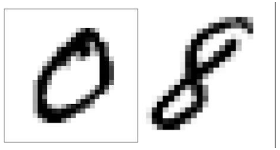
446 Section 

TA: Varun Ananth

Plans for today!

1. This
2. Reminders
3. Neural Networks
   a. Forward/backward passes
   b. Weight initializations

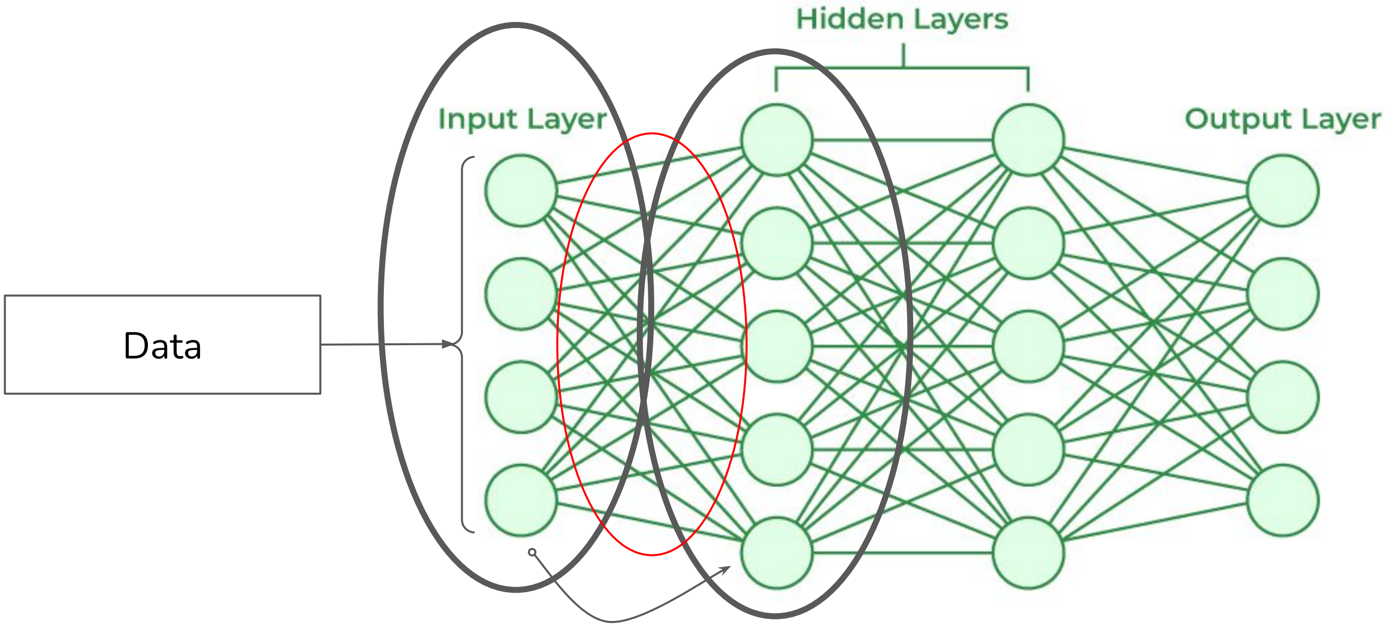
Reminders

- HW3 due Nov 19 @ 11:59 PM
  - Double check your late days!

Nothing else to say… How's life?

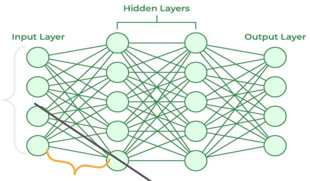
Forward Pass

Passing Data Through the Network



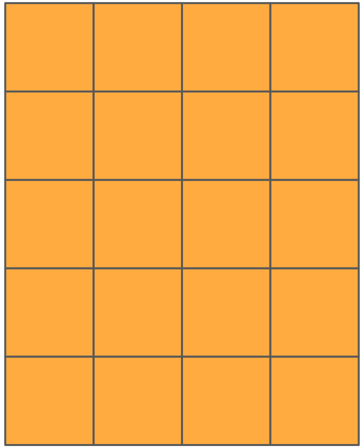
How is this done?

Goal:  x^{i+1}_{5x1}



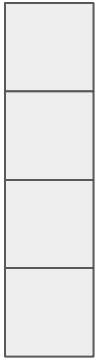
Floats initialized semi-randomly. Learning these = learning the function

W_{5x4}



Weights

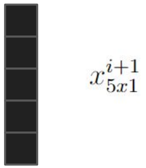
x^{i}_{4x1}



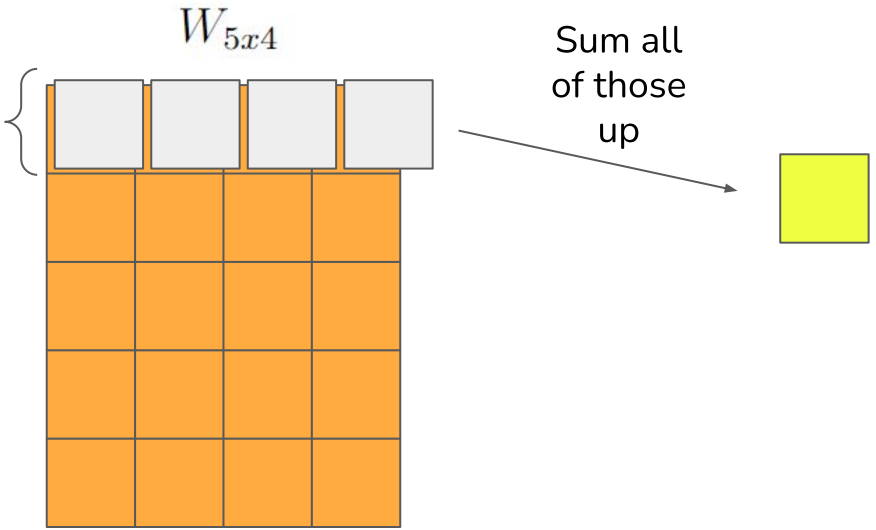
Data

How is this done?

Goal:



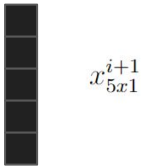
Multiply each square of the input with the square under it

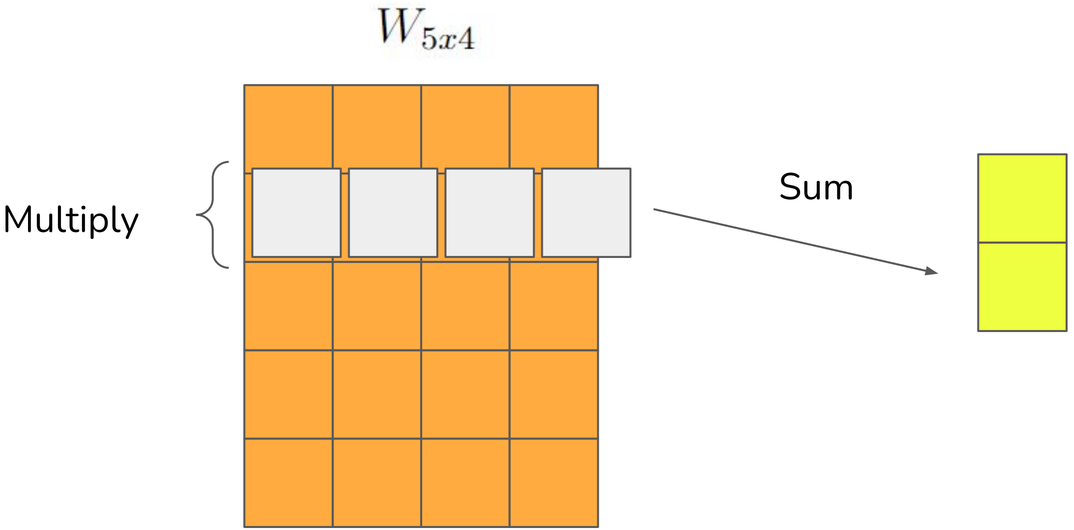


Sum all of those up

How is this done?

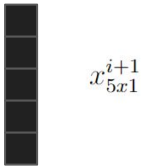
Goal:

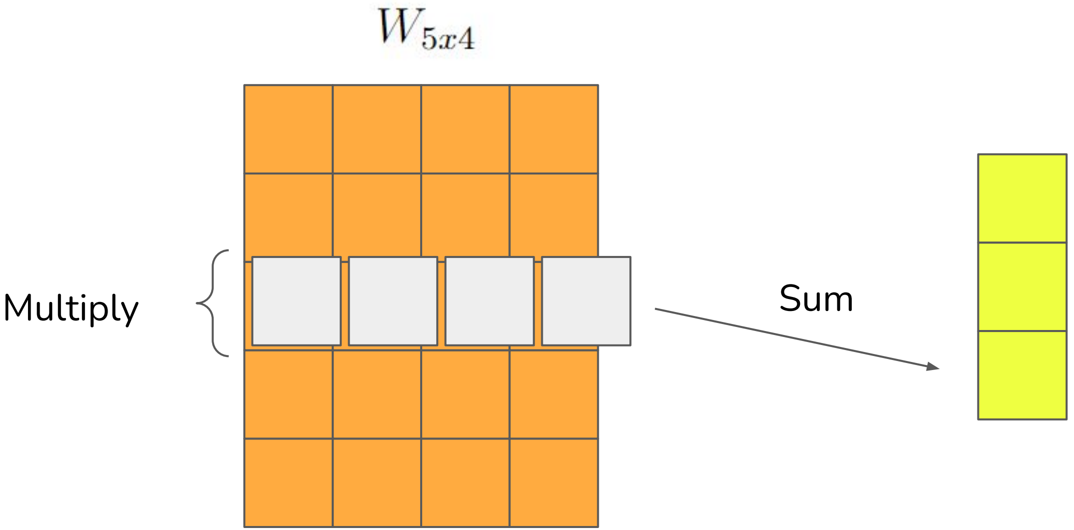




How is this done?

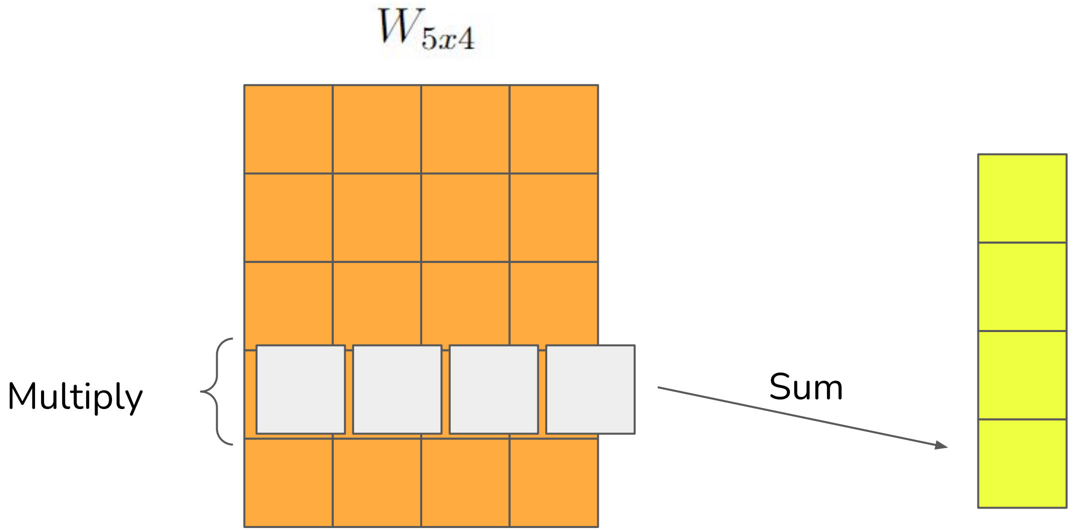
Goal:



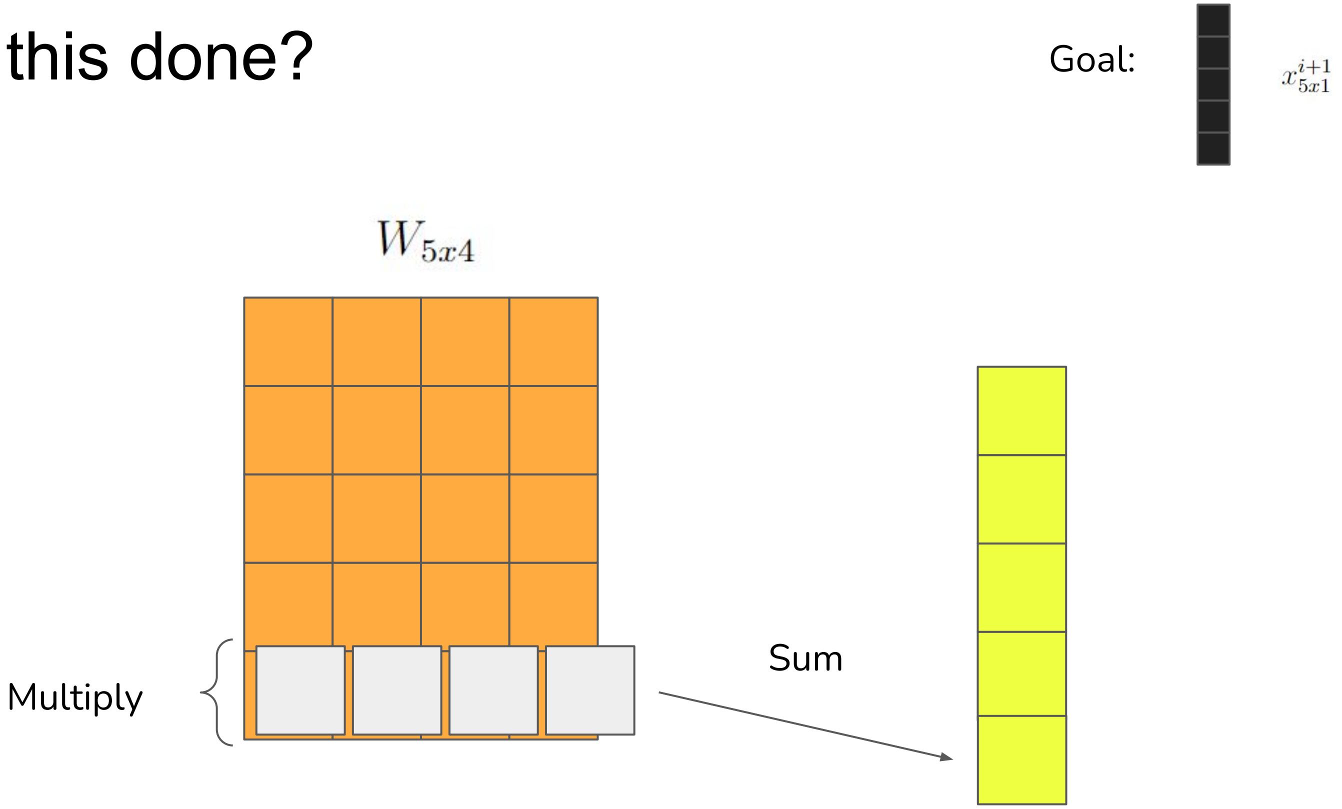


How is this done?

Goal:  x^{i+1}_{5x1}



How is this done?



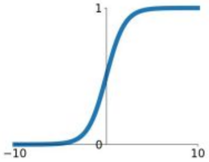
Sigmoid Activation Function

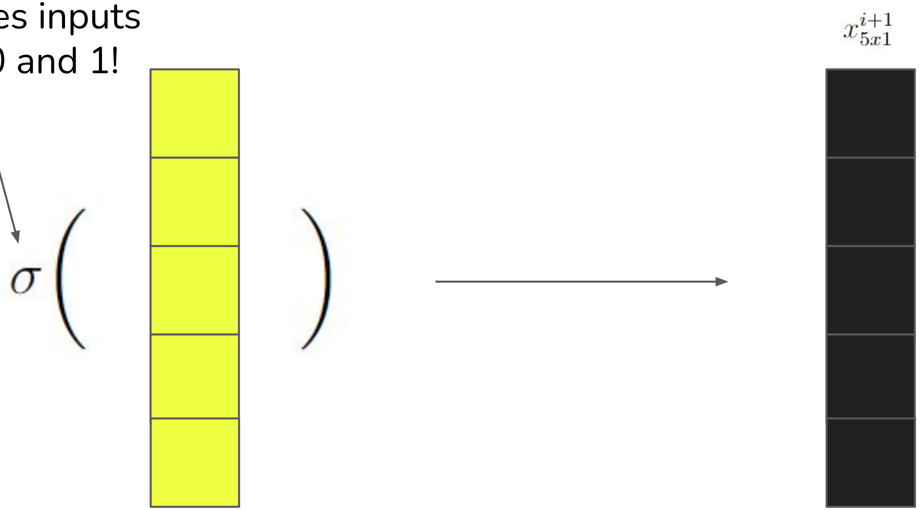
Goal:  x^{i+1}_{5x1}

Sigmoid <u>activation function</u>
that squishes inputs
between 0 and 1!

Sigmoid

$\sigma(x) = \frac{1}{1+e^{-x}}$

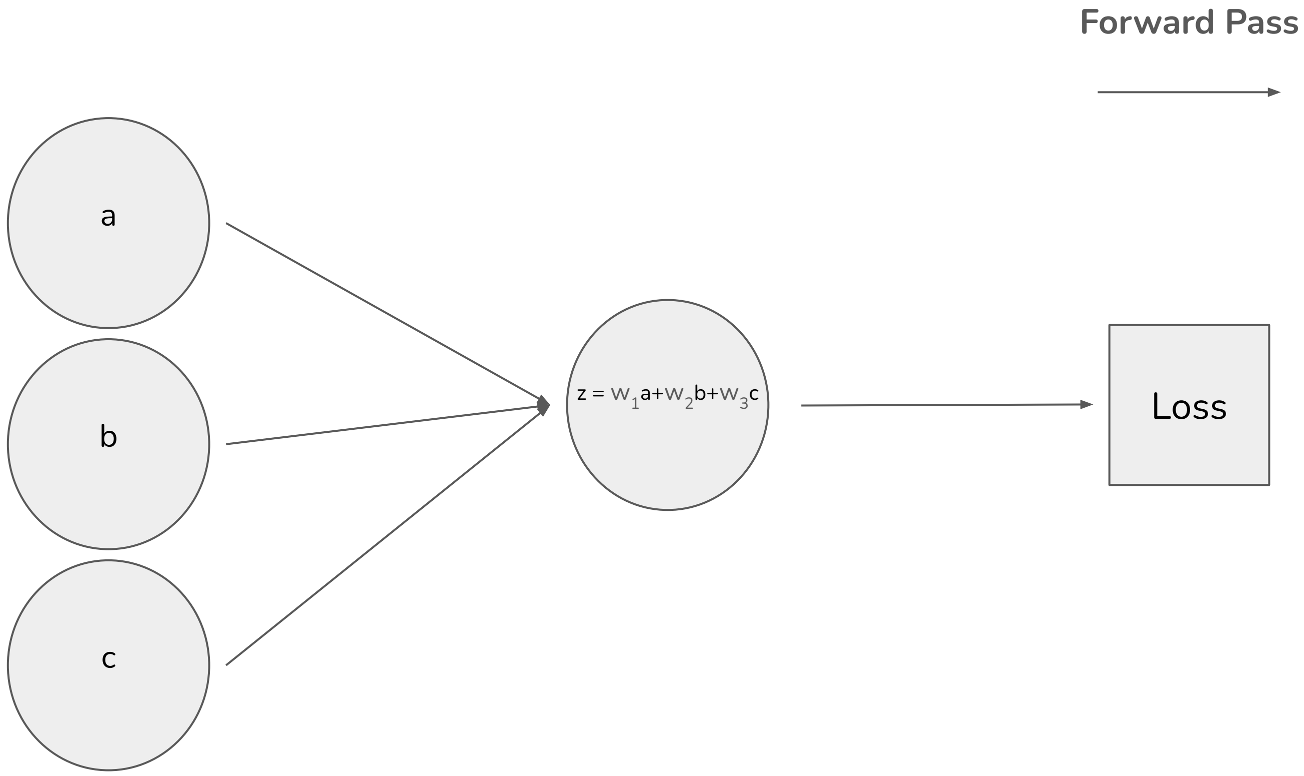




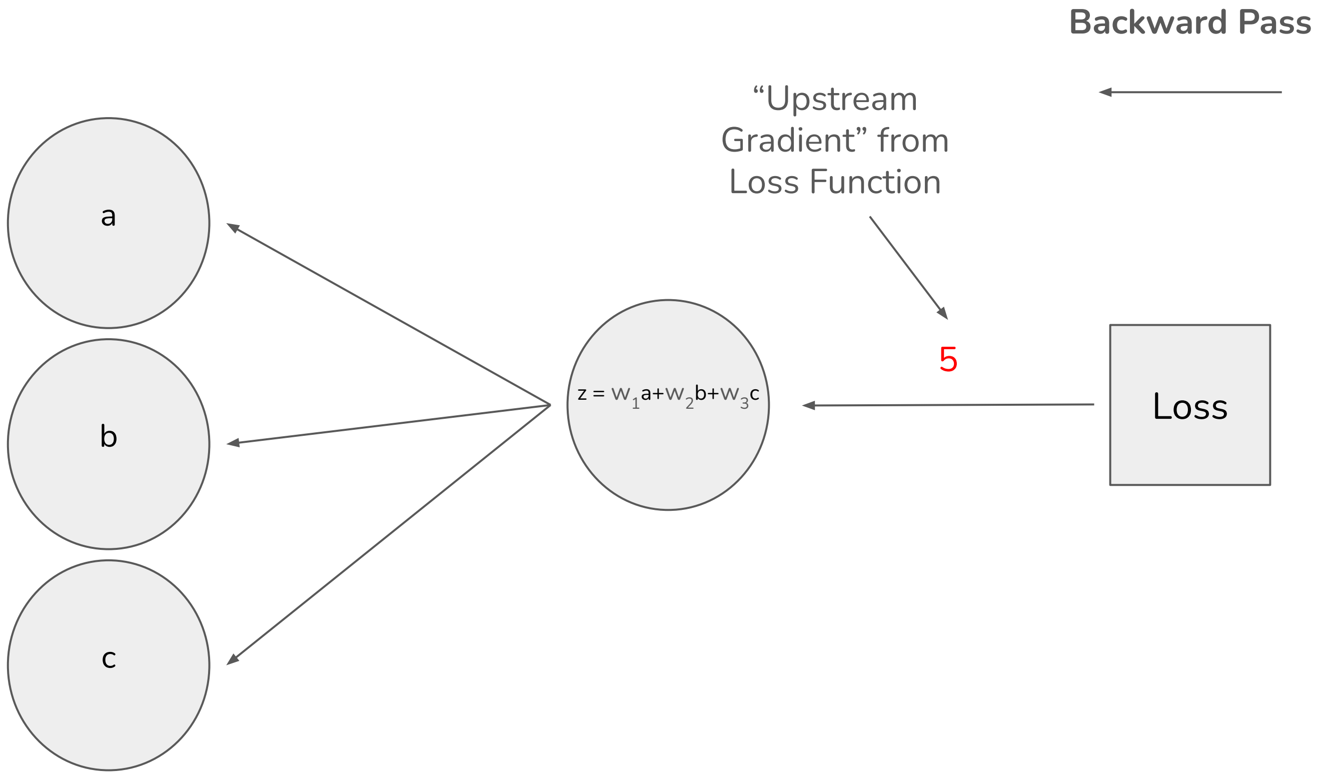
Sigmoid does have some problems
though...

Backpropagation

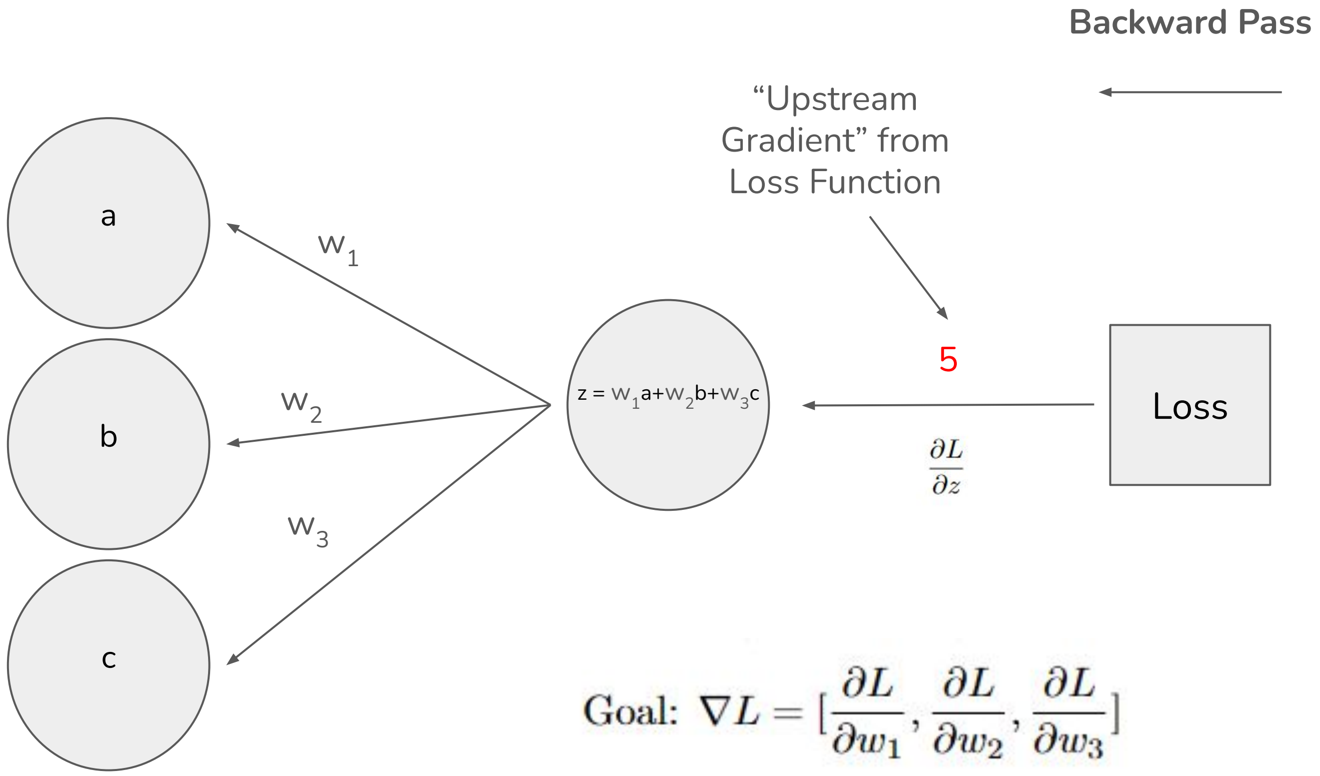
Backpropagation Intuition



Backpropagation Intuition

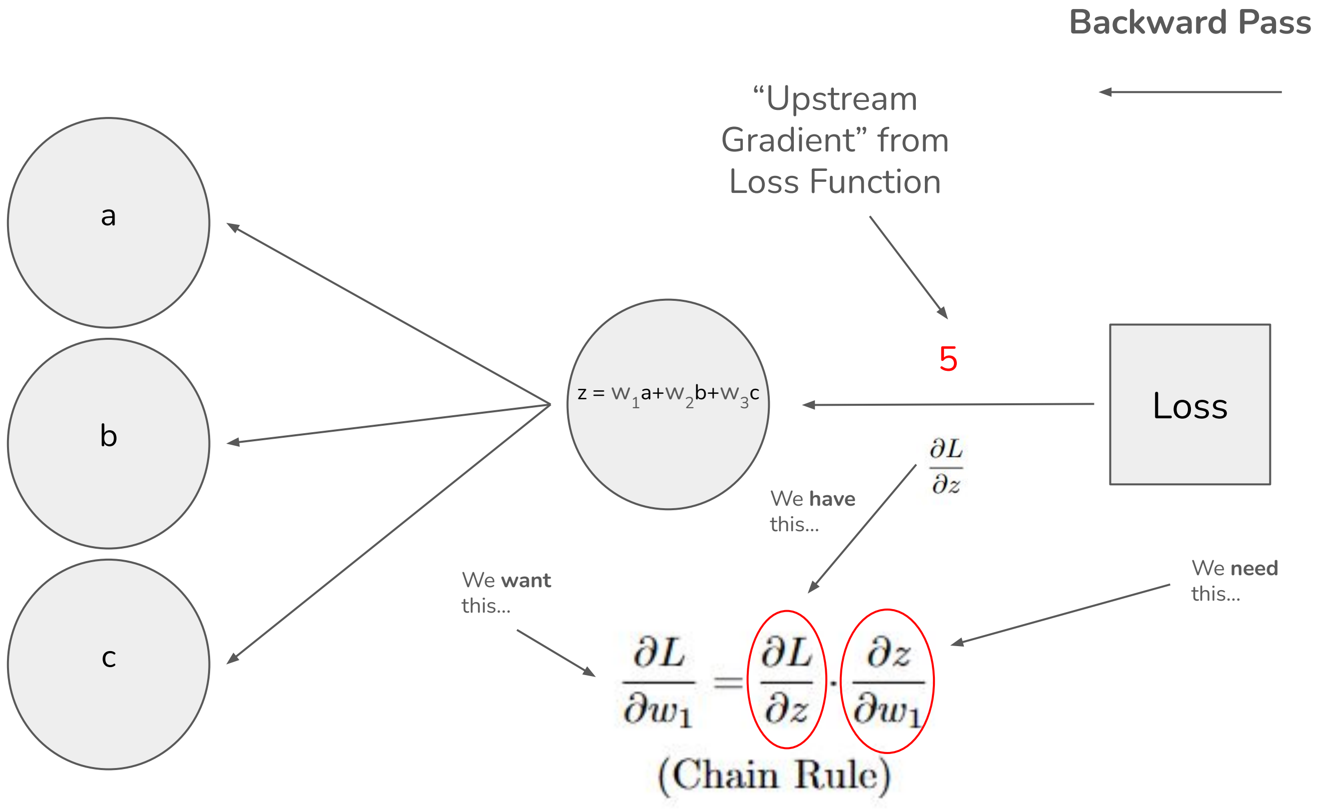


Backpropagation Intuition



$$\text{Goal: } \nabla L = [\frac{\partial L}{\partial w_1}, \frac{\partial L}{\partial w_2}, \frac{\partial L}{\partial w_3}]$$

Backpropagation Intuition



Backpropagation Intuition

$$\frac{\partial}{\partial w_1} z = \frac{\partial}{\partial w_1}(w_1 a + w_2 b + w_3 c)$$

$$\frac{\partial z}{\partial w_1} = a$$

$$\frac{\partial L}{\partial z} = 5$$

$$\frac{\partial L}{\partial w_1} = 5 * a$$

$$\frac{\partial}{\partial w_2} z = \frac{\partial}{\partial w_2}(w_1 a + w_2 b + w_3 c)$$

$$\frac{\partial z}{\partial w_2} = b$$

$$\frac{\partial L}{\partial z} = 5$$

$$\frac{\partial L}{\partial w_2} = 5 * b$$

$$\frac{\partial}{\partial w_3} z = \frac{\partial}{\partial w_3}(w_1 a + w_2 b + w_3 c)$$

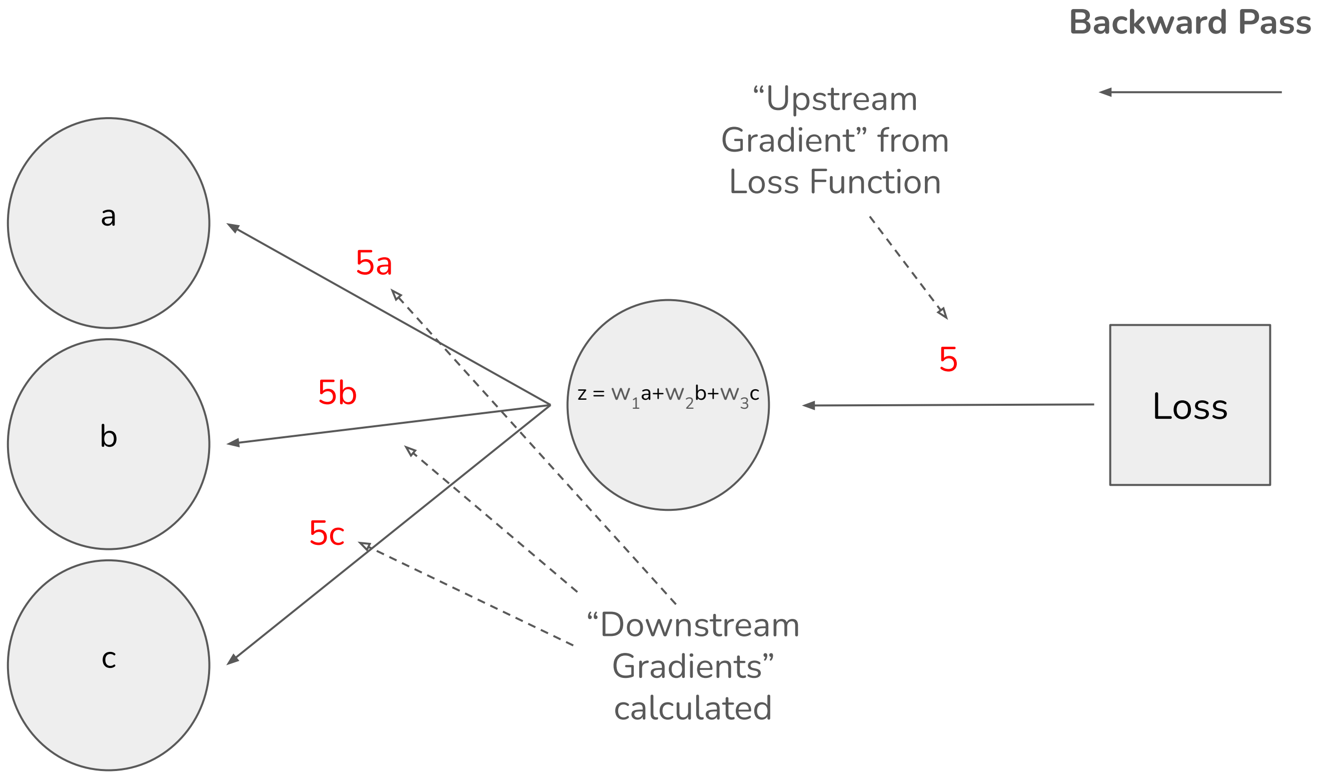
$$\frac{\partial z}{\partial w_3} = b$$

$$\frac{\partial L}{\partial z} = 5$$

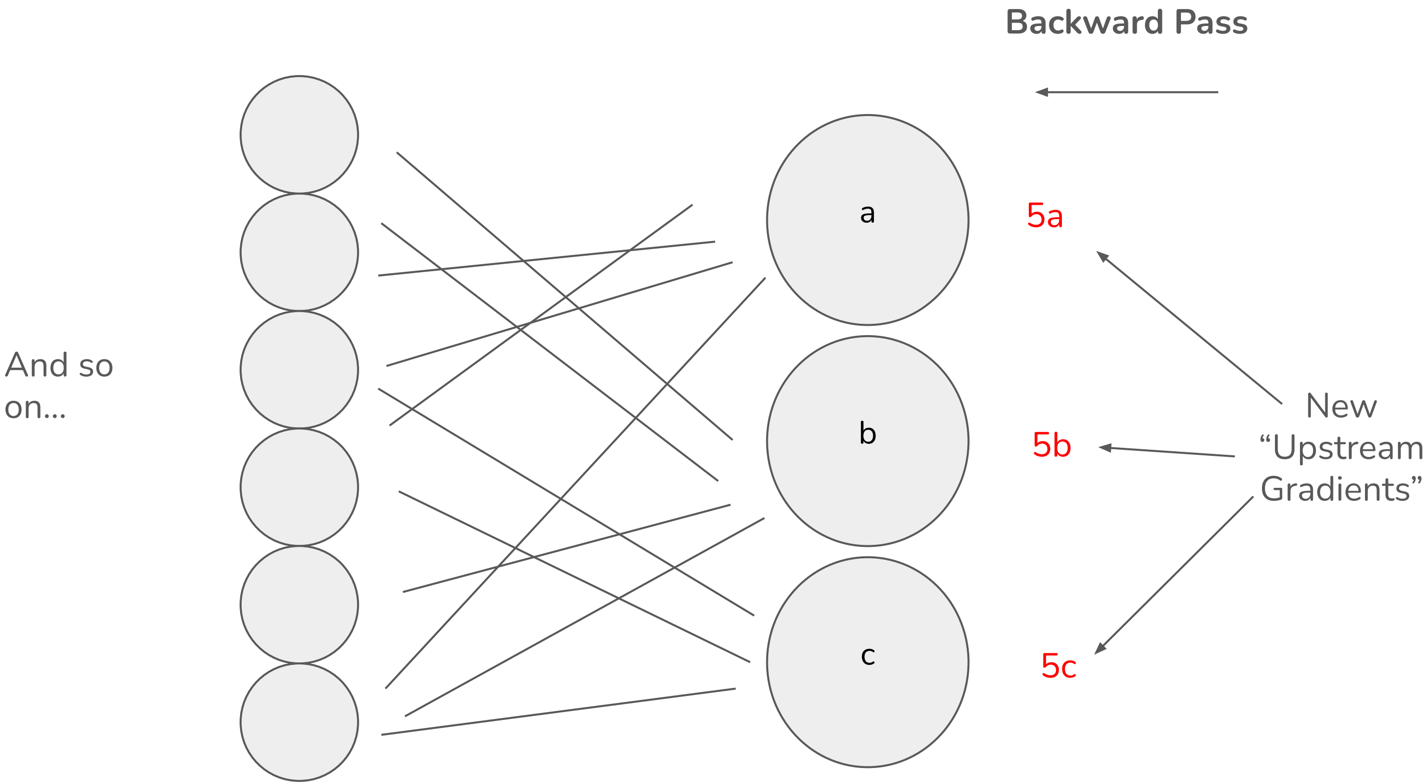
$$\frac{\partial L}{\partial w_3} = 5 * c$$

$$\text{Goal: } \nabla L = [\frac{\partial L}{\partial w_1}, \frac{\partial L}{\partial w_2}, \frac{\partial L}{\partial w_3}]$$

Backpropagation Intuition



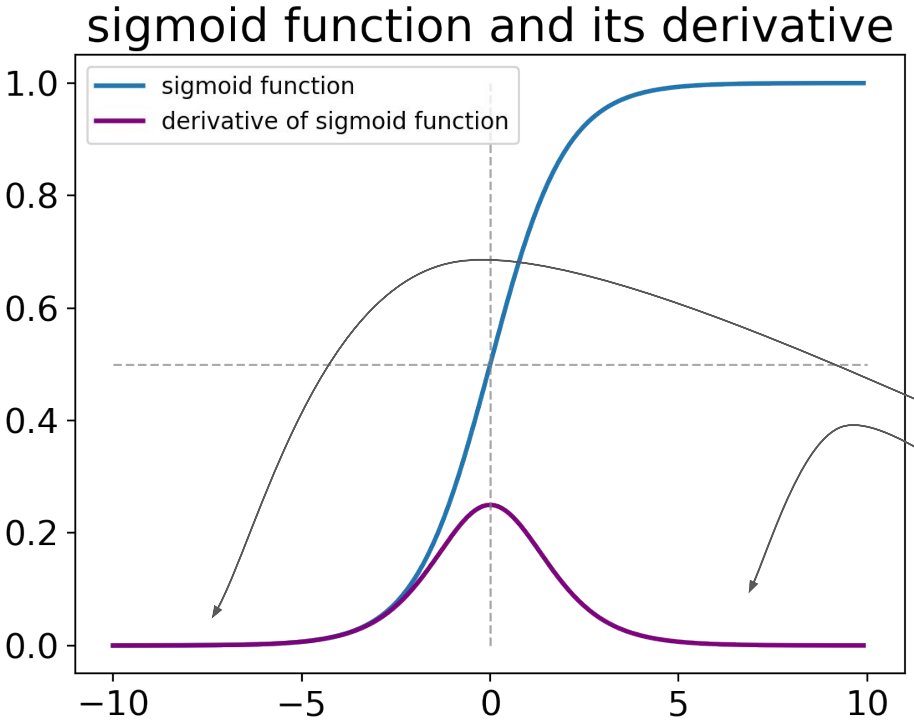
Backpropagation Intuition



Issue with Sigmoid from earlier

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$\frac{d}{dx}\sigma(x) = \sigma'(x) = \sigma(x)(1-\sigma(x))$$

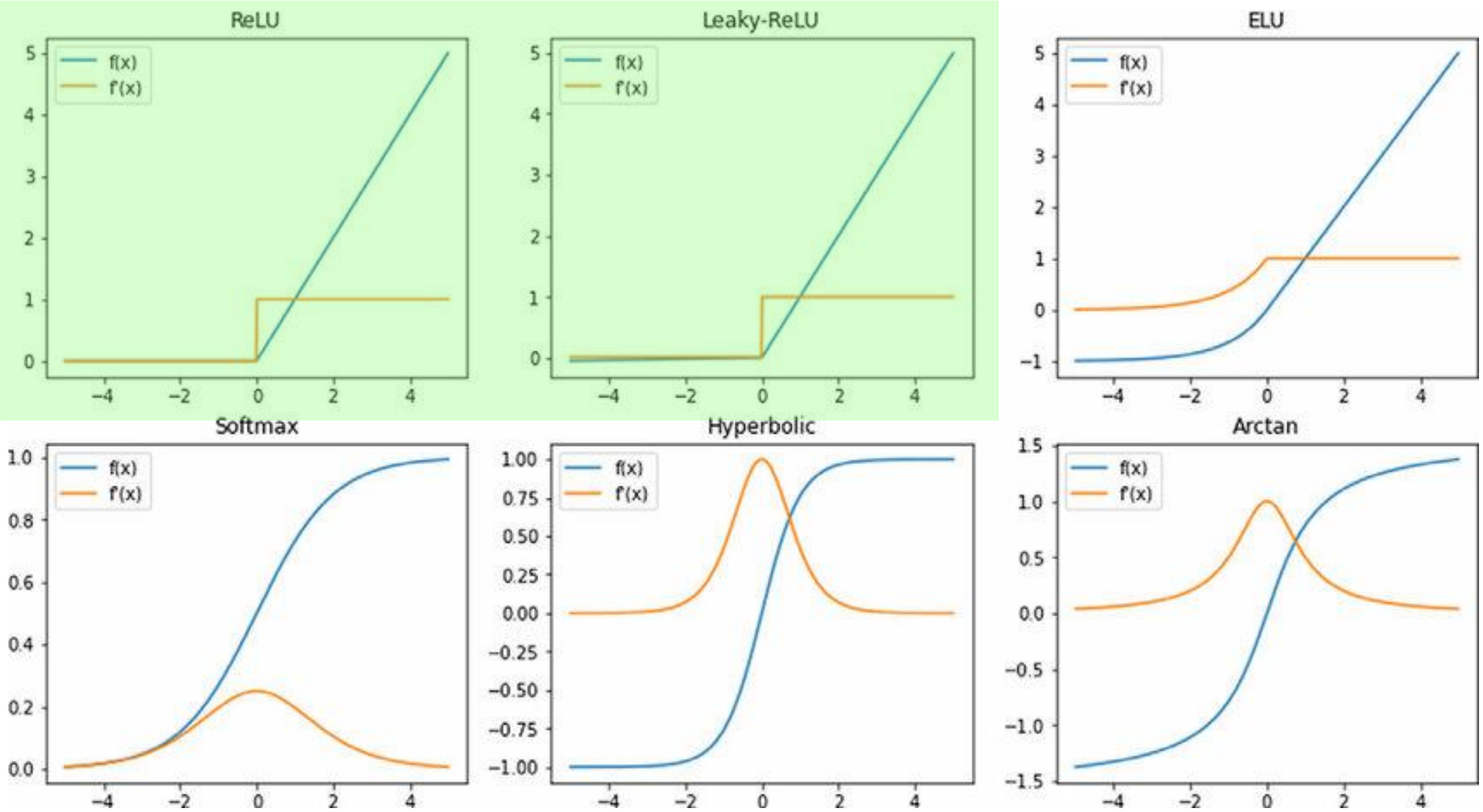


The **derivative of the sigmoid function** is what upstream gradients must pass through to get to layers further back

The sigmoid function has these "dead zones" that can kill information flowing backwards!

Issue with Sigmoid from earlier

ReLU is usually the best default activation function - this is partially why

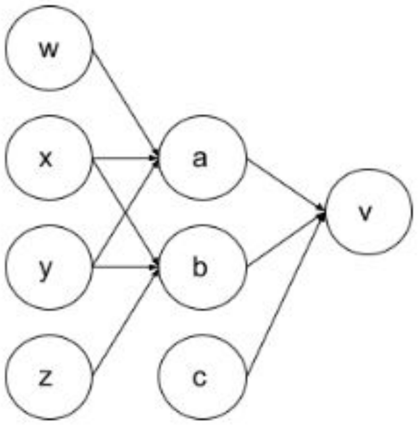


1. Neural Network Chain Rule Warm-Up

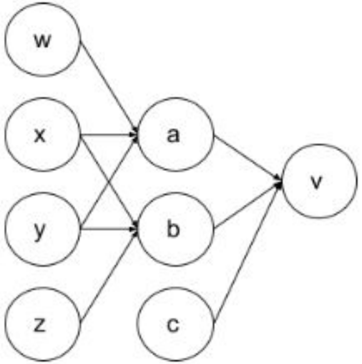
Consider the following equations:

$$v(a,b,c) = c(a-b)^2$$
$$a(w,x,y) = (w+x+y)^2$$
$$b(x,y,z) = (x-y-z)^2$$

The way variables are related to each other can be represented as the network:



(a) Using the multi-variate chain rule(part 1.b), write the derivatives of the output v with respect to each of the input variables: c, w, x, y, z using only partial derivative symbols.



$$v(a,b,c) = c(a-b)^2$$
$$a(w,x,y) = (w+x+y)^2$$
$$b(x,y,z) = (x-y-z)^2$$

(a) Using the multi-variate chain rule(part 1.b), write the derivatives of the output v with respect to each of the input variables: c, w, x, y, z using only partial derivative symbols.

Solution:

$$\frac{\partial v}{\partial c} = \frac{\partial v}{\partial c}$$
$$\frac{\partial v}{\partial w} = \frac{\partial v}{\partial a} \cdot \frac{\partial a}{\partial w}$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial v}{\partial b} \cdot \frac{\partial b}{\partial x}$$
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial a} \cdot \frac{\partial a}{\partial y} + \frac{\partial v}{\partial b} \cdot \frac{\partial b}{\partial y}$$
$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial b} \cdot \frac{\partial b}{\partial z}$$

(b) Compute the values of all the partial derivatives on the RHS of your results to the previous question. Then use them to compute the values on the LHS.

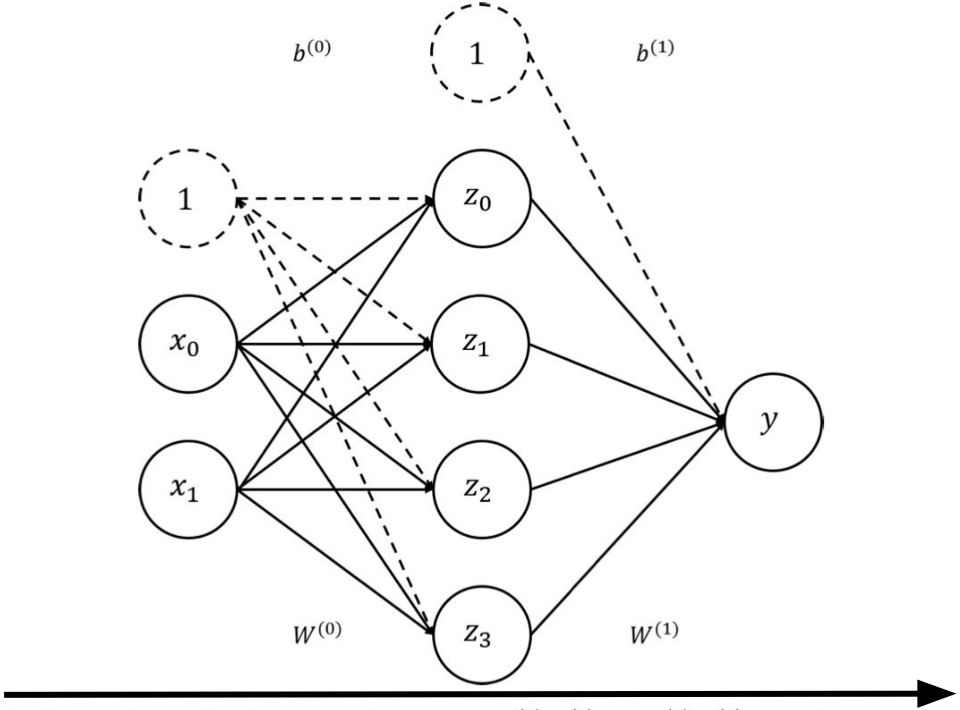
$$\frac{\partial v}{\partial a} = 2c(a-b) \quad \frac{\partial v}{\partial b} = -2c(a-b) \quad \frac{\partial v}{\partial c} = (a-b)^2$$
$$\frac{\partial a}{\partial w} = 2(w+x+y) \quad \frac{\partial a}{\partial x} = 2(w+x+y) \quad \frac{\partial a}{\partial y} = 2(w+x+y)$$
$$\frac{\partial b}{\partial x} = 2(x-y-z) \quad \frac{\partial b}{\partial y} = -2(x-y-z) \quad \frac{\partial b}{\partial z} = -2(x-y-z)$$

$$\frac{\partial v}{\partial c} = (a-b)^2$$
$$\frac{\partial v}{\partial w} = \frac{\partial v}{\partial a} \cdot \frac{\partial a}{\partial w} = 4c(a-b)(w+x+y)$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial v}{\partial b} \cdot \frac{\partial b}{\partial x} = 4c(a-b)(w+x+y) - 4c(a-b)(x-y-z) = 4c(a-b)(w+2y+z)$$
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial a} \cdot \frac{\partial a}{\partial y} + \frac{\partial v}{\partial b} \cdot \frac{\partial b}{\partial y} = 4c(a-b)(w+x+y) + 4c(a-b)(x-y-z) = 4c(a-b)(w+2x-z)$$
$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial b} \cdot \frac{\partial b}{\partial z} = 4c(a-b)(x-y-z)$$

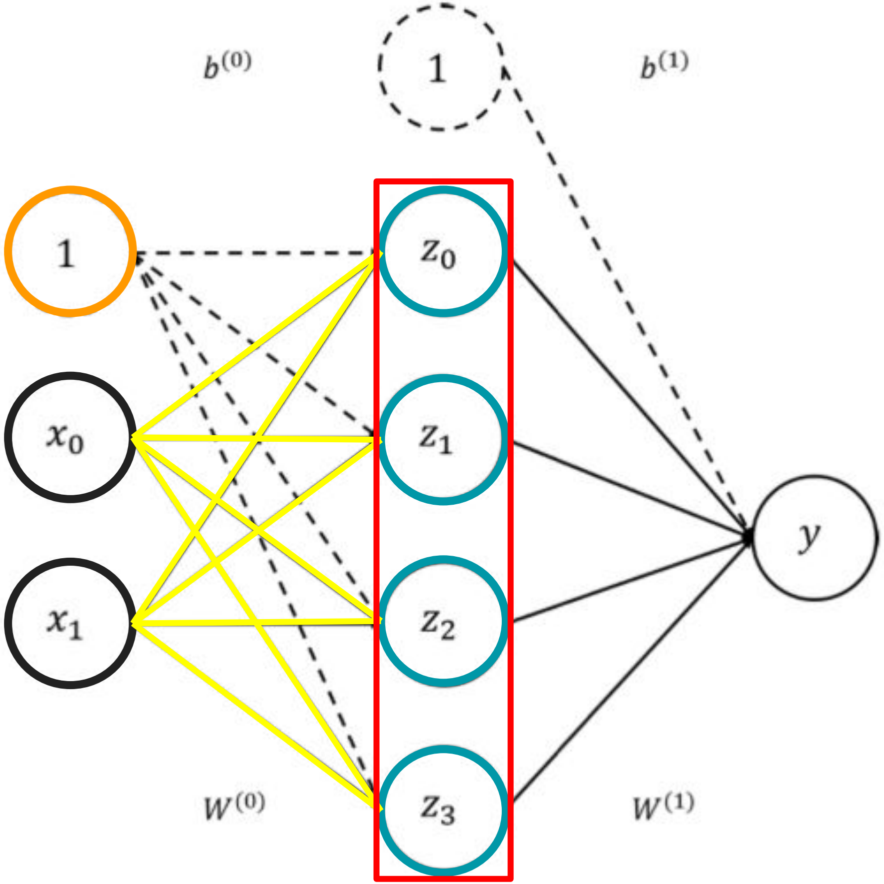
2. 1-Hidden-Layer Neural Network Gradients and Initialization

2.1. Forward and Backward pass

Consider a 1-hidden-layer neural network with a single output unit. Formally the network can be defined by the parameters $W^{(0)} \in \mathbb{R}^{h \times d}$, $b^{(0)} \in \mathbb{R}^{h}$; $W^{(1)} \in \mathbb{R}^{1 \times h}$ and $b^{(1)} \in \mathbb{R}$. The input is given by $x \in \mathbb{R}^{d}$. We will use sigmoid activation for the first hidden layer z and no activation for the output y. Below is a visualization of such a neural network with $d = 2$ and $h = 4$.

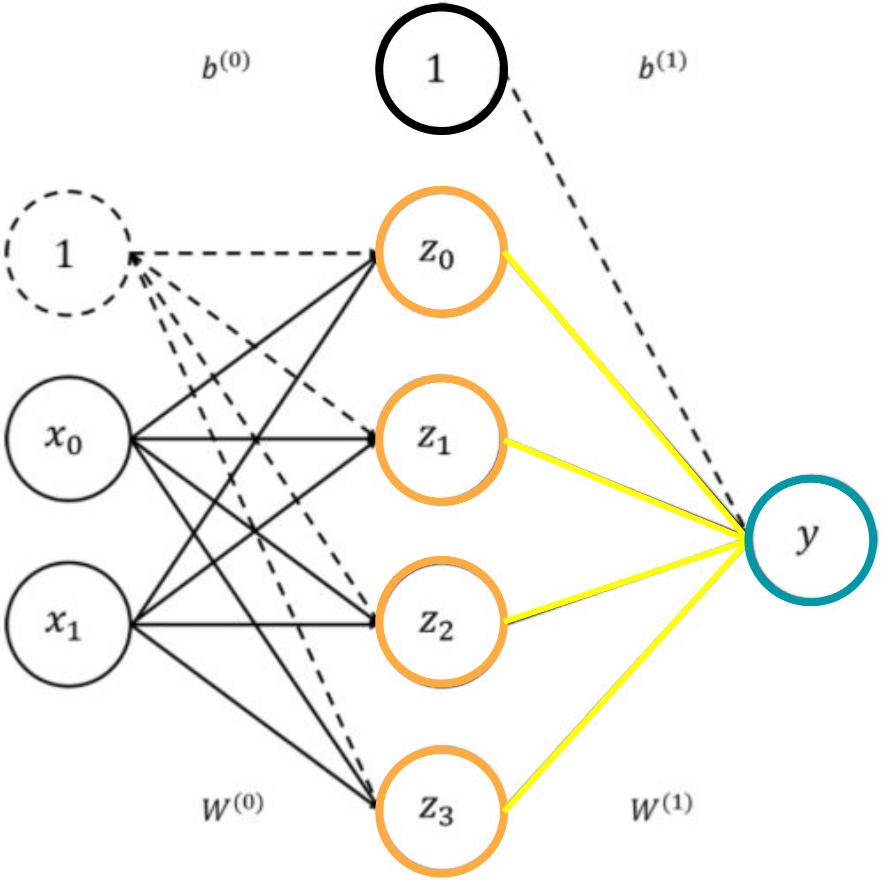


(a) Write out the forward pass for the network using $x, W^{(0)}, b^{(0)}, z, W^{(1)}, b^{(1)}, \sigma$ and y.

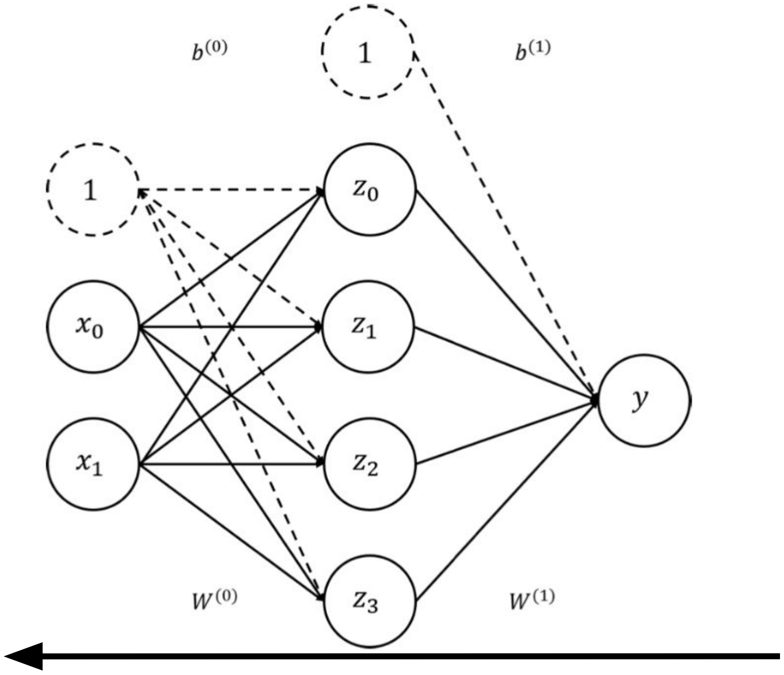


Apply sigmoid activation on z

$$z = \sigma(W^{(0)}x + b^{(0)})$$



$$y = W^{(1)}z + b^{(1)}$$



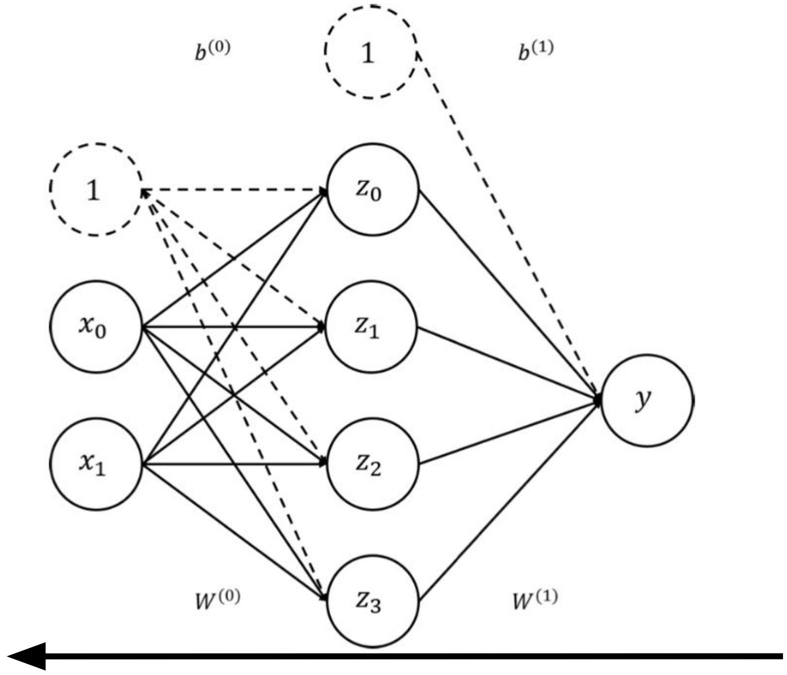
$$z = \boldsymbol{\sigma}(W^{(0)}x + b^{(0)})$$

$$y = W^{(1)}z + b^{(1)}$$

Scalar derivative $f(x) \rightarrow \frac{\mathrm{d}f}{\mathrm{d}x}$			Vector derivative $f(\mathbf{x}) \rightarrow \frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}}$		
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{B}$	$\rightarrow$	$\mathbf{B}$
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{b}$	$\rightarrow$	$\mathbf{b}$
x^2	$\rightarrow$	$2x$	$\mathbf{x}^T\mathbf{x}$	$\rightarrow$	$2\mathbf{x}$
bx^2	$\rightarrow$	$2bx$	$\mathbf{x}^T\mathbf{B}\mathbf{x}$	$\rightarrow$	$2\mathbf{B}\mathbf{x}$

(b) Find the partial derivatives of the output with respect $W^{(1)}$ and $b^{(1)}$, namely $\frac{\partial y}{\partial W^{(1)}}$ and $\frac{\partial y}{\partial b^{(1)}}$.

$$\frac{\partial y}{\partial W^{(1)}} = z \qquad \frac{\partial y}{\partial b^{(1)}} = 1$$



$$z = \sigma(W^{(0)}x + b^{(0)})$$

$$y = W^{(1)}z + b^{(1)}$$

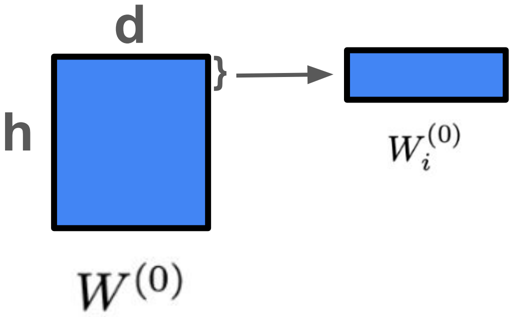
Scalar derivative			Vector derivative		
$f(x)$	$\rightarrow$	$\frac{\mathrm{d}f}{\mathrm{d}x}$	$f(\mathbf{x})$	$\rightarrow$	$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}}$
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{B}$	$\rightarrow$	$\mathbf{B}$
bx	$\rightarrow$	b	$\mathbf{x}^T\mathbf{b}$	$\rightarrow$	$\mathbf{b}$
x^2	$\rightarrow$	$2x$	$\mathbf{x}^T\mathbf{x}$	$\rightarrow$	$2\mathbf{x}$
bx^2	$\rightarrow$	$2bx$	$\mathbf{x}^T\mathbf{B}\mathbf{x}$	$\rightarrow$	$2\mathbf{B}\mathbf{x}$

(c) Now find the partial derivative of the output with respect to the output of the hidden layer z, that is $\frac{\partial y}{\partial z}$

$$\frac{\partial y}{\partial z} = W^{(1)}$$

(d) Finally find the partial derivatives of the output with respect to $W^{(0)}$ and $b^{(0)}$, that is $\frac{\partial y}{\partial W^{(0)}}$ and $\frac{\partial y}{\partial b^{(0)}}$.

$\sigma'(a) = \sigma(a) * (1 - \sigma(a))$



$$z_i = \sigma(\overset{\mathbb{R}^{1\times d}}{W_i^{(0)}}\overset{\mathbb{R}^{d\times 1}}{x} + b_i^{(0)}) \longrightarrow \frac{\partial z_i}{\partial W_i^{(0)}} = z_i(1 - z_i)x^\top$$

$$y = W^{(1)}z + b^{(1)} \longrightarrow \frac{\partial y}{\partial z_i} = W_i^{(1)}$$

$$\frac{\partial y}{\partial W_i^{(0)}} = \frac{\partial y}{\partial z_i}\frac{\partial z_i}{\partial W_i^{(0)}} = W_i^{(1)} \cdot z_i(1 - z_i)x^\top \in \mathbb{R}^{1\times d}$$

Generalizing for all h rows…

$$\frac{\partial y}{\partial W^{(0)}} = \left[{W^{(1)}}^\top \circ z \circ (1 - z)\right] x^\top \in \mathbb{R}^{h\times d}$$

(d) Finally find the partial derivatives of the output with respect to $W^{(0)}$ and $b^{(0)}$, that is $\frac{\partial y}{\partial W^{(0)}}$ and $\frac{\partial y}{\partial b^{(0)}}$.

$\sigma'(a) = \sigma(a) * (1 - \sigma(a))$

Repeat same steps for bias…

$$z_i = \sigma(W_i^{(0)}x + b_i^{(0)}) \longrightarrow \frac{\partial z_i}{\partial b_i^{(0)}} = z_i(1 - z_i)$$

$$y = W^{(1)}z + b^{(1)} \longrightarrow \frac{\partial y}{\partial z_i} = W_i^{(1)}$$

$$\frac{\partial y}{\partial b_i^{(0)}} = \frac{\partial y}{\partial z_i}\frac{\partial z_i}{\partial b_i^{(0)}} = W_i^{(1)} \cdot z_i(1 - z_i) \in \mathbb{R}$$

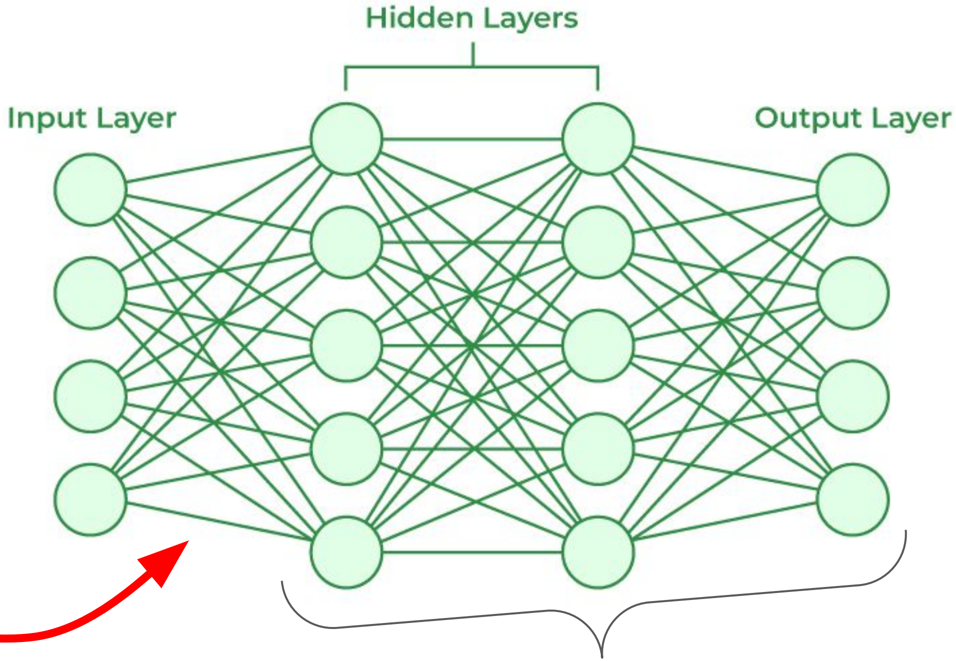
$\Big\downarrow$ Generalizing for all h rows…

$$\frac{\partial y}{\partial b^{(0)}} = {W^{(1)}}^\top \circ z \circ (1 - z) \in \mathbb{R}^h$$

Initializations

Initializations are incredibly important

What happens if we set all weights to 0 initially in a neural network?



No matter the input, all outputs will be the same…

Everything stays the same here too, no learning!

Interactive Initialization Example

IF THIS DOES NOT MAKE SENSE JUST IGNORE IT, THIS IS NOT REQUIRED

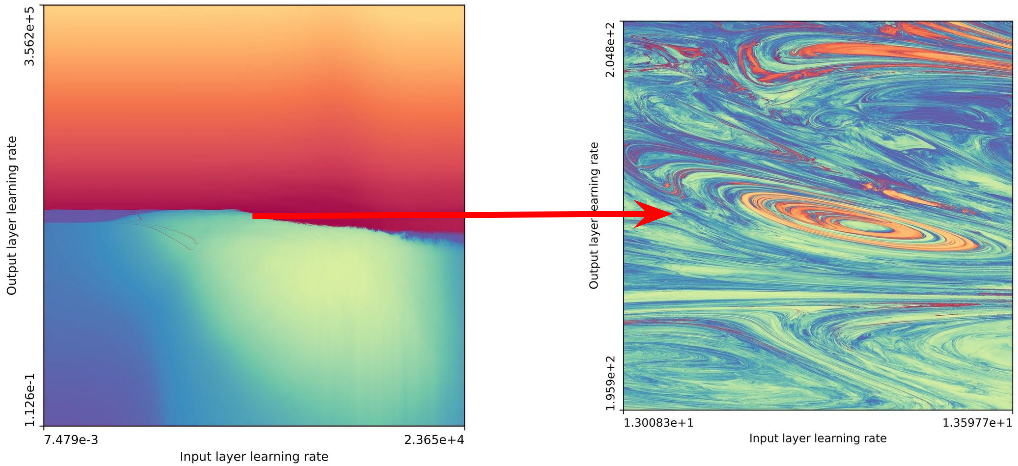
Sensitive dependence in initializations

input layer: 16 neurons,
hidden layer: 16 neurons,
output layer: 1 neuron, tanh activation

I'm not an expert in fractals or chaos theory, but thought this was super cool:

(video links in paper)

[The boundary of neural network trainability is fractal]()



Blue-green = convergence
Red-yellow = failure

Questions/Chat Time!