Randomized Consensus

FLP Impossibility

Theorem: *In an asynchronous environment in which a single process can fail by crashing, there does not exist a protocol which solves binary consensus.*

Paxos doesn't save us. It doesn't guarantee liveness.

Result assumed a **deterministic** computation model.

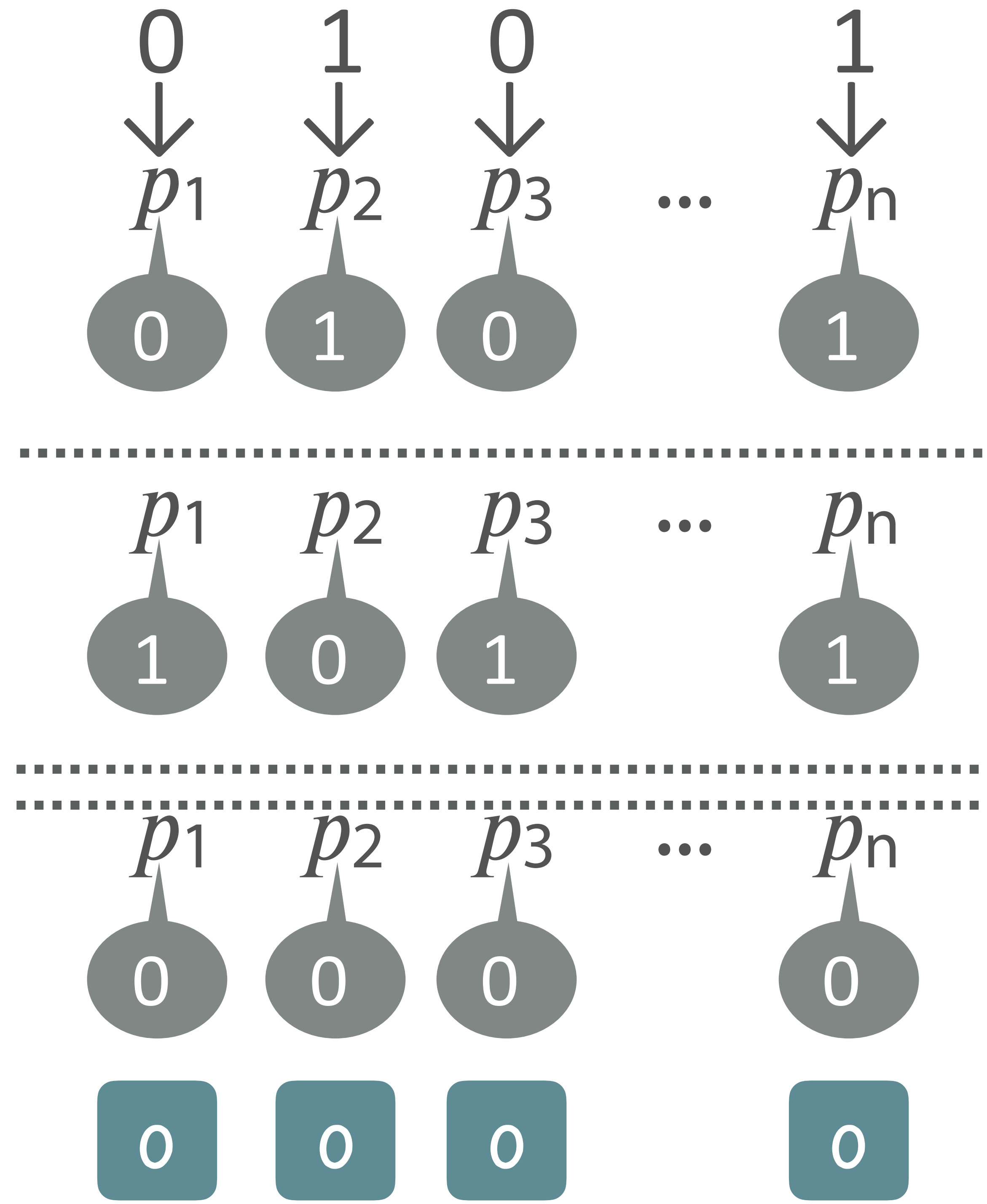
Let's go random!

Ben-Or's algorithm uses randomization to ***guarantee consensus*** for crash failures when $f < n/2$.

A variant even works for Byzantine faults!

Intuition

- At first every process proposes their input value.
- After that, they propose random values.
- When enough processes propose the same value, the value is chosen.
- Eventually, that will happen!



Setup

- Again, we're considering binary consensus.
- Protocol proceeds in **asynchronous rounds**, where each round has two phases.
- For each phase, processes broadcast their input values and wait for $n - f$ messages from the other processes.
- Each message is tagged with the round and phase number. (And messages can be resent to deal with a lossy network. But once a message is sent, that value is locked in for that process for that phase/round.)

Ben-Or Algorithm

Processes send proposals for each phase and then block and wait for the requisite $n - f$ messages (including their own).

During the first phase, processes make a preliminary proposal.

If they receive matching responses from a majority in the first phase, they propose that value in the second phase. Otherwise, they propose $\perp$ (a special null value).

If they get enough non-$\perp$ responses from the second phase, they decide.

```
a←input
loop:
    send_phase1(a)
    A←receive_phase1()
    if (∃a′∈ A : |A_a′| > n/2):
        b←a′
    else:
        b←⊥

    send_phase2(b)
    B←receive_phase2()
    if (∃b′∈ B : b′≠⊥ ∧ |B_b′| > f):
        decide(b′)
    if (∃b′∈ B : b′≠⊥):
        a←b′
    else:
        a←choose_random({0,1})
```

Do We Have Consensus?

- **Agreement:** No two processes decide different values.
- **Integrity:** Every process decides at most one value, and if a process decides a value, some process had it as its input.
- **Termination:** Every correct process eventually decides a value.

$a \leftarrow$ *input*

loop:

  send_phase1(a)

  $A \leftarrow$ *receive_phase1*()

  if ($\exists a' \in A : |A_{a'}| > n/2$):

    $b \leftarrow a'$

  else:

    $b \leftarrow \perp$

  send_phase2(b)

  $B \leftarrow$ *receive_phase2*()

  if ($\exists b' \in B : b' \neq \perp \wedge |B_{b'}| > f$):

    decide(b')

  if ($\exists b' \in B : b' \neq \perp$):

    $a \leftarrow b'$

  else:

    $a \leftarrow$ *choose_random({0,1})*

Integrity I

If both 0 and 1 are input values to processes, integrity is trivially satisfied.

Suppose all processes have the same input value.

- Then, they all send the same phase 1 value in round 1.
- So they all send that same value in phase 2.
- So they all decide that value at the end of round 1.

```
a←input
loop:
    send_phase1(a)
    A←receive_phase1()
    if (∃a′∈ A : |A_a′| > n/2):
        b←a′
    else:
        b←⊥
    send_phase2(b)
    B←receive_phase2()
    if (∃b′∈ B : b′≠⊥ ∧ |B_b′| > f):
        decide(b′)
    if (∃b′∈ B : b′≠⊥):
        a←b′
    else:
        a←choose_random({0,1})
```

Fun Fact

Lemma: *No two processes receive different non-⊥ phase 2 values in the same round.*

Suppose they did. That means that one process received 0s from a majority in phase 1 and another received 1s.

But majorities intersect!

```
a←input
loop:
    send_phase1(a)
    A←receive_phase1()
    if (∃a′∈ A : |A_a′| > n/2):
        b←a′
    else:
        b←⊥
    send_phase2(b)
    B←receive_phase2()
    if (∃b′∈ B : b′≠⊥ ∧ |B_b′| > f):
        decide(b′)
    if (∃b′∈ B : b′≠⊥):
        a←b′
    else:
        a←choose_random({0,1})
```

Agrement + Integrity II

Let round r be the first round any process decides a value, 0 w.l.o.g.

If a process decided a value, it must have received $> f$ 0s in phase 2.

Which means that every process received at least one 0 because they all wait for $n - f$ messages. No process received a 1 by the previous lemma.

Therefore, on round $r + 1$ (and all subsequent rounds), all processes propose 0 and all processes decide 0.

$a \leftarrow$ *input*

loop:

  send_phase1(a)

  $A \leftarrow$ *receive_phase1*()

  if ($\exists a' \in A : |A_{a'}| > n/2$):

    $b \leftarrow a'$

  else:

    $b \leftarrow \bot$

  send_phase2(b)

  $B \leftarrow$ *receive_phase2*()

  if ($\exists b' \in B : b' \neq \bot \wedge |B_{b'}| > f$):

    decide(b')

  if ($\exists b' \in B : b' \neq \bot$):

    $a \leftarrow b'$

  else:

    $a \leftarrow$ *choose_random*({0,1})

Termination

We know that if all processes propose the same value for a round, they all decide that value that round.

At worst, the probability of this happening on any particular round is $1/2^n$.

Why? By the previous lemma, all the non-random values are identical.

Over time, the probability of this happening on **at least one round** converges to 1.

$a \leftarrow$ *input*

loop:

  send_phase1(a)

  $A \leftarrow$ *receive_phase1*$()$

  if $(\exists a' \in A : |A_{a'}| > n/2)$:

    $b \leftarrow a'$

  else:

    $b \leftarrow \perp$

  send_phase2(b)

  $B \leftarrow$ *receive_phase2*$()$

  if $(\exists b' \in B : b' \neq \perp \wedge |B_{b'}| > f)$:

    decide(b')

  if $(\exists b' \in B : b' \neq \perp)$:

    $a \leftarrow b'$

  else:

    $a \leftarrow$ *choose_random({0,1})*

Other Values?

Binary consensus is conceptually simple but not as useful. However, the algorithm can be to support larger domains, even when the processes don't know the domains *a priori* and even when some processes don't receive input values.

- Processes without input values start by proposing ⊥.
- Instead of randomly choosing from {0,1}, processes randomly choose from all non-⊥ values they've seen so far (in any message). Only choose ⊥ as a last resort.

$a \leftarrow$ *input*

loop:

  send_phase1(a)

  $A \leftarrow$ *receive_phase1*()

  if $(\exists a' \in A : |A_{a'}| > n/2)$:

    $b \leftarrow a'$

  else:

    $b \leftarrow \perp$

  send_phase2(b)

  $B \leftarrow$ *receive_phase2*()

  if $(\exists b' \in B : b' \neq \perp \wedge |B_{b'}| > f)$:

    decide(b')

  if $(\exists b' \in B : b' \neq \perp)$:

    $a \leftarrow b'$

  else:

    $a \leftarrow$ *choose_random({0,1})*

Takeaways

- Randomization can actually solve consensus*
- You can structure an asynchronous protocol using rounds. It's potentially useful and certainly an interesting way to think about asynchronous computation.