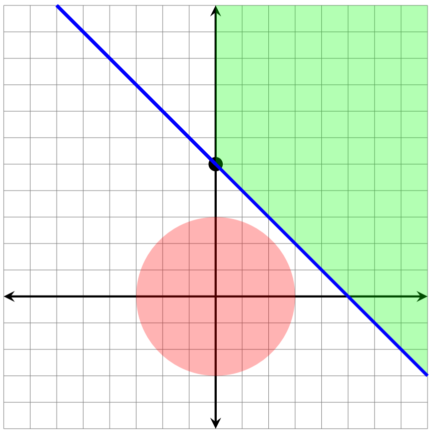


In lecture, we defined the following 2d "robot" transition system.

$$\begin{aligned} S &= \mathbb{Z} \times \mathbb{Z} \\ S_0 &= \{(0,5)\} \\ \rightarrow &= \{((x,y),(x,y+1)) \mid x,y \in \mathbb{Z}\} \cup \\ &\quad \{((x,y),(x+1,y-1)) \mid x,y \in \mathbb{Z}\} \end{aligned}$$

We analyzed this system with the safety property

$$P = \{(x,y) \mid x,y \in \mathbb{Z} \wedge \sqrt{x^2+y^2} > 3\}$$

Solutions are in red font.

Problem 1. Warmup/lecture review

(a) Draw the set of reachable states as a region on the picture above.
See the green region above.

(b) Just visually, why does the safety property hold? Use your drawing.
The green region and the red circle do not overlap.

(c) Justify the safety property a bit more formally by giving an inductive invariant. Argue (informally) that no transition "crosses the boundary" from inside your inductive invariant to outside of it.
One inductive invariant is $x + y \geqslant 5$ (above the blue diagonal). No transition crosses the boundary from inside the region to outside, because north moves go "more above" the diagonal, and southeast moves do not change "how far above" the diagonal the state is.

Problem 2. If we change the initial state to $(0, -5)$, does the safety property still hold? If yes, give a new inductive invariant. If no, give a counterexample to safety.
No. Take two steps north to enter the red circle.

Problem 3. Consider the original system with initial state $(0, 5)$, but now suppose the transitions are to either take a step west or to take a step southeast. Does the safety property still hold? If yes, give a new inductive invariant. If no, give a counterexample to safety.
No. Take one step west and three steps southeast to enter the red circle.

Problem 4. Consider the modified west-and-southeast system but now with the initial state $(0, -5)$. Does the safety property still hold? If yes, give a new inductive invariant. If no, give a counterexample to safety.
Yes. $x + y \leqslant -5$ is an inductive invariant for this system.

Problem 5. In the original system, consider changing the initial state to *any* point. For which points is the system safe?
It is safe for any initial state that cannot reach the red circle through any sequence of steps. Those are the states above the diagonal $x + y = 5$ *or* on or to the right of the line $x = 4$, *or* the special point $(3, 1)$.

Problem 6. Now consider changing the set of bad states (the red circle). Suppose we add one new bad state at the point $(8, 8)$. The system is now unsafe. What is the shortest execution that leads to a bad state?
There are many shortest executions of length 19. One example is to first go southeast 8 times (to the point $(8, -3)$), and then go north 11 times to $(8, 8)$.

Problem 7. (Optional) Consider this pseudo-liveness claim:

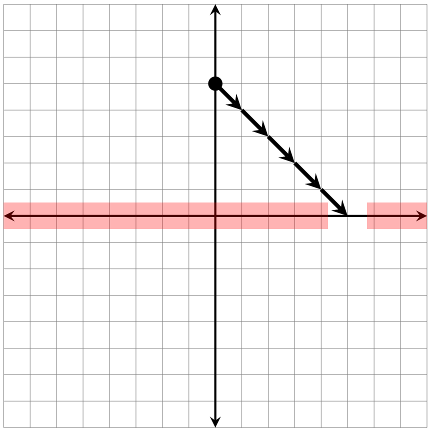
> From any reachable state, there is at least one execution that reaches a point *on* the x-axis.

Is this claim true for this transition system? Why or why not?

True. Any reachable state is either already on the x-axis, above it, or below it. If it's on it, we're done. If it's above it, move southeast until you reach the x-axis. Otherwise, if the point is below the x-axis, move north until you hit it

Consider the following program

```
n = read_input();
x = 0;
y = n;
while (y != 0) {
  x = x + 1;
  y = y - 1;
}
assert x == n;
```



Given an input n, we can model this program's execution as a transition system:

$$
\begin{aligned}
S &= \mathbb{Z} \times \mathbb{Z} \\
S_0 &= \{(0, n)\} \\
\rightarrow &= \{((x, y), (x + 1, y - 1)) \mid x, y \in \mathbb{Z} \land y \neq 0\}
\end{aligned}
$$

The bad states are the ones that cause the assertion to be false. In other words, states where we have exited the loop (i.e., $y = 0$) but $x \neq n$.

Problem 8. Suppose $n = 5$. Draw the bad states as a region on the grid above.
The bad states are all states on the x axis *except* $x = 5$. See the red region on the grid above.

Problem 9. Suppose $n = 5$. Draw the (only) execution of the program on the grid above.
See the black arrows on the grid above.

Problem 10. Is the program safe in this execution? Why or why not?
Yes. It avoids the red region.

Problem 11. The safety property is not inductive. Give a counterexample to induction (i.e., a state that satisfies the safety property but steps in one step to a state that violates the safety property).
The state is $(3, 1)$ is a counterexample to induction because it is outside the red region but steps into the red region in one step. As usual, this state is *not* actually reachable, but we still have to rule it out in our proof.

Problem 12. Strengthen the safety property so that it is an inductive invariant. Explain why it is an inductive invariant.
One such invariant is $x + y = 5$. It is true initially. Also, if you are on this diagonal and you increment x and decrement y, you remain on the diagonal.

Problem 13. Suppose we change the program to increment both x and y in the loop. Model the new program as a transition system. Is the safety property true?
Yes, the program is still safe. It never reaches the x-axis, so the assertion is never even executed.

Problem 14. (Optional) In the original transition system, consider this pseudo-liveness property:

> For any non-negative integer input n, there is an execution of the transition system that reaches a state on the x-axis.

Is this property true? Why or why not?
True. The transition system reaches the x-axis in exactly n steps in the only execution.