Lecture 3

Pixels and Filters

Administrative

A0 is out.
- It is upgraded
- Meant to help you with python and numpy basics
- Learn how to do homeworks and submit them on gradescope.

Administrative

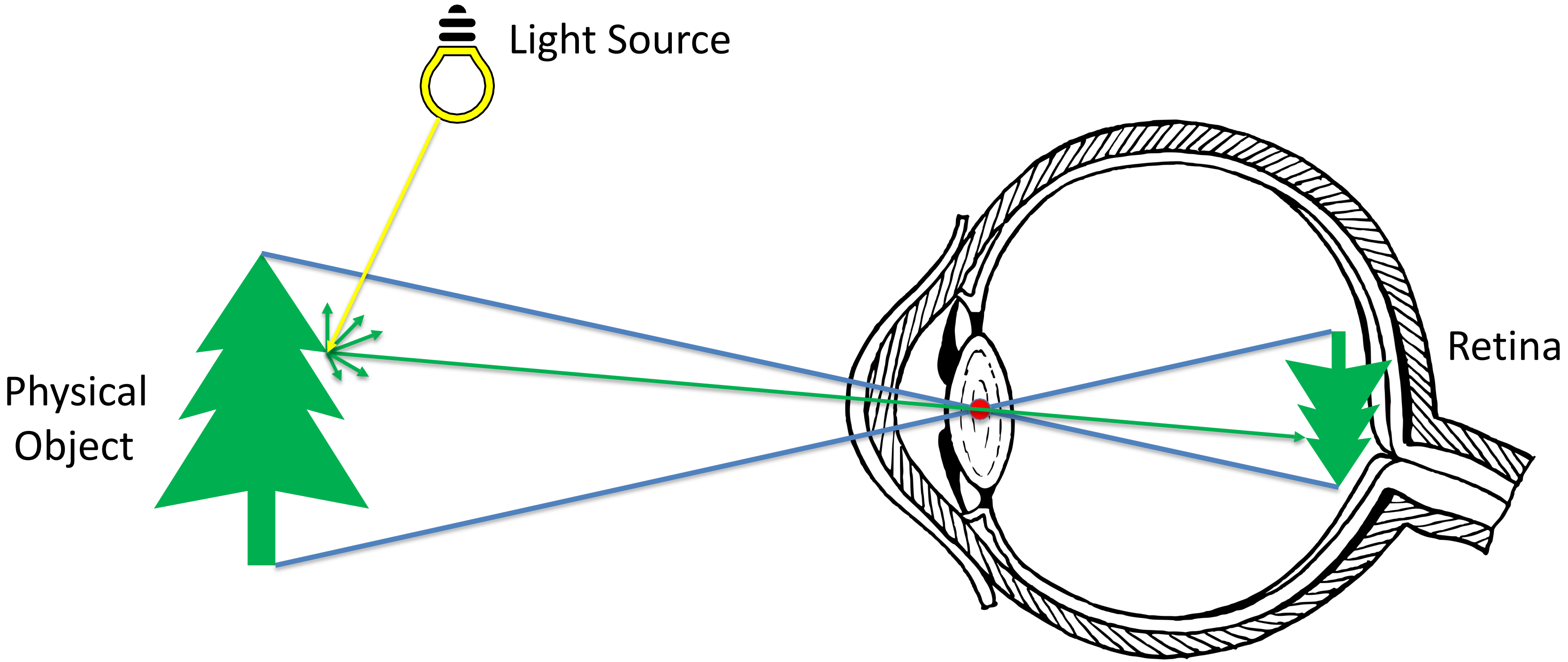
Recitation sections on fridays

- (optional)
- Fridays 12:30pm-1:20pm
- JHU 102
- It will be recorded

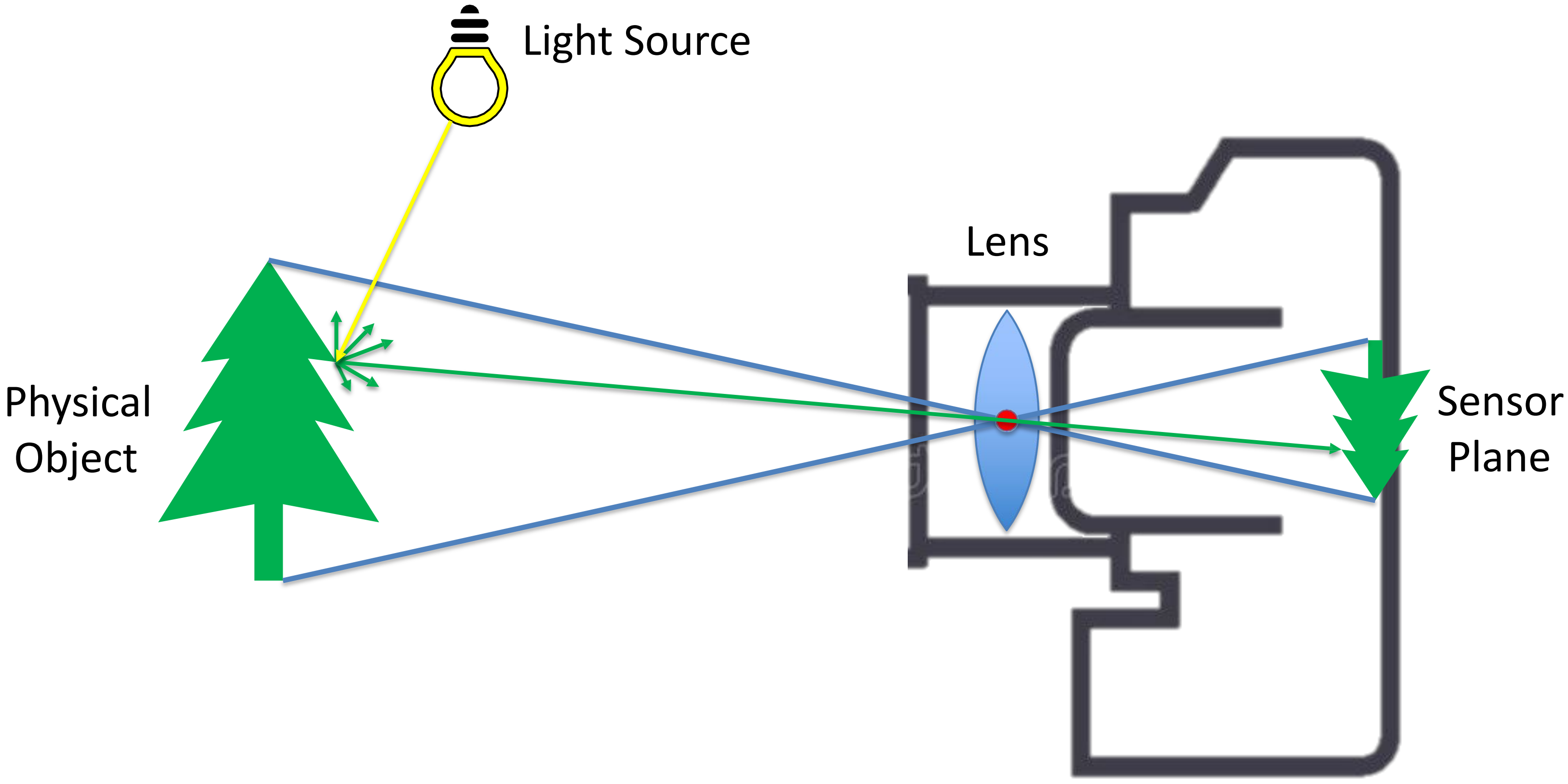
This week:

We will go over Python & Numpy basics

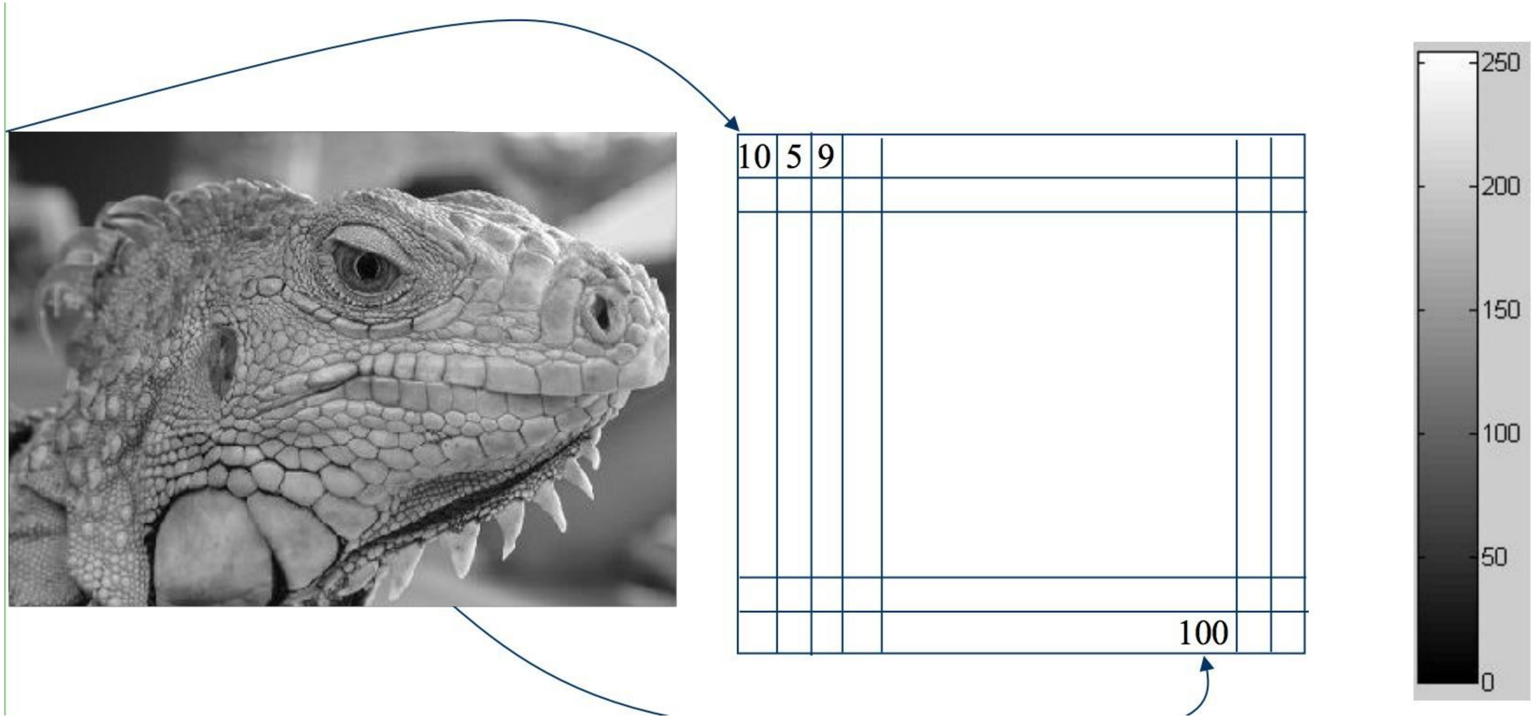
So far: The lossy process through which we see



So far: The lossy process through which images form



So far: Grayscale images can be represented as matrices



So far: Color image can be represented as HxWxC tensors

B channel

G channel

R channel

What we will learn today?

- Image histograms
- Images as functions
- Filters
- Properties of systems

Some background reading:
Forsyth and Ponce, Computer Vision, Chapter 7

What we will learn today?

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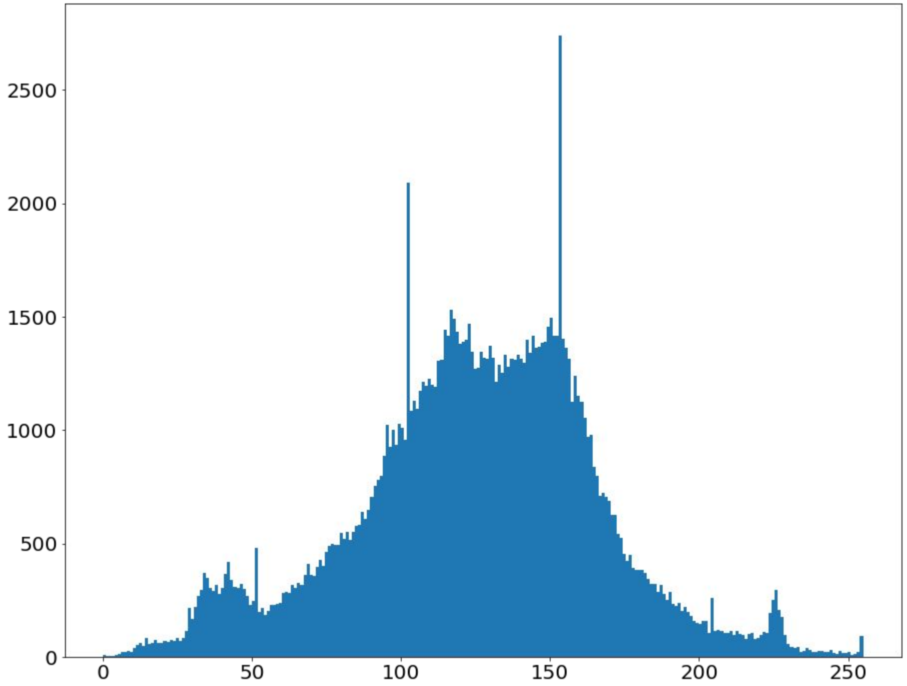
Starting with grayscale images:

- Histogram captures the distribution of gray levels in the image.
- How frequently each gray level occurs in the image



Grayscale histograms in code

- Histogram of an image provides the frequency of the brightness (intensity) value in the image.

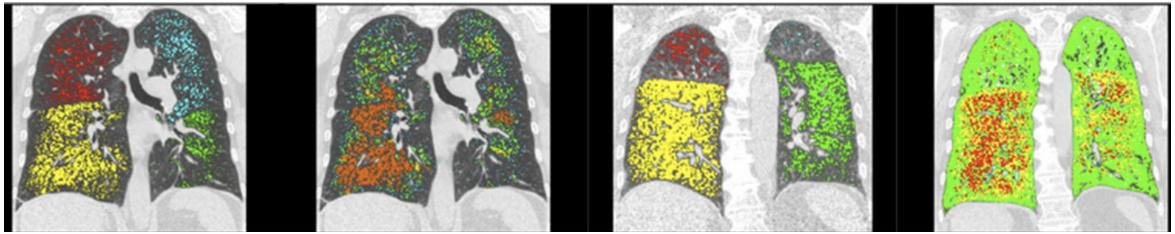
Here is an efficient implementation of calculating histograms:

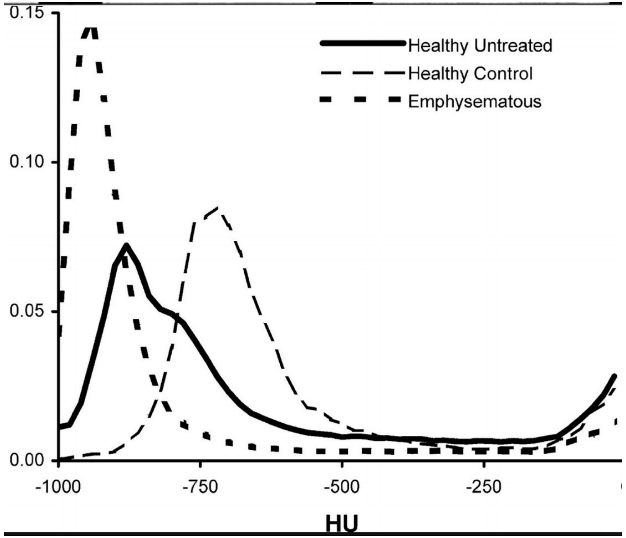
```
def histogram(im):
    h = np.zeros(255)
    for row in im.shape[0]:
        for col in im.shape[1]:
            val = im[row, col]
            h[val] += 1
```

Visualizing Histograms for patches

Slide credit: Dr. Mubarak Shah

Histogram – use case

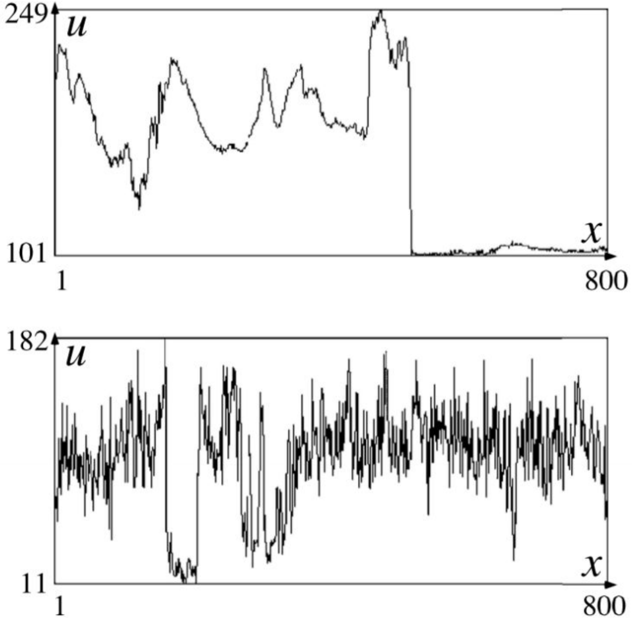




Slide credit: Dr. Mubarak Shah

Another type of Histogram

x axis represents the width of the image.
Each histogram represents the intensity at a specific height of the image
Regions with sharp changes are likely to contain objects



Histograms are a convenient representation to extract information

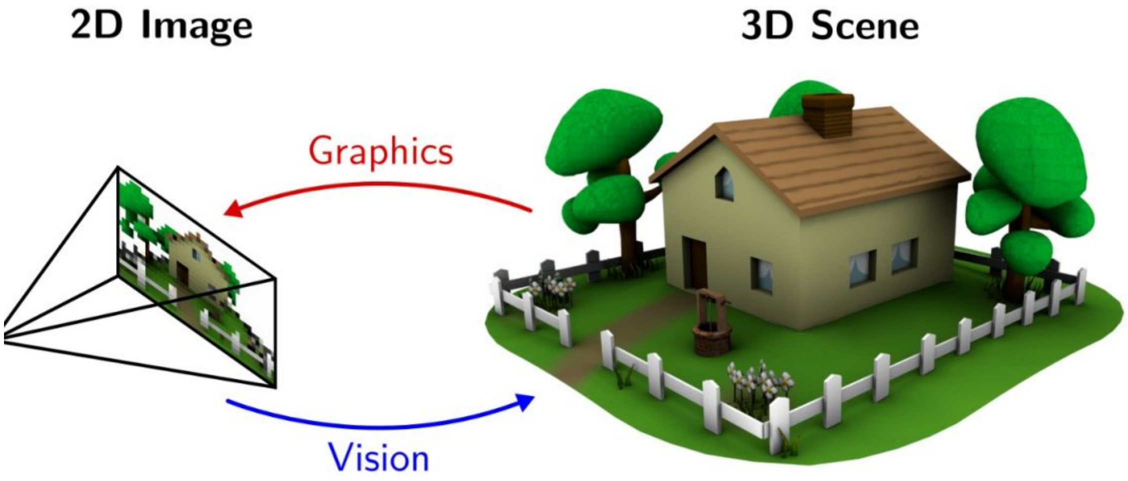
Can we develop better transformations than histograms?

What we will learn today?

- Image histograms
- Images as functions
- Filters
- Properties of systems

Images are a function!!!

This is a new formalism that will allow us to borrow ideas from signal processing to extract meaningful information.



At every pixel location, we get an intensity value for that pixel.

The world captured by the image continues beyond that confines of the image

Images as discrete functions

- Digital images are usually discrete:
  - **Sample** the 2D space on a regular grid

- Represented as a matrix of integer values

pixel intensity

m

n

62	79	23	119	120	05	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

Images as discrete function f

- The input to the image function is a pixel location, [n m]
- The output to the image function is the pixel intensity

pixel intensity

m

n

62	79	23	119	120	05	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
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255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

Images as discrete function f

- The input to the image function is a pixel location, [n m]
- The output to the image function is the pixel intensity

Q1. What is f[0, 0]?

pixel intensity

m

n

62	79	23	119	120	05	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
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166	63	127	17	1	0	99	30

Images as discrete function f

- The input to the image function is a pixel location, [n m]
- The output to the image function is the pixel intensity

Q2. What is f[0, 0]?

pixel intensity

m

n

62	79	23	119	120	05	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
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Images as discrete function f

- The input to the image function is a pixel location, [n m]
- The output to the image function is the pixel intensity

Q2. What is f[0, 4]?

pixel intensity

m

n

62	79	23	119	120	05	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

Images as coordinates

We can represent this function as f.

f[n, m] represents the pixel intensity at that value.

$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \dots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \dots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

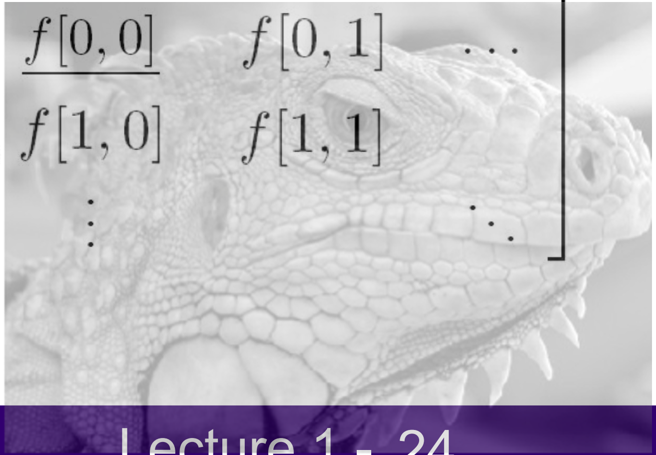
Notation for discrete functions

n and *m* can be any real valued numbers.

Even negative!!

We don’t have the intensity values for negative indices

$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \dots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \dots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

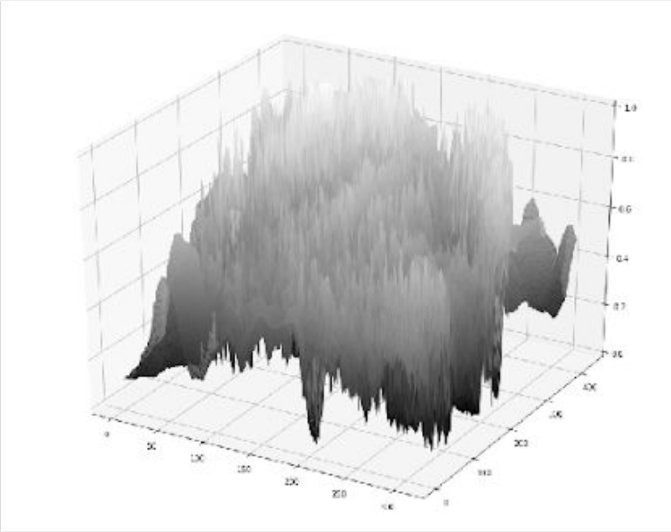


n and *m* can be any real valued numbers.

Even negative!!

Images as functions

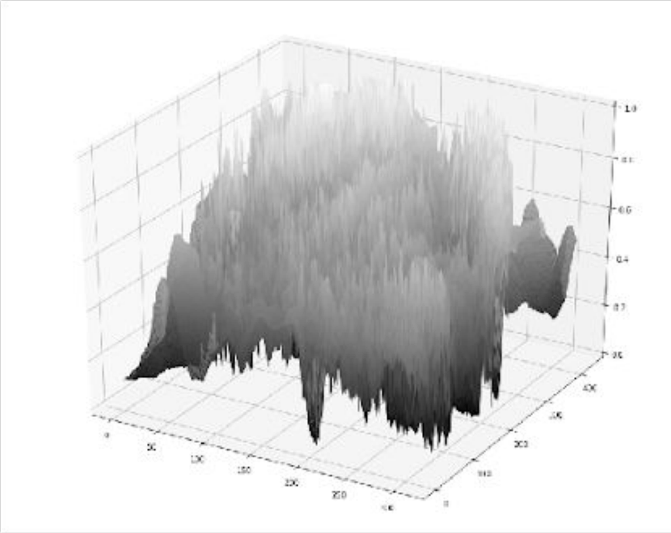
- **An Image** as a function *f* from R^2 to R^C:
  - if grayscale then C=1, if color then C=3



Images as functions

- **An Image** as a function *f* from R^2 to R^C:
  - if grayscale, C=1, if color, C=3
  - f [n, m] gives the intensity at position [n, m]
  - Has values over a rectangle, with a finite range:

$$\underbrace{f: [0,H] \times [0,W]}_{\text{Domain support}} \rightarrow \underbrace{[0,255]}_{\text{range}}$$



Images as functions

- **An Image** as a function *f* from R^2 to R^C:
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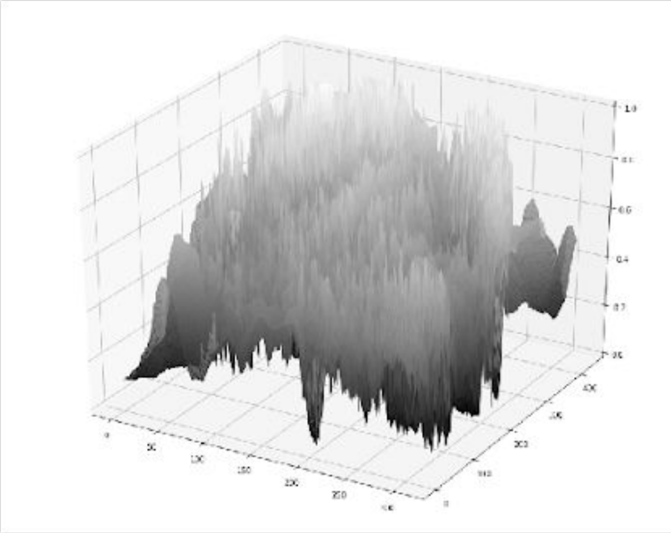
  - f [n, m] gives the intensity at position [n, m]
  - Has values over a rectangle, with a finite range:

    $f: [0,H] \times [0,W] \rightarrow [0,255]$

    Domain support ($[0,H] \times [0,W]$) — range ($[0,255]$)



- Doesn't have values outside of the image rectangle

  $f: [-inf,inf] \times [-inf,inf] \rightarrow [0,255]$

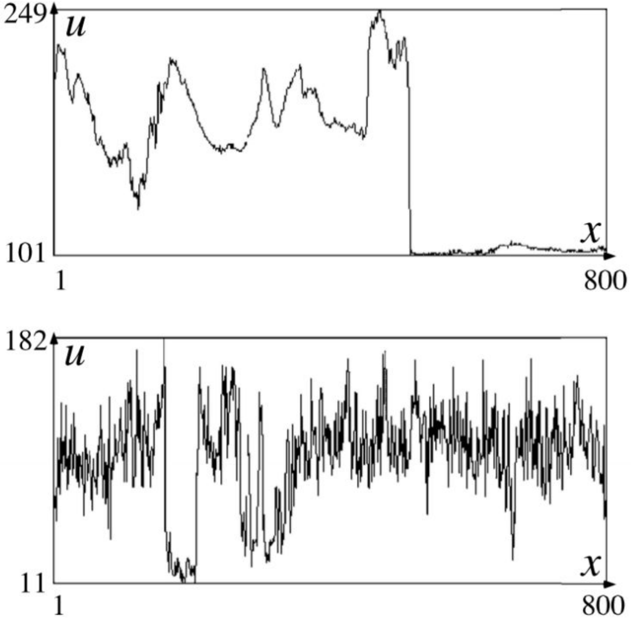
- we assume that f[n, m] = 0 outside of the image rectangle

Images as functions

- **An Image** as a function *f* from R^2 to R^C:
  - f [n, m] gives the intensity at position [n, m]
  - Defined over a rectangle, with a finite range:

$$f: \underbrace{[a,b] \times [c,d]}_{\text{Domain support}} \rightarrow \underbrace{[0,255]}_{\text{range}}$$

Histograms are a type of image function



What we will learn today?

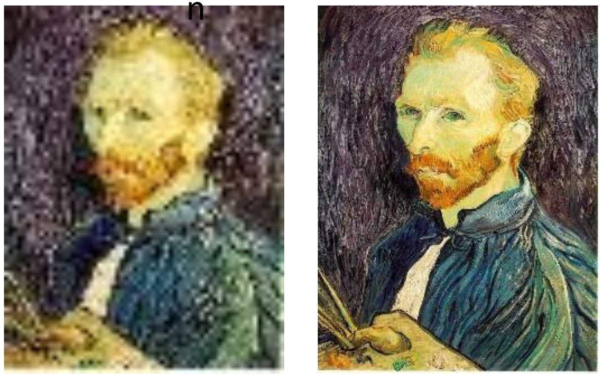
- Image histograms
- Images as functions
- Filters
- Properties of systems

Applications of linear systems or filters

De-noising

Salt and pepper noise

Super-resolution



In-painting

Bertamio et al

Systems and Filters

Filtering:

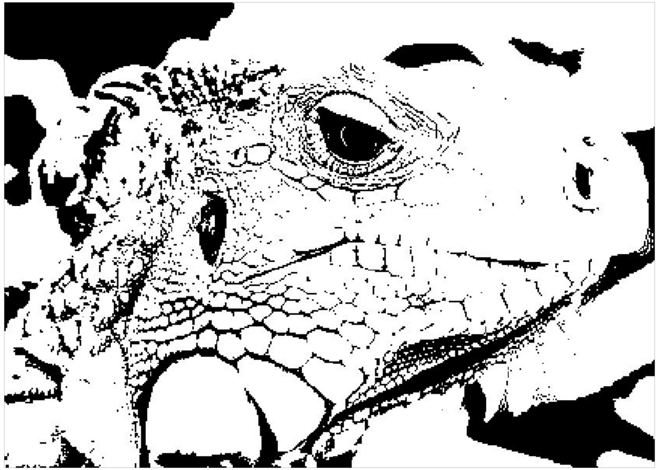
- Forming a new image whose pixel values are transformed from original pixel values

Goals of filters:

- Goal is to extract useful information from images, or transform images into another domain where we can modify/enhance image properties
  - Features (edges, corners, blobs…)
  - super-resolution; in-painting; de-noising

Intuition behind linear systems

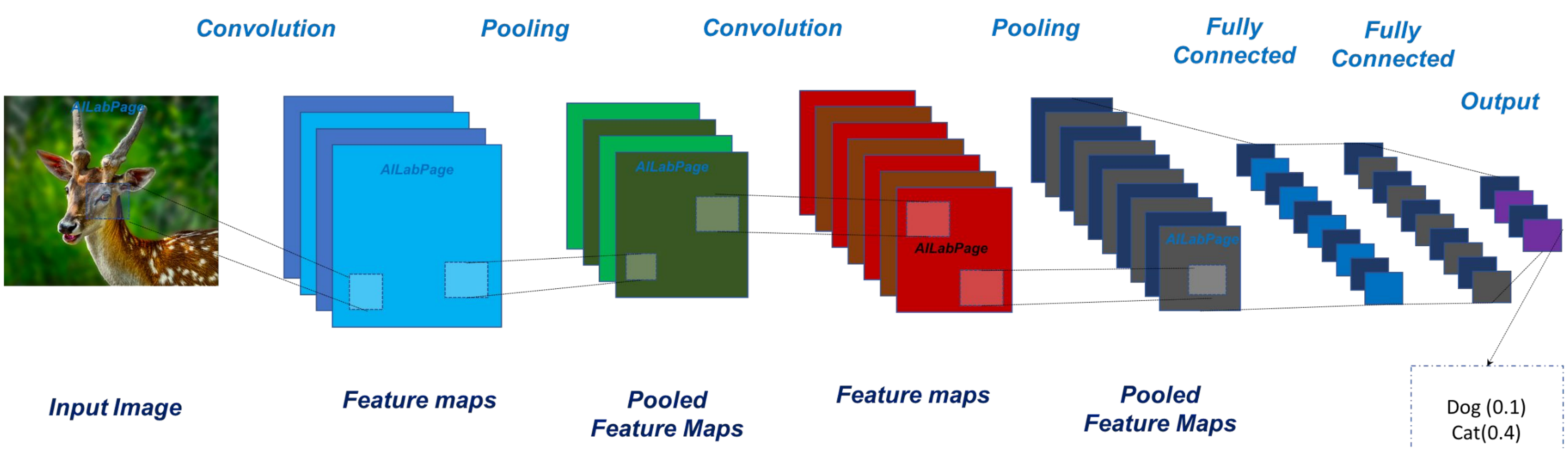
- We will view linear systems as a type of function that operates over images
- Translating an image or multiplying by a constant leaves the semantic content intact
  - but can reveal interesting patterns



Aside

- Neural networks and specifically **convolutional** neural networks are a sequence of filters (except they are a non-linear system) that contains multiple individual linear sub-systems.

- (we will learn more about this later in class)



Systems use Filters

- we define a system as a unit that converts an input function f[n,m] into an output (or response) function g[n,m], where (n,m) are the independent variables.
  - In the case for images, (n,m) represents the **spatial position in the image.**

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

Images produce a 2D matrix with pixel intensities at every location

$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[0,-1] & f[1,-1] & \\ \dots & f[-1,0] & \underline{f[0,0]} & f[1,0] & \dots \\ & f[-1,1] & f[0,1] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

Notation for discrete functions

2D discrete system (filters)

S is the **system operator**, defined as a mapping or assignment of possible inputs f[n,m] to some possible outputs g[n,m].

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

2D discrete system (filters)

S is the **system operator**, defined as a mapping or assignment of possible inputs f[n,m] to some possible outputs g[n,m].

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

Other notations:

$$g = \mathcal{S}[f], \quad g[n,m] = \mathcal{S}\{f[n,m]\}$$

$$f[n,m] \xrightarrow{\mathcal{S}} g[n,m]$$

Filter example #1: Moving Average

Original image

Q. What do you think will happen to the photo if we use a moving average filter?

Assume that the moving average replaces each pixel with an average value of itself and all its neighboring pixels.

Filter example #1: Moving Average

Original image

Smoothed image

Visual interpretation of moving average

A moving average over a 3 × 3 neighborhood window

h is a 3x3 matrix with values 1/9 everywhere.

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

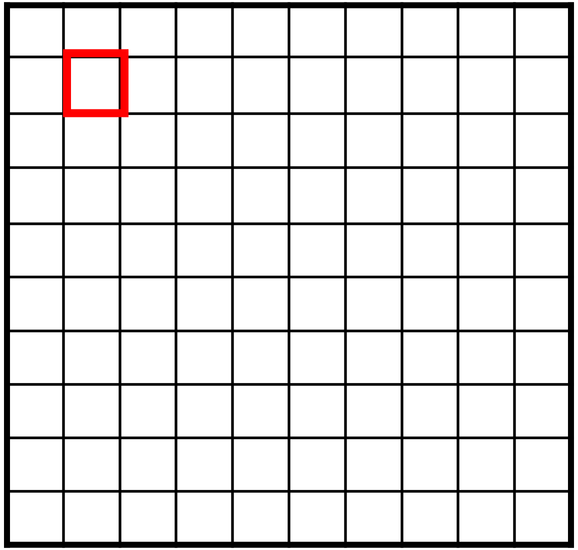
Visualizing what happens with a moving average filter

The red box is the **h** matrix

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$



Courtesy of S. Seitz

Visualizing what happens with a moving average filter

$f[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n, m]$

	0	10							

Visualizing what happens with a moving average filter

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$

	0	10	20						

Visualizing what happens with a moving average filter

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$

	0	10	20	30					

Visualizing what happens with a moving average filter

f[n, m]

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

g[n, m]

	0	10	20	30	30				

Visualizing what happens with a moving average filter

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$

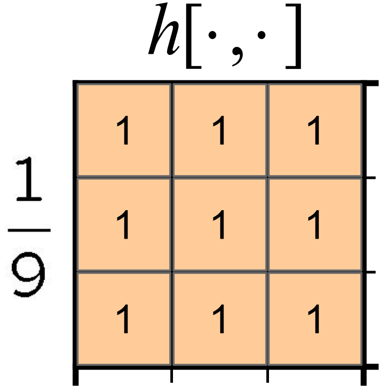
	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Visual interpretation of moving average

A moving average over a 3 × 3 neighborhood window

h is a 3x3 matrix with values 1/9 everywhere.

Q. Why are the values 1/9?



Filter example #1: Moving Average

In summary:

- This filter "Replaces" each pixel with an average of its neighborhood.

- Achieve smoothing effect (remove sharp features)

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Mathematical interpretation of moving average

How do we represent applying this filter mathematically?

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Mathematical interpretation of moving average

How do we represent applying this filter mathematically?

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$g[n,m] = \frac{1}{9} \sum_{k=??}^{??} \sum_{l=??}^{??} f[k,l]$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q. What values will **k** take?

Mathematical interpretation of moving average

How do we represent applying this filter mathematically?

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=??}^{??} f[k,l]$$

$$h[\cdot,\cdot] = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

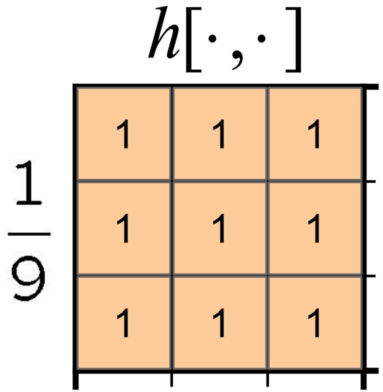
k goes from n-1 to n+1

Mathematical interpretation of moving average

How do we represent applying this filter mathematically?

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=??}^{??} f[k,l]$$



Q. What values will **l** take?

Mathematical interpretation of moving average

How do we represent applying this filter mathematically?

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=m-1}^{m+1} f[k,l]$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

l goes from m-1 to m+1

Math formula for the moving average filter

A moving average over a 3 × 3 neighborhood window

We can write this operation mathematically:

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=m-1}^{m+1} f[k,l]$$

$$h[\cdot,\cdot] = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Rewriting this formula

We are almost done. Let's rewrite this formula a little bit

Let $k' = n - k$

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=m-1}^{m+1} f[k,l]$$

$$h[\cdot,\cdot] \quad \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

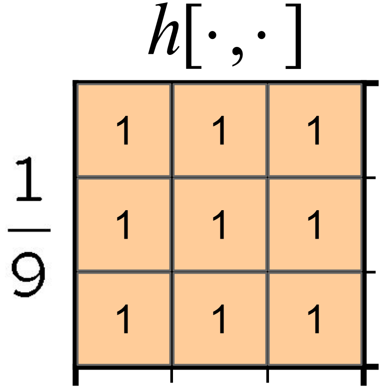
Rewriting this formula

We are almost done. Let's rewrite this formula a little bit

Let $k' = n - k$

therefore, $k = n - k'$

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=m-1}^{m+1} f[k,l]$$



Now we can replace k in the equation above

Rewriting this formula

We are almost done. Let's rewrite this formula a little bit

Let $k' = n - k$

therefore, $k = n - k'$

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=m-1}^{m+1} f[k,l]$$

$$g[n,m] = \frac{1}{9} \sum_{n-k'=n-1}^{n-k'=n+1} \sum_{l=m-1}^{m+1} f[n-k',l]$$

$h[\cdot,\cdot]$

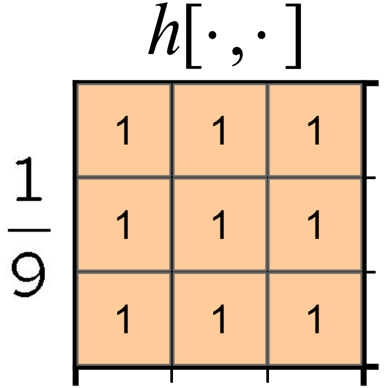
$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

Rewriting this formula

So now we have this:

$$g[n,m] = \frac{1}{9} \sum_{n-k'=n-1}^{n-k'=n+1} \sum_{l=m-1}^{m+1} f[n-k', l]$$



Rewriting this formula

So now we have this:

$$g[n,m] = \frac{1}{9} \sum_{\color{red}{n-k'=n-1}}^{\color{red}{n-k'=n+1}} \sum_{l=m-1}^{m+1} f[n-k',l]$$

We can simply the equations in red:

$$g[n,m] = \frac{1}{9} \sum_{\color{red}{k'=1}}^{\color{red}{k'=-1}} \sum_{l=m-1}^{m+1} f[n-k',l]$$

$h[\cdot,\cdot]$

$\frac{1}{9}$

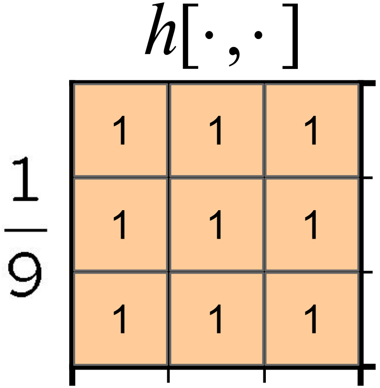
1	1	1
1	1	1
1	1	1

Rewriting this formula

So now we have this:

$$g[n,m] = \frac{1}{9} \sum_{k'=1}^{k'=-1} \sum_{l=m-1}^{m+1} f[n-k', l]$$

Remember that summations are just for-loops!!



Rewriting this formula

So now we have this:

$$g[n,m] = \frac{1}{9} \sum_{k'=1}^{k'=-1} \sum_{l=m-1}^{m+1} f[n-k', l]$$

Remember that summations are just for-loops!!

$$g[n,m] = \frac{1}{9} \sum_{k'=-1}^{1} \sum_{l=m-1}^{m+1} f[n-k', l]$$

$h[\cdot,\cdot]$

$\frac{1}{9}$

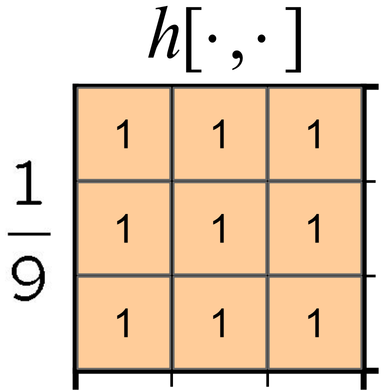
1	1	1
1	1	1
1	1	1

Rewriting this formula

One last change: since there are no more k and only k',
let's just write k' as k

$$g[n,m] = \frac{1}{9} \sum_{k'=-1}^{1} \sum_{l=m-1}^{m+1} f[n-k', l]$$

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=m-1}^{m+1} f[n-k, l]$$



Mathematical interpretation of moving average

Let’s repeat for l, just like we did for k

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=m-1}^{m+1} f[k,l]$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

$$h[\cdot,\cdot] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Filter example #1: Moving Average

Original image

Smoothed image

Filter example #2: Image Segmentation

Q. How would you use pixel values to design a filter to segment an image so that you only keep around the edges?

Filter example #2: Image Segmentation

- Use a simple pixel threshold:

$$g[n,m] = \begin{cases} 255, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$

Summary so far

- Discrete Systems convert input discrete signals and convert them into something more meaningful.
- There are an infinite number of possible filters we can design.
- What are ways we can category the space of possible systems?

What we will learn today?

- Image histograms
- Images as functions
- Filters
- Properties of systems

Properties of systems

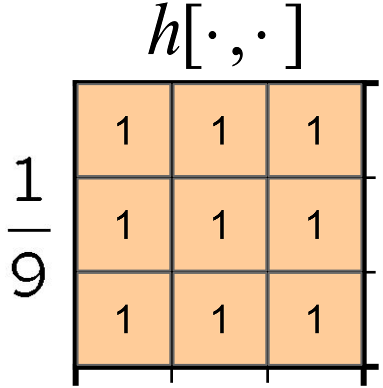
- Amplitude properties:
  - Additivity

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Example question:

Q. Is the moving average filter additive?

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$



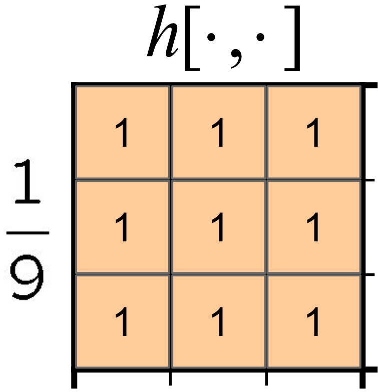
How would you prove it?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

Example question:

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$



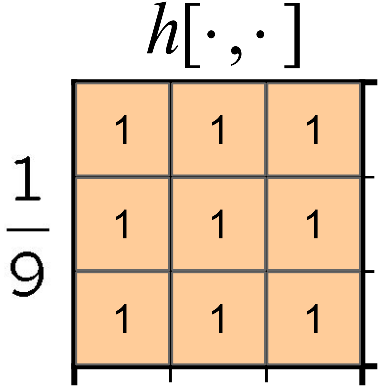
$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

Example question:

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f'[n,m]]$$



$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Example question:

$$h[\cdot\,,\cdot\,] \quad \frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

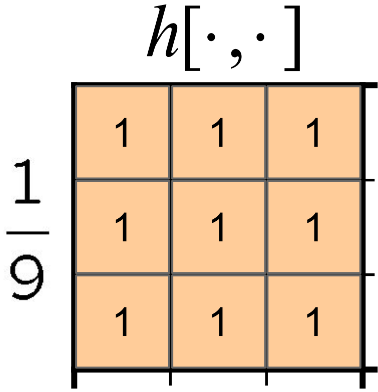
$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$\begin{aligned}\mathcal{S}[f_i[n,m] + f_j[n,m]] &= \mathcal{S}[f'[n,m]] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k, m-l]\end{aligned}$$

Example question:



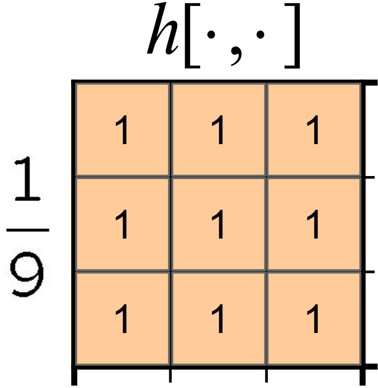
$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$\begin{aligned}
\mathcal{S}[f_i[n,m] + f_j[n,m]] &= \mathcal{S}[f'[n,m]] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k, m-l] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} [f_i[n-k, m-l] + f_j[n-k, m-l]]
\end{aligned}$$

Example question:



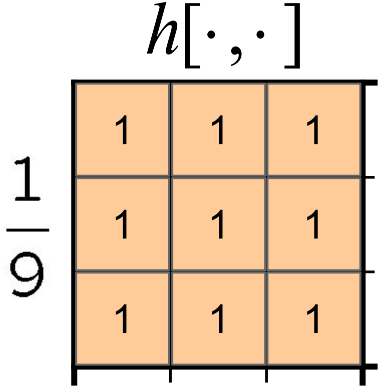
$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k,m-l]$$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$\begin{aligned}\mathcal{S}[f_i[n,m] + f_j[n,m]] &= \mathcal{S}[f'[n,m]] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k,m-l] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} [f_i[n-k,m-l] + f_j[n-k,m-l]] \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_i[n-k,m-l] + \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_j[n-k,m-l]]\end{aligned}$$

Example question:



$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Let $f'[n,m] = f_i[n,m] + f_j[n,m]$

$$\begin{aligned}
\mathcal{S}[f_i[n,m] + f_j[n,m]] &= \mathcal{S}[f'[n,m]] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f'[n-k, m-l] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} [f_i[n-k, m-l] + f_j[n-k, m-l]] \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_i[n-k, m-l] + \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f_j[n-k, m-l]] \\
&= \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]
\end{aligned}$$

Properties of systems

- Amplitude properties:
    - Additivity

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

Properties of systems

- Amplitude properties:
  - Additivity

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

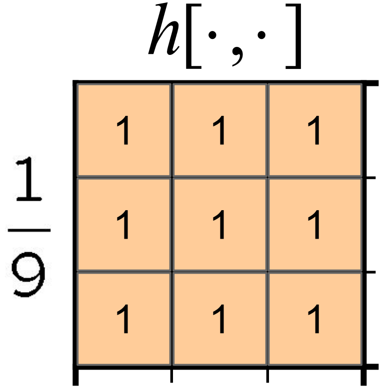
  - Homogeneity

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$

Another question:

Q. Is the moving average filter homogeneous?

$$\mathcal{S}[\alpha f[n,m]] = \alpha \mathcal{S}[f[n,m]]$$



Practice proving it at home using:

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

Properties of systems

- Amplitude properties:
  - Additivity

$$\mathcal{S}[f_i[n,m] + f_j[n,m]] = \mathcal{S}[f_i[n,m]] + \mathcal{S}[f_j[n,m]]$$

  - Homogeneity

$$\mathcal{S}[\alpha f[n,m]] = \alpha\mathcal{S}[f[n,m]]$$

  - Superposition

$$\mathcal{S}[\alpha f_i[n,m] + \beta f_j[n,m]] = \alpha\mathcal{S}[f_i[n,m]] + \beta\mathcal{S}[f_j[n,m]]$$

This is an important property. Make sure you know how to prove if any system has this property

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m, \ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m, \ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Q. Is the moving average filter stable?

$$\text{If } \forall n, m, \ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m, \ |f[n,m]| \leq k$$

If $\forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck$ for some constant c and k

Proof of stability

Let $\forall n, m,\ |f[n,m]| \leq k$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]|$$

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]|$$

$$\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k$$

$$\text{If } \forall n, m,\ |f[n,m]| \le k \implies |\mathcal{S}[f[n,m]]| \le ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \le k$$

$$|\mathcal{S}f[n,m]| = |\frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]|$$

$$\le \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} |f[n-k, m-l]|$$

$$\le \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} k$$

$$\le \frac{1}{9}(3)(3)k$$

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$\begin{aligned}
|\mathcal{S}f[n,m]| &= |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k \\
&\leq \frac{1}{9}(3)(3)k \\
&\leq k
\end{aligned}$$

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

Proof of stability

$$\text{Let } \forall n, m,\ |f[n,m]| \leq k$$

$$\begin{aligned}
|\mathcal{S}f[n,m]| &= |\frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} |f[n-k, m-l]| \\
&\leq \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} k \\
&\leq \frac{1}{9}(3)(3)k \\
&\leq k \\
&\leq ck, \text{ where } c = 1
\end{aligned}$$

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m, \ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

  - Invertibility

$$\mathcal{S}^{-1}\mathcal{S}[f[n,m]] = f[n,m]$$

Properties of systems

- Amplitude properties:
  - Stability

$$\text{If } \forall n, m,\ |f[n,m]| \leq k \implies |\mathcal{S}[f[n,m]]| \leq ck \text{ for some constant c and k}$$

  - Invertibility

$$\mathcal{S}^{-1}\mathcal{S}[f[n,m]] = f[n,m]$$

Q. Is the 3x3 moving average filter invertible?

Properties of systems

- Spatial properties
  - Causality

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

Is the moving average filter casual?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$f[n,m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n,m]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

⬅ HINT

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

Properties of systems

- Spatial properties
  - Causality

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

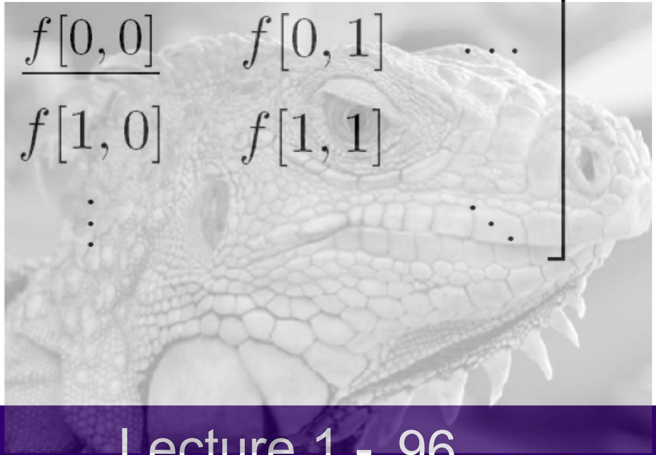
  - Shift invariance:

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

What does shifting an image look like?

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

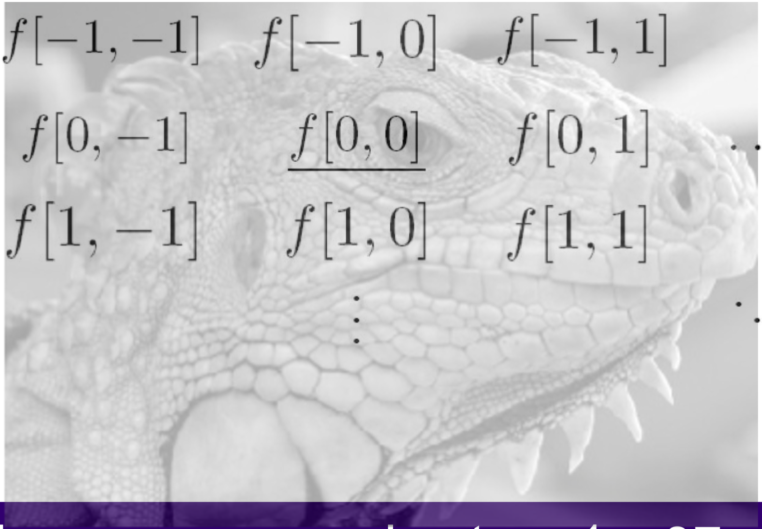
$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \dots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \dots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$



Original image

What does shifting an image look like?

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$



$$f[n,m] = \begin{bmatrix} \ddots & & \vdots & & \\ & f[-1,-1] & f[-1,0] & f[-1,1] & \\ \dots & f[0,-1] & \underline{f[0,0]} & f[0,1] & \dots \\ & f[1,-1] & f[1,0] & f[1,1] & \\ & & \vdots & & \ddots \end{bmatrix}$$

Shifted image

$n_0 = 1$

$m_0 = 1$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

$f[n, m]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$g[n, m]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$g[n-n_0, m-m_0] = g[n', m']$$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

Is the moving average system is shift invariant?

$$g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

Let $n' = n - n_0$ and $m' = m - m_0$

$$g[n-n_0, m-m_0] = g[n', m']$$

$$= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n'-k, m'-l]$$

$$f[n-n_0, m-m_0] \xrightarrow{\mathcal{S}} g[n-n_0, m-m_0]$$

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What we covered today?

- Image histograms
- Images as functions
- Filters
- Properties of systems

Next time:

Linear systems and convolutions