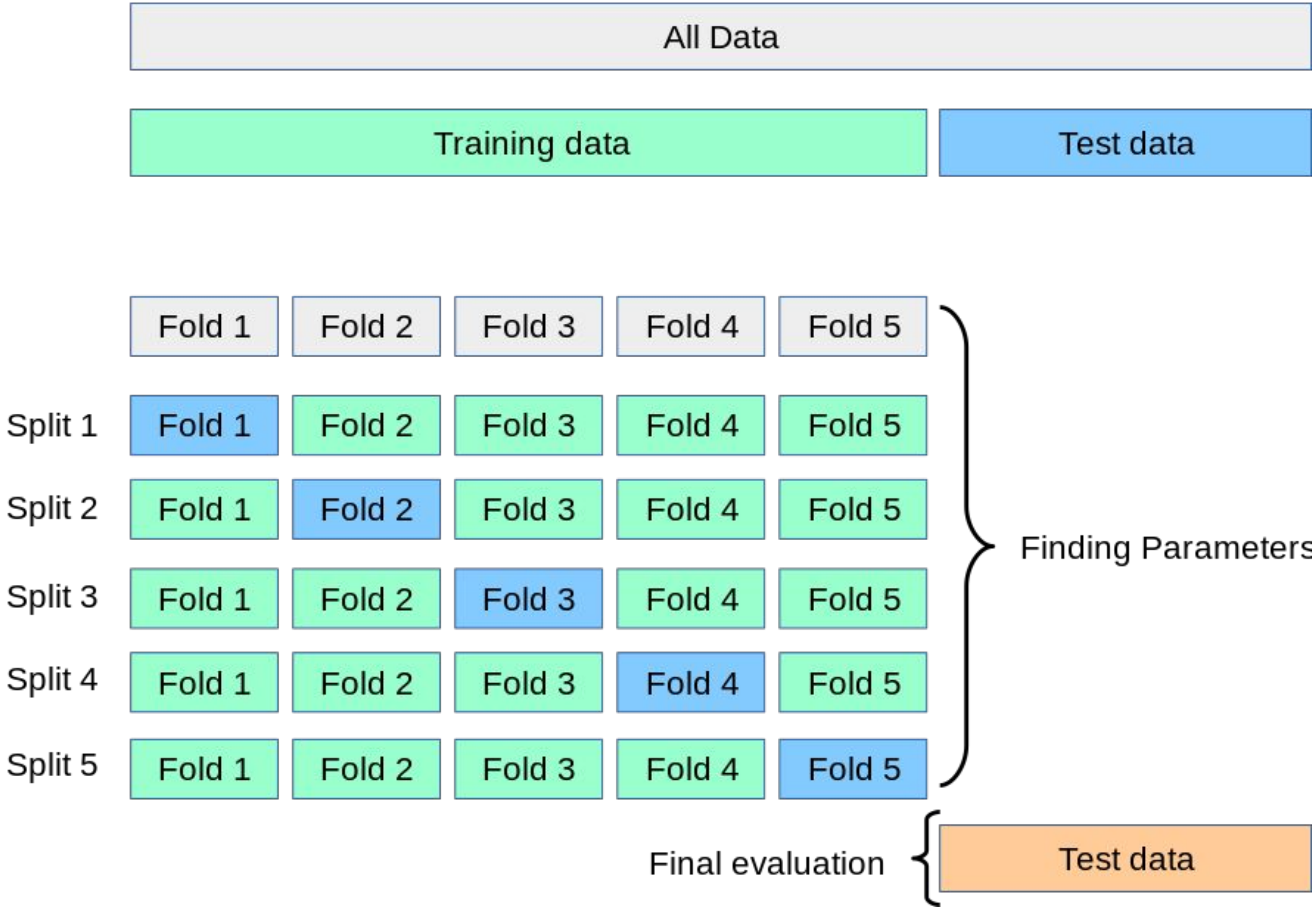


Cross validation



Cross validation

Cross validation

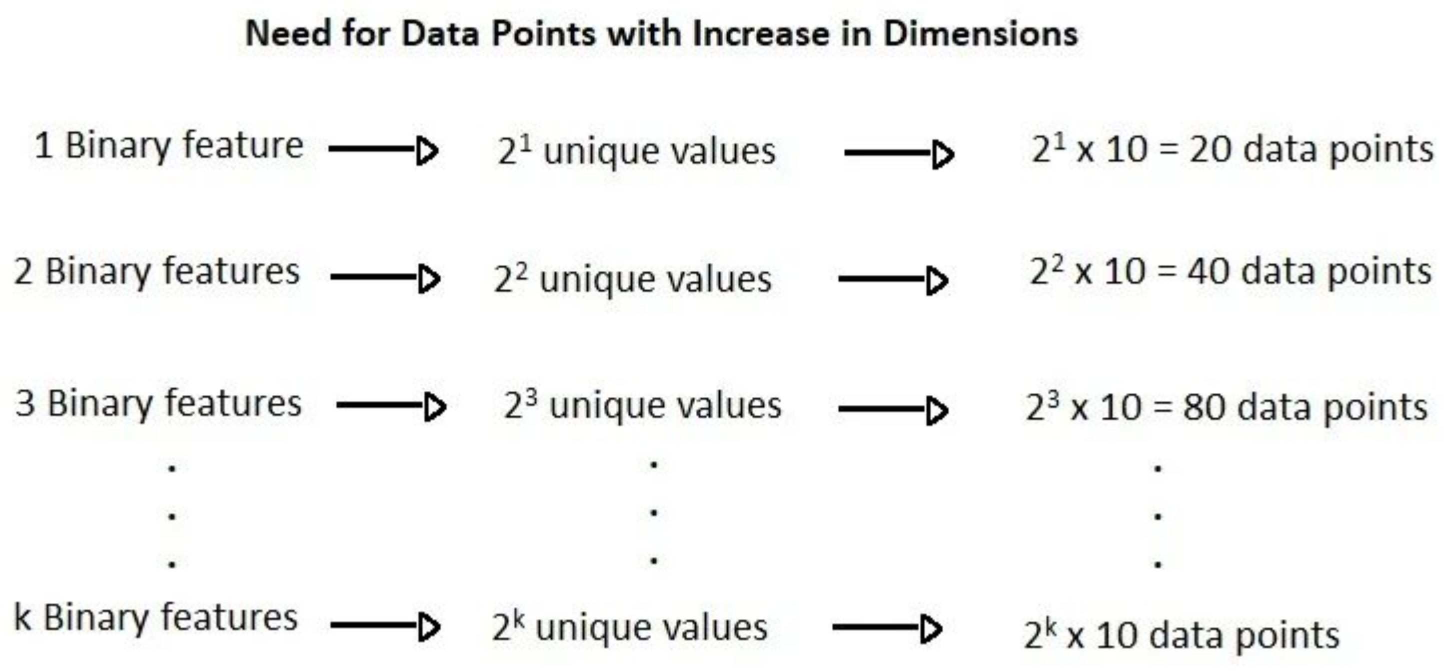


K-NN: issues to keep in mind

- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes
  - Solution: cross validate!
- Curse of Dimensionality

Curse of dimensionality

- As the dimensionality increases, the number of data points required for good performance increases exponentially.
- Let's say that for a model to perform well, we need at least 10 data points for each combination of feature values.



K-NN: issues to keep in mind

- Choosing the value of k:
  - If too small, sensitive to noise points
  - If too large, neighborhood may include points from other classes
  - Solution: cross validate!
- Curse of Dimensionality
  - Solution: dimensionality reduction

What we will learn today

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- A simple Object Recognition pipeline
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- Dimensionality reduction
- Principal Component Analysis (PCA)

Singular Value Decomposition (SVD)

$$\mathbf{U\Sigma V^T = A}$$

- Where **U** and **V** are rotation matrices, and **Σ** is a scaling matrix. For example:

$$\overset{U}{\begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}}$$

Singular Value Decomposition (SVD)

- Beyond 2x2 matrices:
  - In general, if **A** is *m* x *n*, then **U** will be *m* x *m*, **Σ** will be *m* x *n*, and $\mathbf{V}^T$ will be *n* x *n*.
  - (Note the dimensions work out to produce *m* x *n* after multiplication)

$$\overset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

Singular Value Decomposition (SVD)

- **U** and **V** are always rotation matrices.
  - Geometric rotation may not be an applicable concept, depending on the matrix. So we call them “unitary” matrices – each column is a unit vector.
- **Σ** is a diagonal matrix
  - The number of nonzero entries = rank of **A**
  - The algorithm always sorts the entries high to low

$$\overset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

SVD Applications

- We’ve discussed SVD in terms of geometric transformation matrices
- But SVD of an image matrix can also be very useful
- To understand this, we’ll look at a less geometric interpretation of what SVD is doing

What is SVD actually doing for images?

$$\overset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

- Look at how the multiplication works out, left to right:
- Column 1 of **U** gets scaled by the first value from **Σ**.

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}}$$

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$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- The resulting vector gets scaled by row 1 of **V**$^{\mathbf{T}}$ to produce a contribution to the columns of **A**

SVD is a type dimensionality reduction

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

$$+ \overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}}$$

$$= \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- Each product of (*column i of* ***U***)·(*value i from* ***Σ***)·(*row i of* ***V****^{T}*) produces a component of the final **A**.

SVD is a type dimensionality reduction

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}}$$

- We're building **A** as a linear combination of the columns of ***U***
- Using all columns of ***U***, we'll rebuild the original matrix perfectly
- But, in real-world data, often we can just use the first few columns of ***U*** and we'll get something close (e.g. the first $\boldsymbol{A_{partial}}$, above)

SVD is a type dimensionality reduction

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}}$$

- We can call those first few columns of ***U*** the Principal Components of the data
- They show the major patterns that can be added to produce the columns of the original matrix
- The rows of $\mathbf{V}^\mathbf{T}$ show how the *principal components* are mixed to produce the columns of the matrix

SVD is a type dimensionality reduction

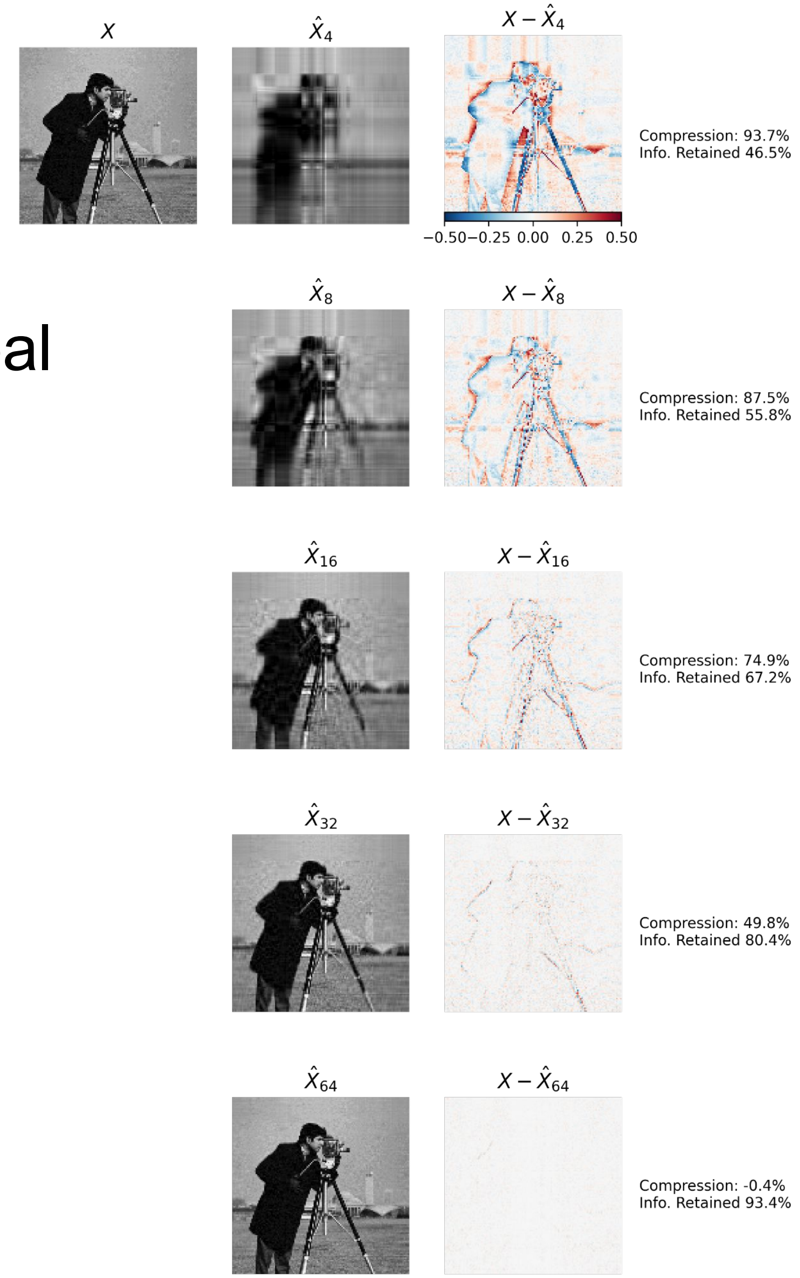
$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

We can look at **Σ** to see that the first column has a large effect

while the second column has a much smaller effect in this example

Image compression

- For this image, using **only the first 16** of 300 principal components produces a recognizable reconstruction
- Using the first 64 almost perfectly reconstructs the image



SVD for symmetric matrices

- If A is a symmetric matrix, it can be decomposed as the following:

$$A = \Phi \Sigma \Phi^T$$

- Compared to a traditional V^T and is an orthogonal matrix.

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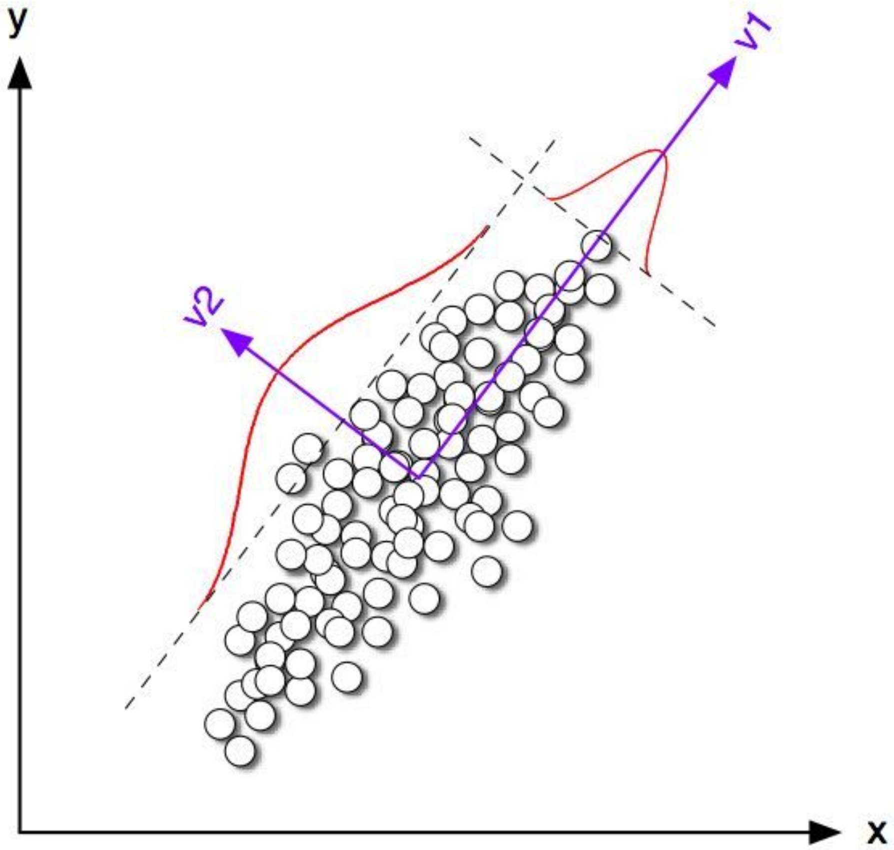
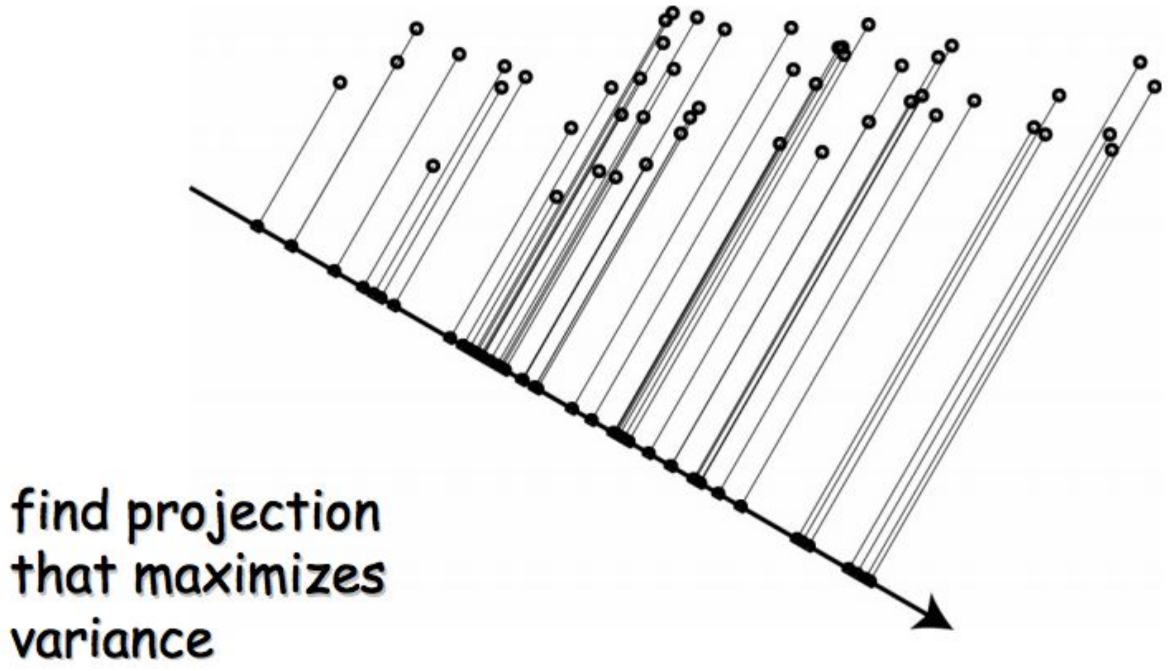
Intuition behind PCA: high dimensional data usually lives in some lower dimensional space

Covariance between the two dimensions of features is high. Can we reduce the number of dimensions to just 1?

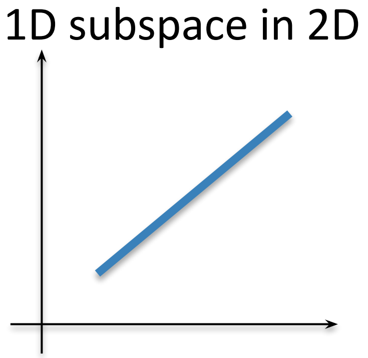


Geometric interpretation of PCA



Geometric interpretation of PCA

- Let’s say we have a set of 2D data points x. But we see that all the points lie on a line in 2D.
- So, 2 dimensions are redundant to express the data. We can express all the points with just one dimension.



PCA: Principle Component Analysis

- Given a dataset of images, can we compressed them like we can compress a single image?
  - Yes, the trick is to look into the correlation between the points
  - The tool for doing this is called PCA

PCA can be used to compress image RGB pixel values or also be used to compress their features!

PCA by SVD

- To relate this to PCA, we consider the image (or feature) matrix

$$X = \begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix}$$

- The sample mean of this dataset (or in plain english, the average image) is:

$$\mu = \frac{1}{n}\sum_i x_i = \frac{1}{n}\begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{n}X1$$

PCA by SVD

- Center the data by subtracting the mean to each column of X
- The <u>centered dataset matrix</u> is

$$X_c = \begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix} - \begin{bmatrix} | & & | \\ \mu & \dots & \mu \\ | & & | \end{bmatrix}$$

PCA by SVD

- The sample covariance matrix is

$$C = \frac{1}{n}\sum_i (x_i - \mu)(x_i - \mu)^T = \frac{1}{n}\sum_i x_i^c (x_i^c)^T$$

where x_i^c is the i^{th} column of X_c

- This can be written as

$$C = \frac{1}{n}\begin{bmatrix} | & & | \\ x_1^c & \dots & x_n^c \\ | & & | \end{bmatrix}\begin{bmatrix} - & x_1^c & - \\ & \vdots & \\ - & x_n^c & - \end{bmatrix} = \frac{1}{n} X_c X_c^T$$

PCA by SVD

- The matrix

$$X_c^T = \begin{bmatrix} - & x_1^c & - \\ & \vdots & \\ - & x_n^c & - \end{bmatrix}$$

is real (n x d). Assuming n>d it has SVD decomposition

$$X_c^T = U\Sigma V^T \qquad U^T U = I \qquad V^T V = I$$

and

$$C = \frac{1}{n} X_c X_c^T = \frac{1}{n} U\Sigma V^T (U\Sigma V^T)^T = \frac{1}{n} U\Sigma V^T V\Sigma U^T = \frac{1}{n} U\Sigma^2 U^T$$

PCA by SVD

$$C = \frac{1}{n} U \Sigma^2 U^T$$

- Note that U is (d x d) and orthonormal, and Σ^2 is diagonal. This is just the eigenvalue decomposition of C
- It follows that
  - The eigenvectors of C are the columns of U
  - The eigenvalues of C are the diagonal entries of Σ^2: λ_i^2

PCA by SVD

- In summary, computation of PCA by SVD
- Given X with one image (or feature) per column
  - Create the centered data matrix

$$X_c = \begin{bmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{bmatrix} - \begin{bmatrix} | & & | \\ \mu & \dots & \mu \\ | & & | \end{bmatrix}$$

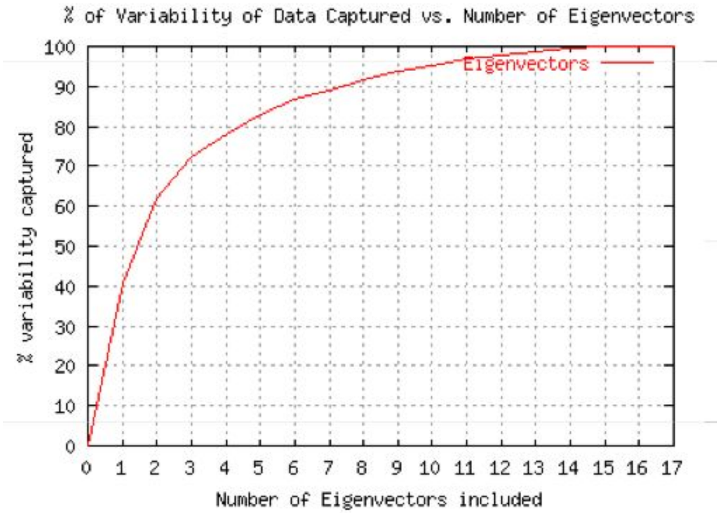
  - Compute its SVD

$$X_c^T = U \Sigma V^T$$

  - Principal components are columns of its covariance is the squared diagonal entries of Σ^2

To compress an image dataset, pick the largest eigenvalues and their corresponding eigenvectors

- Pick the eigenvectors that explain p% of the image data variability
  - Can be done by plotting the ratio r_k as a function of k



$$r_k = \sum_{i=1}^{k} \lambda_i^2 \Big/ \sum_{i=1}^{n} \lambda_i^2$$

  - E.g. we need k=3 eigenvectors to cover 70% of the variability of this dataset

What exactly is the covariance

- Variance and Covariance are a measure of the **"spread"** of a set of points around their center of mass (mean)
- **Variance** – measure of the deviation from the mean for points in one dimension e.g. heights
- **Covariance** as a measure of how much each of the dimensions vary from the mean with respect to each other.
- Covariance is measured between 2 dimensions to see if there is a relationship between the 2 dimensions e.g. number of hours studied & marks obtained.
- The covariance between one dimension and itself is the variance

Covariance

$$\text{covariance}(X,Y) = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{(n-1)}$$

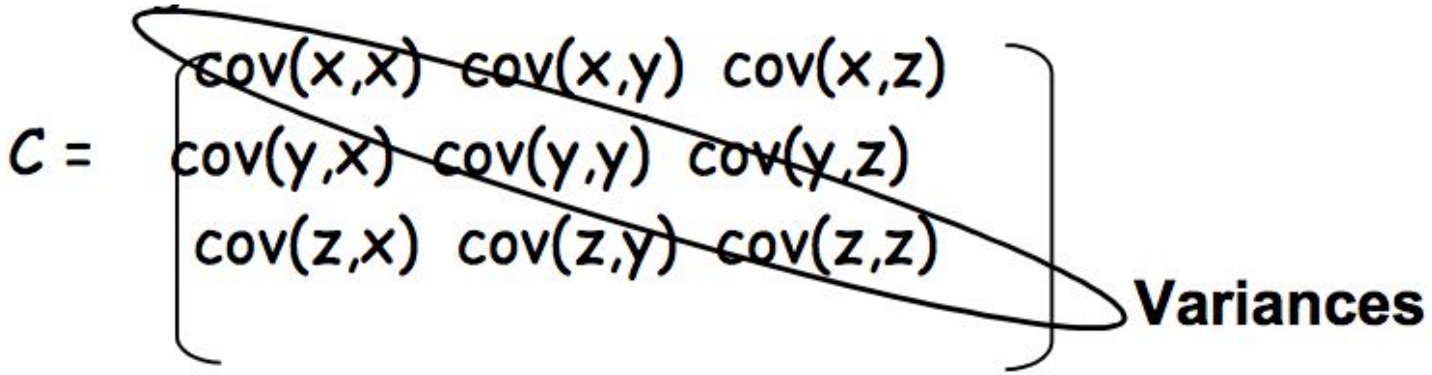
- So, if you had a 3-dimensional data set (x,y,z), then you could measure the covariance between the x and y dimensions, the y and z dimensions, and the x and z dimensions. Measuring the covariance between x and x , or y and y , or z and z would give you the variance of the x , y and z dimensions respectively

Covariance

- What is the interpretation of covariance calculations?
  - e.g.: 3 dimensional data set
  - **x**: number of hours studied for a subject
  - **y**: marks obtained in that subject
  - **z**: number of lectures attended
  - covariance value between **x and y is say: 104.53**
  - what does this value mean?

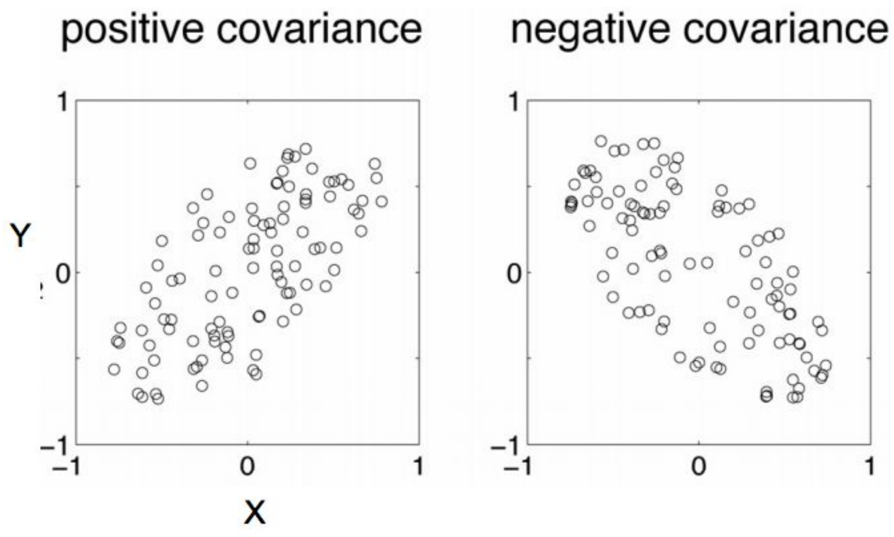
Visualizing this covariance matrix

- Representing Covariance between dimensions as a matrix e.g. for 3 dimensions



- Diagonal is the **variances** of x, y and z
- cov(x,y) = cov(y,x) hence matrix is symmetrical about the diagonal
- N-dimensional data will result in NxN covariance matrix

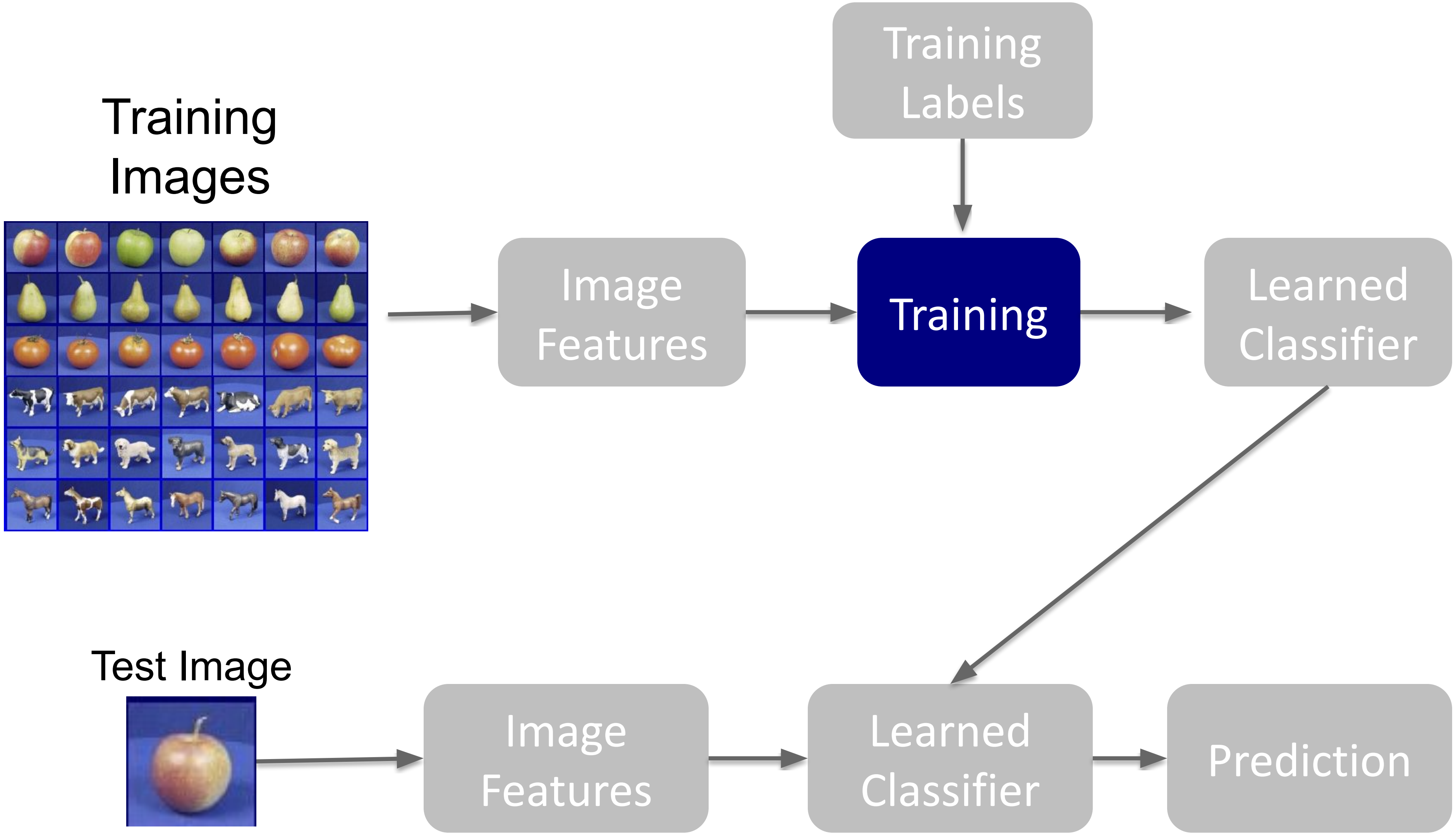
Covariance interpretation



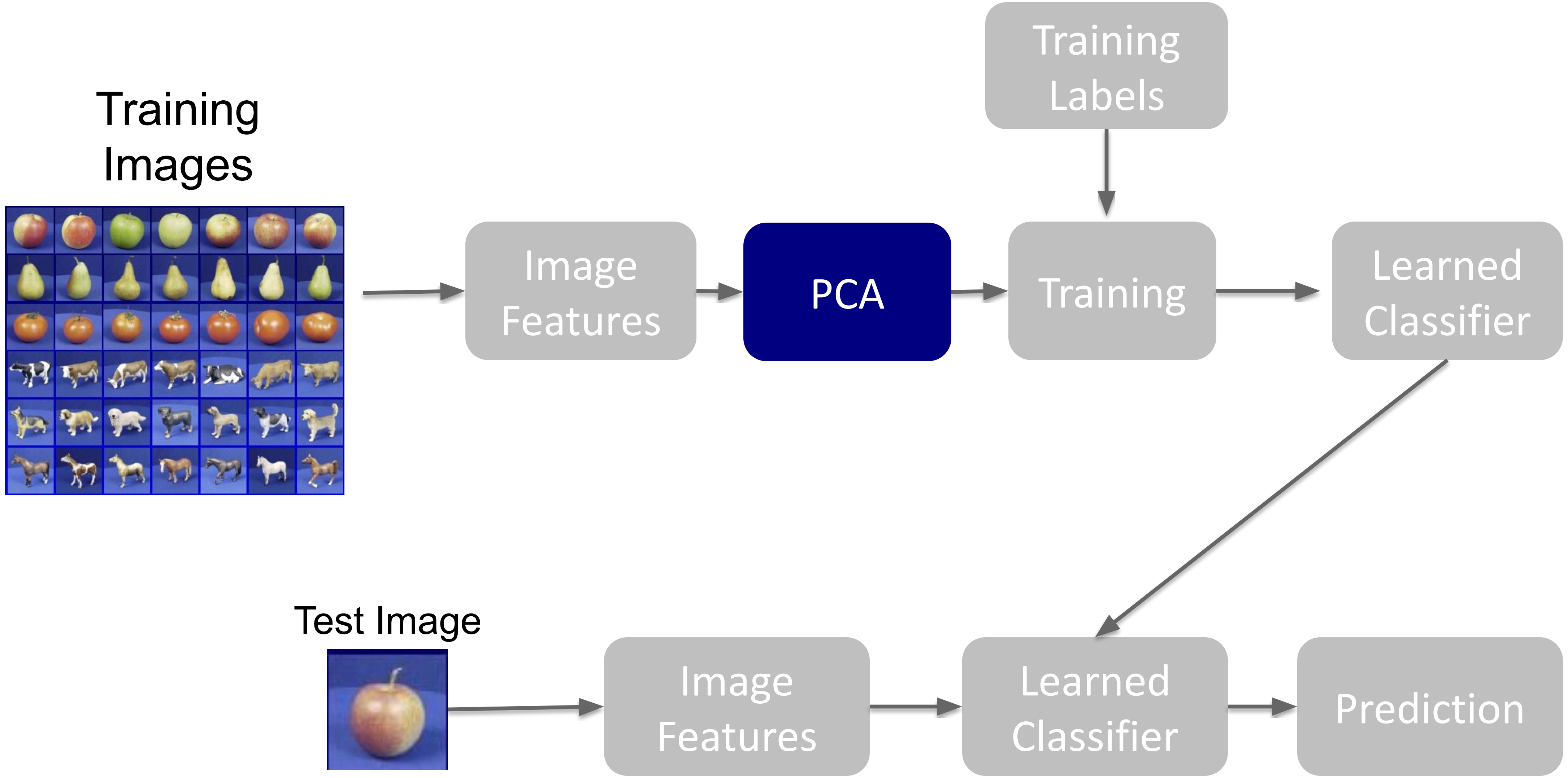
Covariance interpretation

- Exact value is not as important as it's sign.
- A **positive value** of covariance indicates both dimensions increase or decrease together e.g. as the number of hours studied increases, the marks in that subject increase.
- A **negative value** indicates while one increases the other decreases, or vice-versa e.g. active social life at PSU vs performance in CS dept.
- If **covariance is zero**: the two dimensions are independent of each other e.g. heights of students vs the marks obtained in a subject

What happens with PCA during training?



What happens with PCA during training?



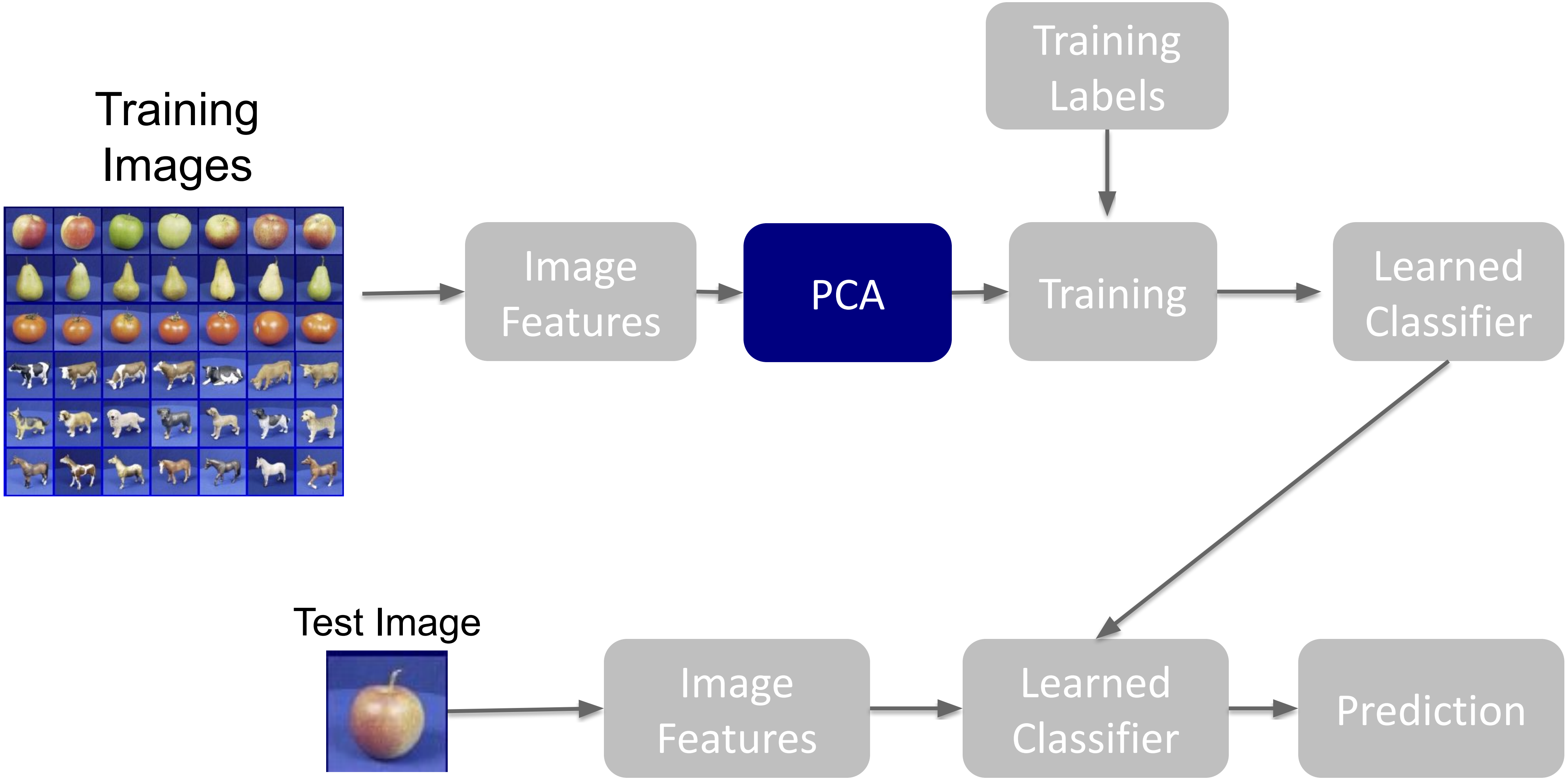
PCA during training

Let's say that we choose k top eigenvalues and their corresponding eigenvectors: $[u_1, \ldots, u_k]$

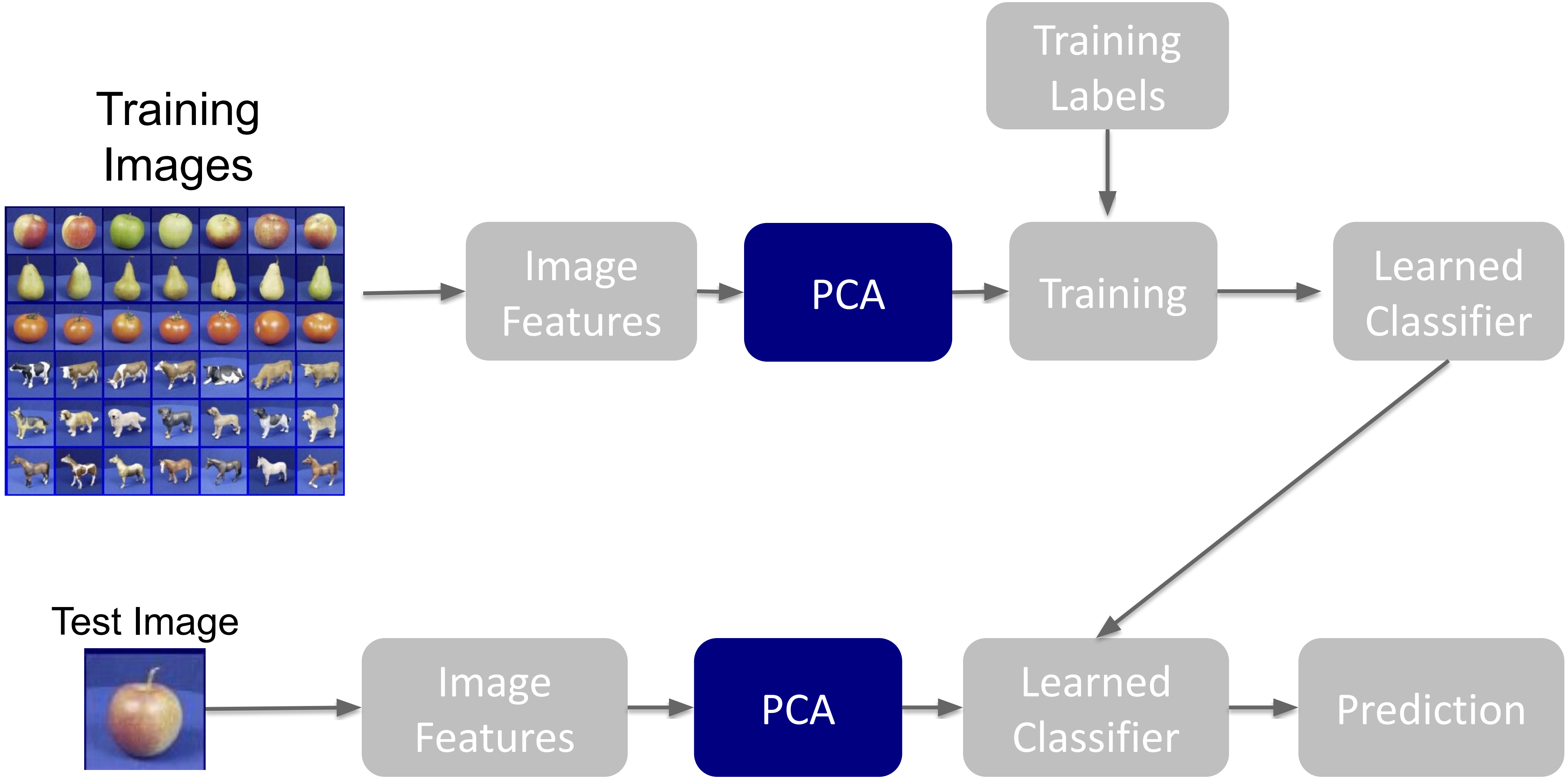
Replace all image features x with:

$$\hat{x} = \sum_{i=1}^{k} u_i^T x$$

What happens with PCA during testing?



What happens with PCA during testing?



What we have learned today?

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- A training algorithm: kNN
- Testing an algorithm
- Challenges with kNN
- Dimensionality reduction
- Principal Component Analysis (PCA)

Next lecture

Extra slides (out of scope)

for those of you curious about how SVD is calculated
and what PCA is usually used for outside of computer vision

Principal Component Analysis

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- Remember, columns of ***U*** are the *Principal Components* of the data: the major patterns that can be added to produce the columns of the original matrix
- One use of this is to construct a matrix where each column is a separate data sample
- Run SVD on that matrix, and look at the first few columns of ***U*** to see patterns that are common among the columns
- This is called *Principal Component Analysis* (or PCA) of the data samples

Principal Component Analysis

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

- Often, raw data samples have a lot of redundancy and patterns
- PCA can allow you to represent data samples as weights on the principal components, rather than using the original raw form of the data
- By representing each sample as just those weights, you can represent just the “meat” of what’s different between samples.
- This minimal representation makes machine learning and other algorithms much more efficient

How is SVD computed?

- For this class: tell PYTHON to do it. Use the result.
- But, if you're interested, one computer algorithm to do it makes use of Eigenvectors!

Eigenvector definition

- Suppose we have a square matrix **A**. We can solve for vector x and scalar λ such that Ax= λx
- In other words, find vectors where, if we transform them with **A**, the only effect is to scale them with no change in direction.
- These vectors are called eigenvectors (German for “self vector” of the matrix), and the scaling factors λ are called eigenvalues
- An *m* x *m* matrix will have ≤ *m* eigenvectors where λ is nonzero

Finding eigenvectors

- Computers can find an x such that Ax= λx using this iterative algorithm:

    - X = random unit vector
    - while(x hasn't converged)
        - X = Ax
        - normalize x

- x will quickly converge to an eigenvector
- Some simple modifications will let this algorithm find all eigenvectors

Finding SVD

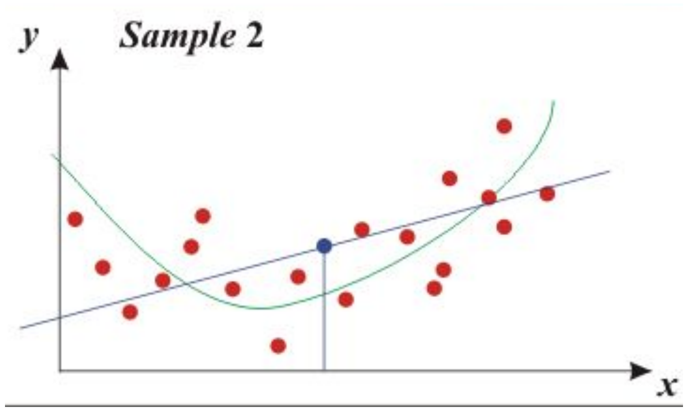
- Eigenvectors are for square matrices, but SVD is for all matrices
- To do svd(A), computers can do this:
  - Take eigenvectors of AA^T (matrix is always square).
    - These eigenvectors are the columns of **U**.
    - Square root of eigen*values* are the singular values (the entries of $\mathbf{\Sigma}$).
  - Take eigenvectors of A^TA (matrix is always square).
    - These eigenvectors are columns of **V** (or rows of $\mathbf{V^T}$)

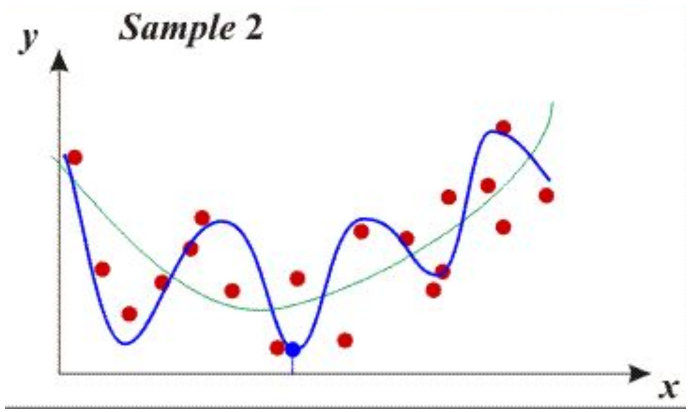
Finding SVD

- Moral of the story: SVD is fast, even for large matrices
- It’s useful for a lot of stuff
- There are also other algorithms to compute SVD or part of the SVD
    - Python’s np.linalg.svd() command has options to efficiently compute only what you need, if performance becomes an issue

A detailed geometric explanation of SVD is here:
http://www.ams.org/samplings/feature-column/fcarc-svd

Bias-Variance Trade-off





- Models with too few parameters are inaccurate because of a large bias (not enough flexibility).

- Models with too many parameters are inaccurate because of a large variance (too much sensitivity to the sample).

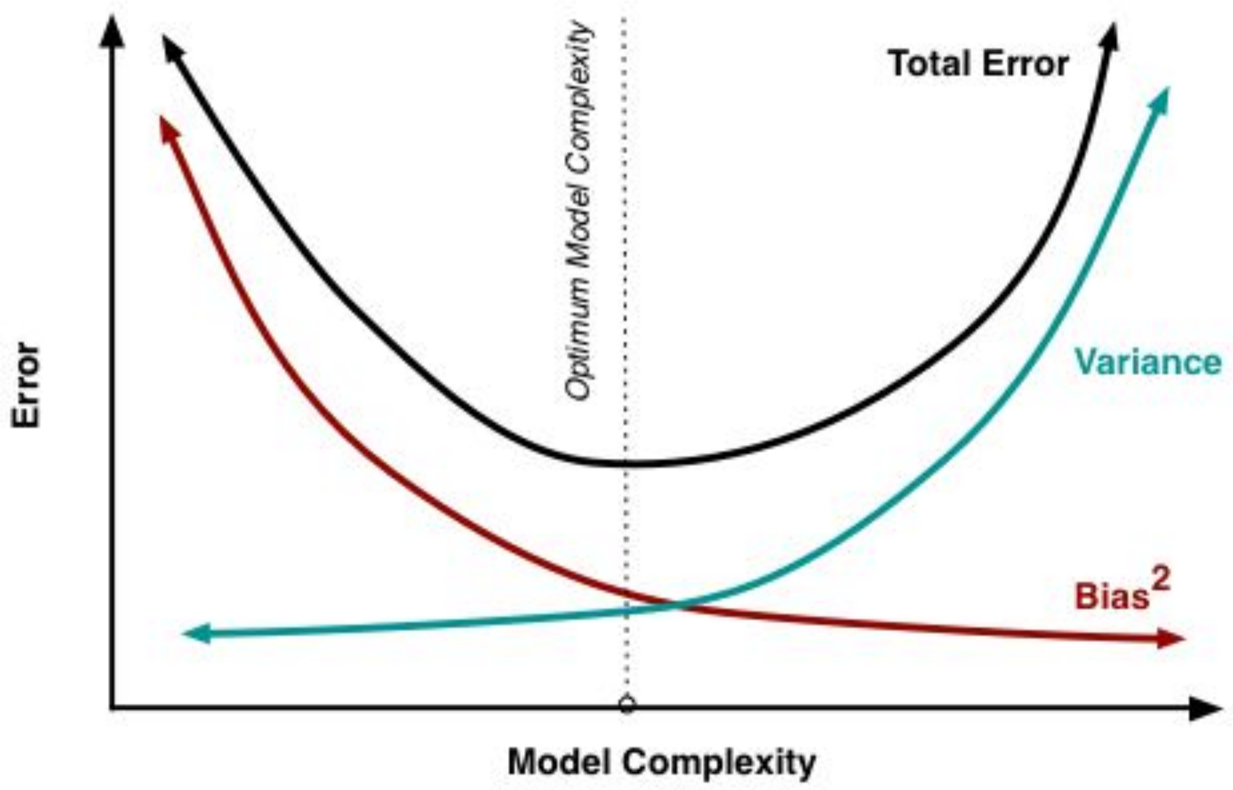
Slide credit: D. Hoiem

Bias versus variance

- Components of generalization error
  - **Bias:** how much the average model over all training sets differ from the true model?
    - Error due to inaccurate assumptions/simplifications made by the model
  - **Variance:** how much models estimated from different training sets differ from each other
- **Underfitting:** model is too “simple” to represent all the relevant class characteristics
  - High bias and low variance
  - High training error and high test error
- **Overfitting:** model is too “complex” and fits irrelevant characteristics (noise) in the data
  - Low bias and high variance
  - Low training error and high test error

Slide credit: L. Lazebnik

Bias versus variance trade off



No Free Lunch Theorem

In a supervised learning setting, we can't tell which classifier will have best generalization

Remember…

- No classifier is inherently better than any other: you need to make assumptions to generalize

- Three kinds of error
  - Inherent: unavoidable
  - Bias: due to over-simplifications
  - Variance: due to inability to perfectly estimate parameters from limited data

Slide credit: D. Hoiem

How to reduce variance?

- Choose a simpler classifier
- Regularize the parameters
- Get more training data

How do you reduce bias?

Slide credit: D. Hoiem

Last remarks about applying machine learning methods to object recognition

- There are machine learning algorithms to choose from
- Know your data:
  - How much supervision do you have?
  - How many training examples can you afford?
  - How noisy?
- Know your goal (i.e. task):
  - Affects your choices of representation
  - Affects your choices of learning algorithms
  - Affects your choices of evaluation metrics
- Understand the math behind each machine learning algorithm under consideration!

PCA by SVD

- An alternative manner to compute the principal components, based on singular value decomposition
- Quick reminder: SVD
  - Any real n x m matrix (n>m) can be decomposed as

$$A = M\Pi N^T$$

  - Where M is an (n x m) column orthonormal matrix of left singular vectors (columns of M)
  - Π is an (m x m) diagonal matrix of singular values
  - N^T is an (m x m) row orthonormal matrix of right singular vectors (columns of N)

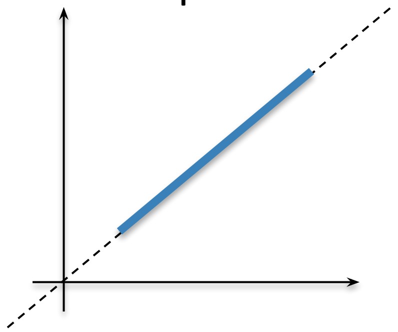
$$M^T M = I \qquad N^T N = I$$

Slide inspired by N. Vasconcelos

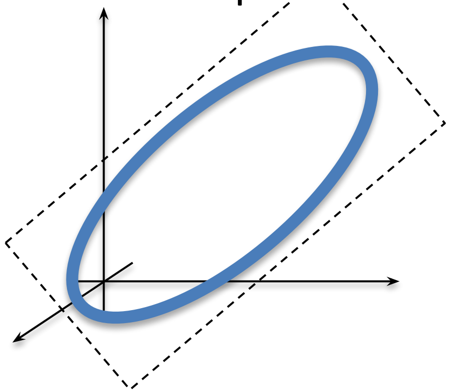
PCA Formulation

- Basic idea:
  - If the images (or their features) live in a subspace, it is going to look very flat when viewed from the full feature space, e.g.

1D subspace in 2D



2D subspace in 3D



Slide inspired by N. Vasconcelos

Alternative PCA Formulation

- Assume images x is Gaussian with covariance Σ.

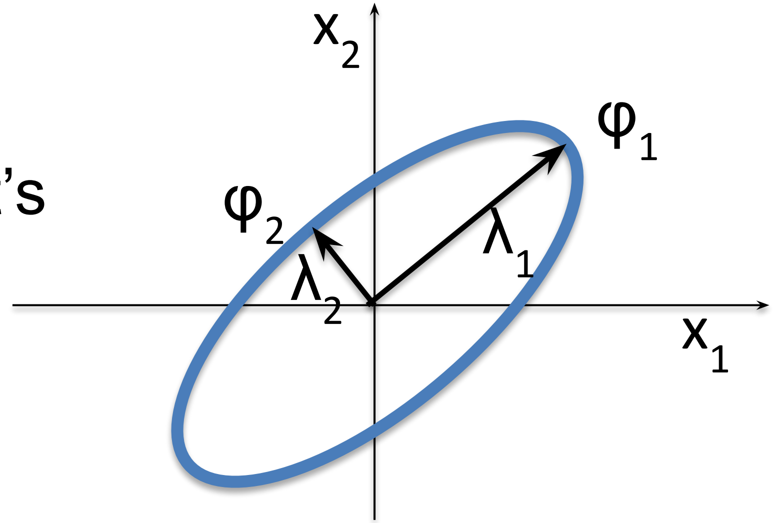
- Recall that a gaussian is defined with it's mean and variance:

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Recall that μ and Σ of a gaussian are defined as:

$$\boldsymbol{\mu} = \mathbf{E}[\mathbf{X}]$$

$$\boldsymbol{\Sigma} =: \mathbf{E}[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}}] = [\mathrm{Cov}[X_i, X_j]; 1 \le i, j \le k]$$



Alternative PCA formulation

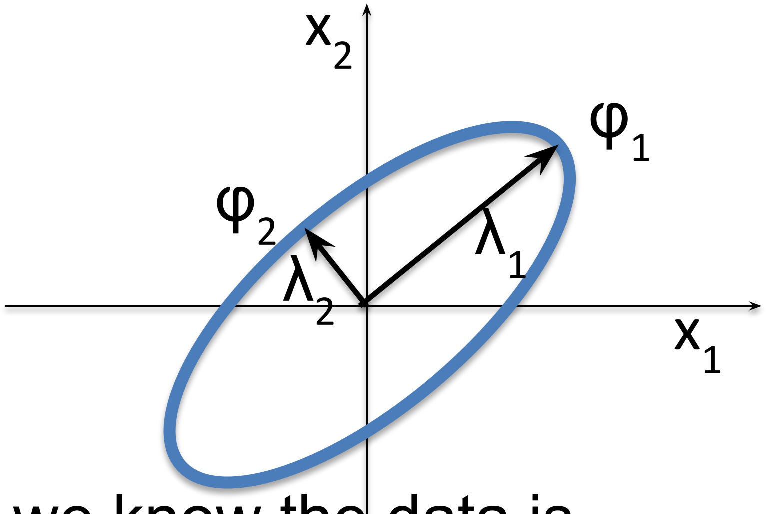
- Since gaussians are symmetric, it's covariance matrix is also a symmetric matrix. So we can express it as:
  - $\mathbf{\Sigma} = \mathbf{U\Lambda U}^T = \mathbf{U\Lambda}^{1/2}(\mathbf{U\Lambda}^{1/2})^T$

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma}) \iff \mathbf{X} \sim \boldsymbol{\mu} + \mathbf{U}\mathbf{\Lambda}^{1/2}\mathcal{N}(0, \mathbf{I})$$

$$\iff \mathbf{X} \sim \boldsymbol{\mu} + \mathbf{U}\mathcal{N}(0, \mathbf{\Lambda}).$$

Alternative PCA Formulation

- If x is Gaussian with covariance Σ,
  - Principal components φ_i are the eigenvectors of Σ
  - Principal lengths λ_i are the eigenvalues of Σ



- by computing the eigenvalues we know the data is
  - Not flat if $\lambda_1 \approx \lambda_2$
  - Flat if $\lambda_1 >> \lambda_2$

Alternative PCA Algorithm (training)

- Given sample $\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}, \; x_i \in \mathcal{R}^d$
  - compute sample mean: $\hat{\mu} = \frac{1}{n}\sum_i(\mathbf{x}_i)$
  - compute sample covariance: $\hat{\Sigma} = \frac{1}{n}\sum_i(\mathbf{x}_i - \hat{\mu})(\mathbf{x}_i - \hat{\mu})^T$
  - compute eigenvalues and eigenvectors of $\hat{\Sigma}$

$$\hat{\Sigma} = \Phi\Lambda\Phi^T, \;\; \Lambda = diag(\sigma_1^2, \ldots, \sigma_n^2) \;\; \Phi^T\Phi = I$$

  - order eigenvalues $\sigma_1^2 > \ldots > \sigma_n^2$
  - if, for a certain k, $\sigma_k << \sigma_1$ eliminate the eigenvalues and eigenvectors above k.

Alternative PCA Algorithm (testing)

- Given principal compoeenents $\phi_i, i \in 1, \ldots, k$ and a test sample $\mathcal{T} = \{\mathbf{t}_1, \ldots, \mathbf{t}_n\},\ \ t_i \in \mathcal{R}^d$
  - subtract mean to each point $\mathbf{t}'_i = \mathbf{t}_i - \widehat{\mu}$
  - project onto eigenvector space $\mathbf{y}_i = \mathbf{A}\mathbf{t}'_i$ where

$$\mathbf{A} = \begin{bmatrix} \phi_1^T \\ \vdots \\ \phi_k^T \end{bmatrix}$$

  - use $\mathcal{T}' = \{\mathbf{y}_1, \ldots \mathbf{y}_n\}$ to estimate class conditional densities and do all further processing on **y**.